A High-Order Conservative Shallow Water Model on the Cubed-Sphere

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Outline

- **Discontinuous Galerkin Method (DGM)**
  - ⋆ **Motivation & Algorithm**
  - ⋆ **DGM in Cartesian Geometry**
- **Extension of DGM to Spherical Geometry**
  - ⋆ **Cubed-Sphere**
- **Flux form SW Model**
- **Numerical experiments & Results**
- **Summary**

DGM - Motivation

- **Advantage:**
  - ⋆ **Inherently conservative (Monotonic option)**
  - ⋆ **High-order accuracy & High parallel efficiency**
  - ⋆ **"Local" method & AMR capable**
- **Potential: Application in climate and atmospheric chemistry modeling.**
- **DGM may be considered as a hybrid approach combining the finite-volume and finite-element methods.**
- **Popular in CFD and other engineering applications *(Cockburn and Shu 1989-98, Bassi & Rebay 1997)*. Global SW model (*Giraldo et al. 2002*).**

DGM in Cartesian Geometry

- **2D scalar conservation law:**

$$\frac{\partial U}{\partial t} + \nabla \cdot \vec{\mathcal{F}}(U) = 0, \quad \text{in} \quad \Omega \times (0, T); \, \forall \, (x, y) \in \Omega$$

where $U = U(x, y, t)$, $\nabla \equiv (\partial/\partial x, \partial/\partial y)$, and $\vec{\mathcal{F}} = (F, G)$ is the flux function.

- **Domain: The domain Ω is partitioned into $N_x \times N_y$ rectangular non-overlapping elements Ω_{ij} such that**

$$\Omega_{ij} = \{(x, y) \,|\, x \in [x_{i-1/2}, x_{i+1/2}], \, y \in [y_{j-1/2}, y_{j+1/2}]\},$$
$$\text{for} \quad i = 1, 2, \ldots, N_x; \quad j = 1, 2, \ldots, N_y.$$

DGM - Weak Galerkin Formulation

- **Consider an element Ω_{ij} and an approximate solution U_h in the finite dimensional vector space $\mathcal{V}_h(\Omega)$.**

- **Multiplication of the basic equation by a test function $\varphi_h \in \mathcal{V}_h$ and integration over the element Ω_{ij} by parts, results in a weak Galerkin formulation of the problem:**

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_{\Omega_{ij}} U_h\,\varphi_h\,d\Omega \quad &- \quad \int_{\Omega_{ij}} \vec{\mathcal{F}}(U_h)\,\cdot\,\nabla\varphi_h)\,d\Omega \\
&+ \quad \int_{\partial\Omega_{ij}} \vec{\mathcal{F}}(U_h)\,\cdot\,\vec{n}\,\varphi_h\,ds = 0,
\end{aligned}
$$

where $\vec{\mathcal{F}}(U_h)\cdot\vec{n}$ is analytic flux and $\vec{n}$ is the outward-facing unit normal vector on the element boundary $\partial\Omega_{ij}$.

DGM - Flux Term

- **Along the boundaries of an element $\partial\Omega_{ij}$, the function U_h is discontinuous.**
- **Therefore, the analytic flux $\mathcal{F}(U_h) \cdot \vec{n}$ must be replaced by a numerical flux $\widehat{\mathcal{F}}\left(U_h^-, U_h^+\right)$**
- **For simplicity, the Lax-Friedrichs numerical flux is used:**

$$\widehat{\mathcal{F}}(U_h^-, U_h^+) = \frac{1}{2}\left[(\mathcal{F}(U_h^-) + \mathcal{F}(U_h^+)) \cdot \vec{n} - \alpha(U_h^+ - U_h^-)\right],$$

  - $\star$ **where U_h^- and U_h^+ are the left and right limits of the discontinuous function U_h**
  - $\star$ **α is the upper bound for the absolute value of eigenvalues of the flux Jacobian $\mathcal{F}'(U)$ in the direction $\vec{n}$.**

DGM - Discretization

- **Orthogonal basis: A modal basis set** $\mathcal{B} = \{P_\ell(\xi), \ell = 0, 1, \ldots, k\}$ **consists of Legendre polynomials.**

- **Reference element: Map** $(x, y) \Rightarrow (\xi, \eta) \in [-1, 1] \otimes [-1, 1]$

