A High-Order Conservative Shallow Water Model on the Cubed-Sphere

Ramachandran D. Nair

(rnair@ucar.edu)

Scientific Computing Division
National Center for Atmospheric Research
Boulder, CO 80305, USA

http://www.ucar.edu/
Outline

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- Summary
DGM - Motivation

- **Advantage:**
  - Inherently conservative (Monotonic option)
  - High-order accuracy & High parallel efficiency
  - “Local” method & AMR capable

- **Potential:** Application in climate and atmospheric chemistry modeling.

- DGM may be considered as a hybrid approach combining the finite-volume and finite-element methods.

- Popular in CFD and other engineering applications (*Cockburn and Shu 1989-98, Bassi & Rebay 1997*). Global SW model (*Giraldo et al. 2002*).
DGM in Cartesian Geometry

- **2D scalar conservation law:**

\[
\frac{\partial U}{\partial t} + \nabla \cdot \vec{F}(U) = 0, \quad \text{in} \quad \Omega \times (0, T); \quad \forall (x, y) \in \Omega
\]

where \( U = U(x, y, t) \), \( \nabla \equiv \left( \partial / \partial x, \partial / \partial y \right) \), and \( \vec{F} = (F, G) \) is the flux function.

- **Domain:** The domain \( \Omega \) is partitioned into \( N_x \times N_y \) rectangular non-overlapping elements \( \Omega_{ij} \) such that

\[
\Omega_{ij} = \{(x, y) \mid x \in [x_{i-1/2}, x_{i+1/2}], \ y \in [y_{j-1/2}, y_{j+1/2}]\},
\]

for \( i = 1, 2, \ldots, N_x; \quad j = 1, 2, \ldots, N_y. \)
DGM - Weak Galerkin Formulation

- Consider an element $\Omega_{ij}$ and an approximate solution $U_h$ in the finite dimensional vector space $\mathcal{V}_h(\Omega)$.

- Multiplication of the basic equation by a test function $\varphi_h \in \mathcal{V}_h$ and integration over the element $\Omega_{ij}$ by parts, results in a weak Galerkin formulation of the problem:

$$
\frac{\partial}{\partial t} \int_{\Omega_{ij}} U_h \varphi_h \, d\Omega - \int_{\Omega_{ij}} \tilde{F}(U_h) \cdot \nabla \varphi_h \, d\Omega + \int_{\partial \Omega_{ij}} \tilde{F}(U_h) \cdot \vec{n} \varphi_h \, ds = 0,
$$

where $\tilde{F}(U_h) \cdot \vec{n}$ is analytic flux and $\vec{n}$ is the outward-facing unit normal vector on the element boundary $\partial \Omega_{ij}$.
DGM - Flux Term

- Along the boundaries of an element $\partial \Omega_{ij}$, the function $U_h$ is discontinuous.

- Therefore, the analytic flux $\mathcal{F}(U_h) \cdot \vec{n}$ must be replaced by a numerical flux $\hat{\mathcal{F}}(U_h^-, U_h^+)$

- For simplicity, the Lax-Friedrichs numerical flux is used:

$$
\hat{\mathcal{F}}(U_h^-, U_h^+) = \frac{1}{2} \left[ (\mathcal{F}(U_h^-) + \mathcal{F}(U_h^+)) \cdot \vec{n} - \alpha (U_h^+ - U_h^-) \right],
$$

- where $U_h^-$ and $U_h^+$ are the left and right limits of the discontinuous function $U_h$
- $\alpha$ is the upper bound for the absolute value of eigenvalues of the flux Jacobian $\mathcal{F}'(U)$ in the direction $\vec{n}$. 
DGM - Discretization

- **Orthogonal basis:** A modal basis set $\mathcal{B} = \{P_\ell(\xi), \ell = 0, 1, \ldots, k\}$ consists of Legendre polynomials.

- **Reference element:** Map $(x, y) \Rightarrow (\xi, \eta) \in [-1, 1] \otimes [-1, 1]$

- **Expand approximate solution** $U_{ij}$ in terms of $P_\ell(\xi) P_m(\eta)$:

  $$U_{ij}(\xi, \eta, t) = \sum_{\ell=0}^{k} \sum_{m=0}^{k} \hat{U}_{ij \ell m}(t) P_\ell(\xi) P_m(\eta) \quad \text{for} \quad -1 \leq \xi, \eta \leq 1$$

- **Evaluate the integrals using** GLL quadrature rule.

- **Solve the resulting ODE**

  $$\frac{d}{dt} U = L(U) \quad \text{in} \quad (0, T)$$
Cubed-Sphere Geometry

• The sphere is decomposed into six identical regions, using the central (gnomonic) projection (Sadourny, 1972):
  \[ x = a \tan \lambda, \quad y = a \tan \theta \sec \lambda, \quad 2a \text{ is the side of the cube.} \]

- Local coordinate systems are free of singularities
- Have identical metric terms
- Creates a non-orthogonal curvilinear coordinate system

• Metric tensor of the transformation is defined as \( G_{ij} \equiv a_i \cdot a_j, i, j \in \{1, 2\} \).