- **Expand approximate solution** U_{ij} **in terms of** $P_\ell(\xi)\, P_m(\eta)$**:**

$$U_{ij}(\xi, \eta, t) = \sum_{\ell=0}^{k} \sum_{m=0}^{k} \hat{U}_{ij\ell m}(t)\, P_\ell(\xi)\, P_m(\eta) \quad \text{for} \quad -1 \leq \xi, \eta \leq 1$$

- **Evaluate the integrals using GLL quadrature rule.**

- **Solve the resulting ODE** $\boxed{\frac{d}{dt} U = L(U) \quad \text{in} \quad (0, T)}$

Cubed-Sphere Geometry

- **The sphere is decomposed into six identical regions, using the central (gnomonic) projection (*Sadourny, 1972*):**

  $\boxed{x = a\tan\lambda,\ \ y = a\tan\theta\,\sec\lambda,}$ $2a$ **is the side of the cube.**

  ⋆ **Local coordinate systems are free of singularities**
  ⋆ **have identical metric terms**
  ⋆ **creates a non-orthogonal curvilinear coordinate system**

- **Metric tensor of the transformation is defined as** $G_{ij} \equiv \mathbf{a}_i \cdot \mathbf{a}_j$**,** $i, j \in \{1, 2\}$**.**

- **The components of the covariant vectors (**u_i**) and the contravariant vectors (**u^i**) are related through:**

  $$\boxed{u_i = G_{ij}u^j,\ u^i = G^{ij}u_j, \quad G^{ij} = (G_{ij})^{-1}}$$

Cubed-Sphere Geometry

- **Equidistant Projection:** **Use** $(x, y) \in [-a, a]$ **as independent variables. The metric tensor of the transformation is**

$$G_{ij} = \frac{R^2}{r^4} \begin{bmatrix} a^2 + y^2 & -x\,y \\ -x\,y & a^2 + x^2 \end{bmatrix}$$

where R **is the radius of the sphere,** $r^2 = a^2 + x^2 + y^2$**.**

- **Equiangular Projection:** **Central angles** $(\alpha, \beta) \in [-\pi/4, \pi/4]$ **are the independent variables. The metric is**

$$G_{ij} = \frac{R^2}{\rho^4\, \cos^2\alpha\, \cos^2\beta} \begin{bmatrix} 1 + \tan^2\alpha & -\tan\alpha\, \tan\beta \\ -\tan\alpha\, \tan\beta & 1 + \tan^2\beta \end{bmatrix}$$

where $\rho^2 = 1 + \tan^2\alpha + \tan^2\beta$

Shallow Water Equations on the Cubed-Sphere

- **In curvilinear coordinates, the continuity and momentum equations for the flux form shallow water system can be written as follows *(Sadourny 1972; Rancic et al. 1996)***

$$
\begin{aligned}
\frac{\partial}{\partial t}(\sqrt{G}\,h) + \frac{\partial}{\partial x^1}(\sqrt{G}\,u^1 h) + \frac{\partial}{\partial x^2}(\sqrt{G}\,u^2 h) &= 0, \\
\frac{\partial u_1}{\partial t} + \frac{\partial}{\partial x^1} E &= -\sqrt{G}\,u^2 (f + \zeta), \\
\frac{\partial u_2}{\partial t} + \frac{\partial}{\partial x^2} E &= \sqrt{G}\,u^1 (f + \zeta),
\end{aligned}
$$

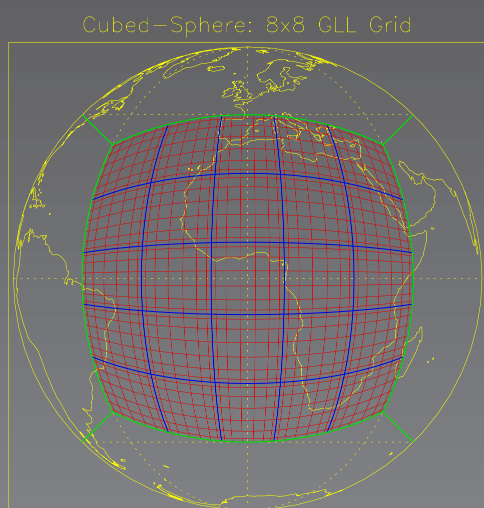
where

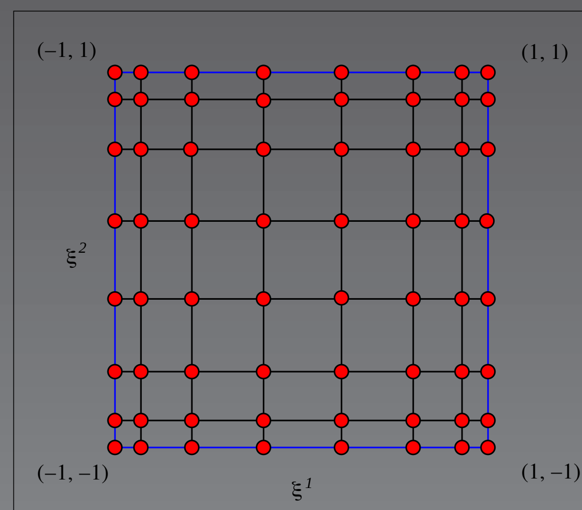
$$
G = \det(G_{ij}),\ E = \Phi + \frac{1}{2}(u_1 u^1 + u_2 u^2),\ \zeta = \frac{1}{\sqrt{G}}\left[\frac{\partial u_2}{\partial x^1} - \frac{\partial u_1}{\partial x^2}\right]
$$

DGM for SW model

- **Domain:** **Each face of the cubed-sphere is partitioned into $N_e \times N_e$ rectangular non-overlapping elements Ω_{ij}.**

- **Each element is mapped onto $[-1, 1] \otimes [-1, 1]$**





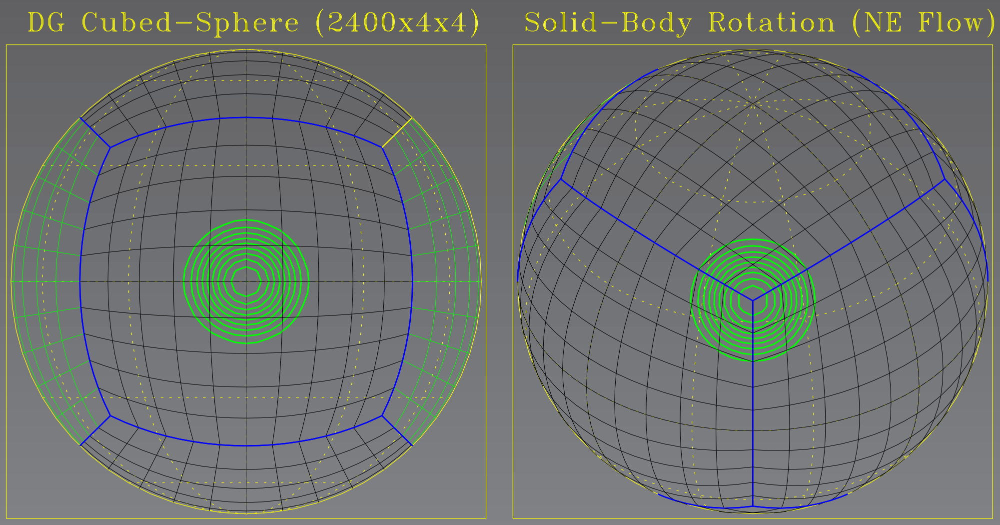
Cubed-Sphere ($N_e = 5$) with 8×8 Gauss-Lobatto-Legendre points

DG-SW : Numerical Experiment

- **Time Integration: A third-order total variation diminishing (TVD) Runge-Kutta scheme without a filter or limiter.**

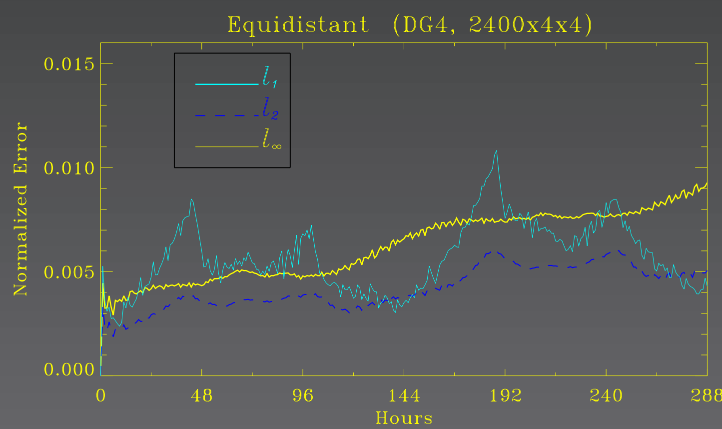
- **Numerical Flux: Lax-Friedrichs; eigenvalues of $\mathcal{F}'(U)$**