• The components of the covariant vectors \( (u_i) \) and the contravariant vectors \( (u^i) \) are related through:
  \[ u_i = G_{ij} u^j, \quad u^i = G^{ij} u_j, \quad G^{ij} = (G_{ij})^{-1} \]
Cubed-Sphere Geometry

- **Equidistant Projection:** Use \((x, y) \in [-a, a]\) as independent variables. The metric tensor of the transformation is

\[
G_{ij} = \frac{R^2}{r^4} \begin{bmatrix}
  a^2 + y^2 & -x y \\
  -x y & a^2 + x^2
\end{bmatrix}
\]

where \(R\) is the radius of the sphere, \(r^2 = a^2 + x^2 + y^2\).

- **Equiangular Projection:** Central angles \((\alpha, \beta) \in [-\pi/4, \pi/4]\) are the independent variables. The metric is

\[
G_{ij} = \frac{R^2}{\rho^4 \cos^2 \alpha \cos^2 \beta} \begin{bmatrix}
  1 + \tan^2 \alpha & -\tan \alpha \tan \beta \\
  -\tan \alpha \tan \beta & 1 + \tan^2 \beta
\end{bmatrix}
\]

where \(\rho^2 = 1 + \tan^2 \alpha + \tan^2 \beta\).
Shallow Water Equations on the Cubed-Sphere

- In curvilinear coordinates, the continuity and momentum equations for the flux form shallow water system can be written as follows (Sadourny 1972; Rancic et al. 1996)

\[
\frac{\partial}{\partial t}(\sqrt{G} h) + \frac{\partial}{\partial x^1}(\sqrt{G} u^1 h) + \frac{\partial}{\partial x^2}(\sqrt{G} u^2 h) = 0,
\]

\[
\frac{\partial u_1}{\partial t} + \frac{\partial}{\partial x^1} E = -\sqrt{G} u^2 (f + \zeta),
\]

\[
\frac{\partial u_2}{\partial t} + \frac{\partial}{\partial x^2} E = \sqrt{G} u^1 (f + \zeta),
\]

where

\[
G = \text{det}(G_{ij}), \quad E = \Phi + \frac{1}{2} (u_1 u^1 + u_2 u^2), \quad \zeta = \frac{1}{\sqrt{G}} \left[ \frac{\partial u_2}{\partial x^1} - \frac{\partial u_1}{\partial x^2} \right]
\]
DGM for SW model

- **Domain:** Each face of the cubed-sphere is partitioned into $N_e \times N_e$ rectangular non-overlapping elements $\Omega_{ij}$.

- Each element is mapped onto $[-1, 1] \otimes [-1, 1]$.

Cubed-Sphere ($N_e = 5$) with $8 \times 8$ Gauss-Lobatto-Legendre points
DG-SW : Numerical Experiment

- **Time Integration:** A third-order total variation diminishing (TVD) Runge-Kutta scheme without a filter or limiter.

- **Numerical Flux:** Lax-Friedrichs; eigenvalues of $\mathcal{F}'(U)$

  \[
  \alpha^1 = \max \left( |u^1| + \sqrt{\Phi G^{11}} \right), \quad \alpha^2 = \max \left( |u^2| + \sqrt{\Phi G^{22}} \right)
  \]

**SW Test case-1:** Solid-body rotation of a cosine-bell ($\alpha = \pi/4$)
DG-SW: Test Case-1 ($\alpha = \pi/4$)

Equidistant vs Equiangular

DGM vs SEM
DG-SW Deformational Flow:

Deforming vortex field \( \psi(\lambda', \theta', t) = 1 - \tanh \left[ \frac{\rho'(\theta')}{\gamma} \sin(\lambda' - \omega' t) \right] \)

Idealized Cyclogenesis (Doswell 1985; Nair, Côté & Satniforth, 1999). Max error is \( \mathcal{O}(10^{-6}) \).
DG-SW: Test Case-2

Steady state geostrophic flow ($\alpha = \pi/4$). Max height error is $O(10^{-6})$ m.
DG-SW: Test Case-5

Zonal flow over a mountain: \((864 \times 4 \times 4)\) grid, after 5 and 15 days of integration
DG-SW Test: Rossby-Haurwitz Wave

(DG 864x4x4): Rossby–Haurwitz Wave (Day–0)

(DG 864x4x4): Rossby–Haurwitz Wave (Day–7)

(DG 864x4x4): Rossby–Haurwitz Wave (Day–14)

(864 × 4 × 4) Grid.
Summary

• Discontinuous Galerkin Method (DGM) based flux form shallow water model has been developed on the cubed-sphere (Nair, Thomas & Loft 2004 MWR, submitted).

• The standard relative error metrics are significantly smaller for the equiangular as opposed to the equidistant projection.

• Numerical results either comparable or better than a standard spectral element method.

• DG scheme exhibits exponential convergence for SW test case-2

• DG solutions of the SW test cases are much better than those of a spectral model (Jacob-Chien et al. 1995) for a given spatial resolution.
For high-order spatial discretization, the solution do not exhibit spurious oscillation for the flow over a mountain test case.

DG model conserves mass to machine precision. Conservation of total energy and enstrophy is better preserved than the finite-volume models (Lin & Rood 1997; Thuburn 1996).

Future work: Time integration scheme, limiters. Parallel implementation of the DG model in the NCAR SE modeling framework.