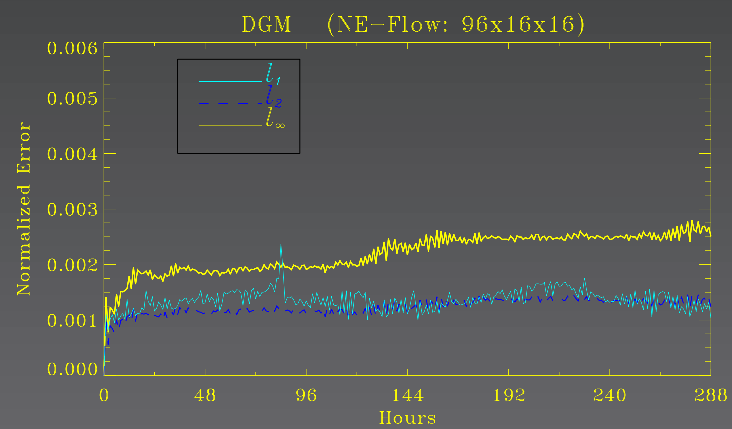
$$\alpha^1 = \max\left(|u^1| + \sqrt{\Phi\, G^{11}}\right), \quad \alpha^2 = \max\left(|u^2| + \sqrt{\Phi\, G^{22}}\right)$$

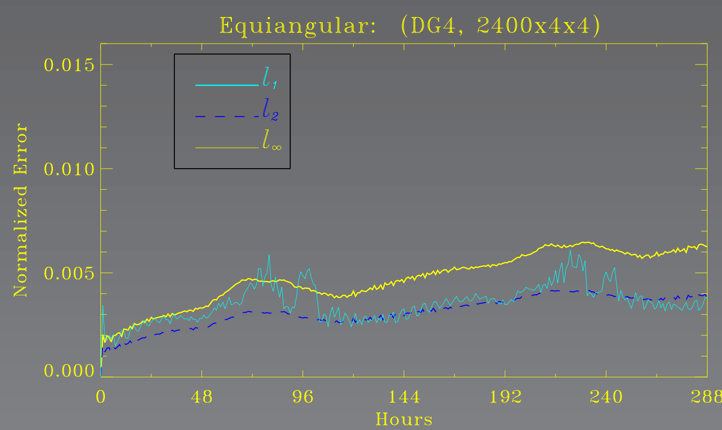


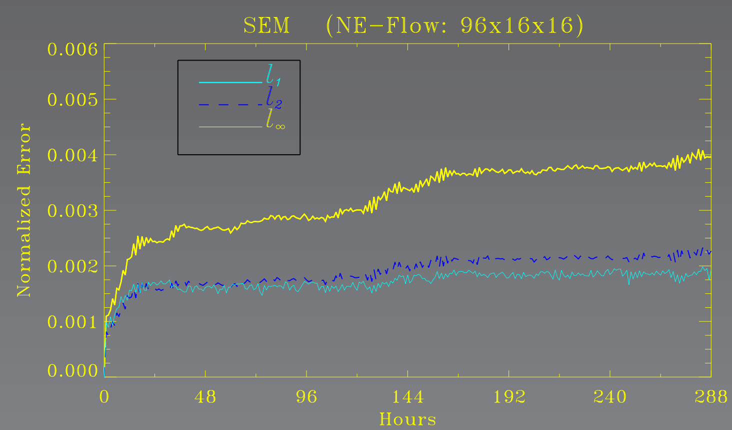
SW Test case-1: Solid-body rotation of a cosine-bell $(\alpha = \pi/4)$

DG-SW: Test Case-1 ($\alpha = \pi/4$)







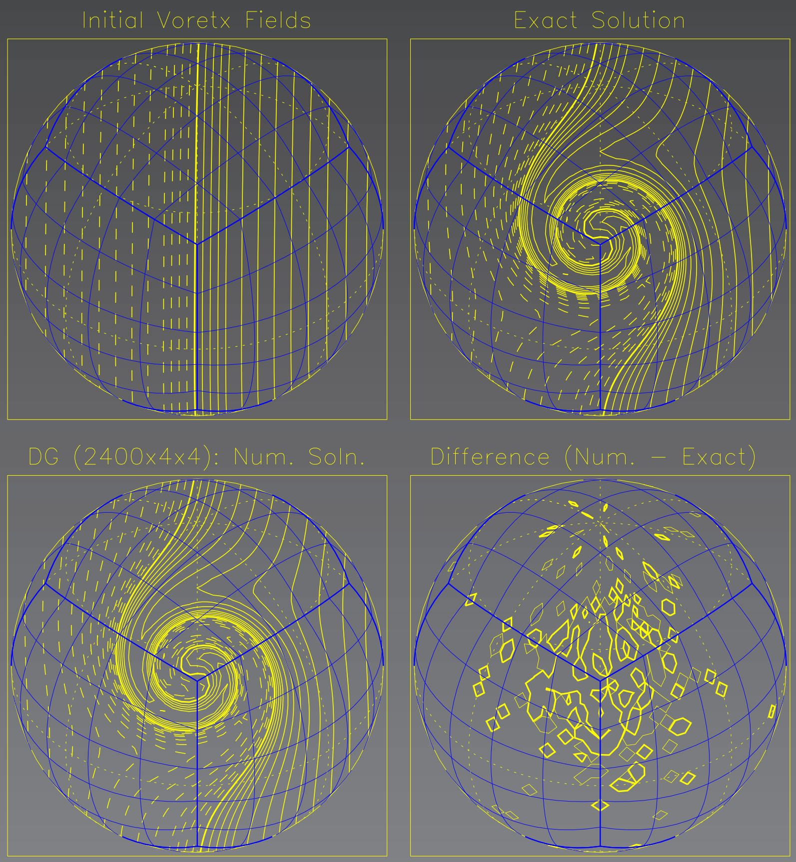


Equidistant vs Equiangular

DGM vs SEM

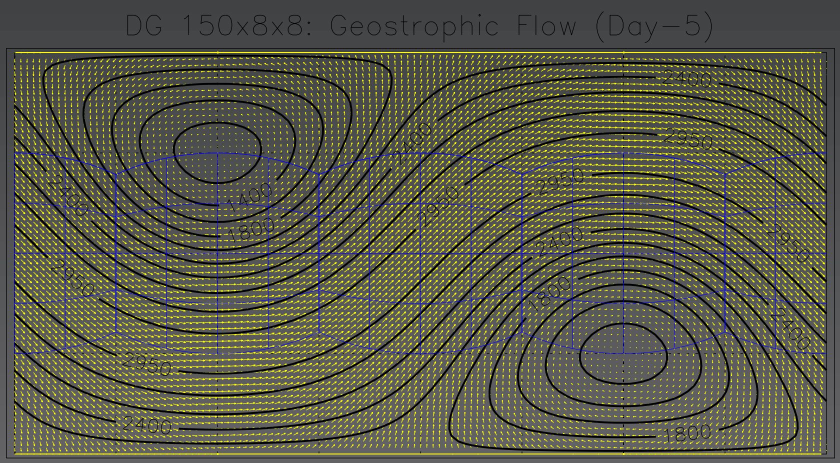
DG-SW Deformational Flow:

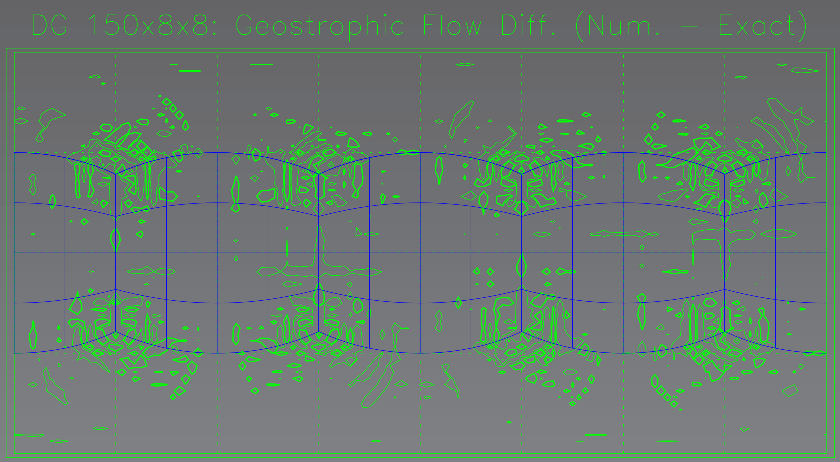
Deforming vortex field $\psi(\lambda', \theta', t) = 1 - \tanh\left[\frac{\rho'(\theta')}{\gamma}\sin(\lambda' - \omega' t)\right]$

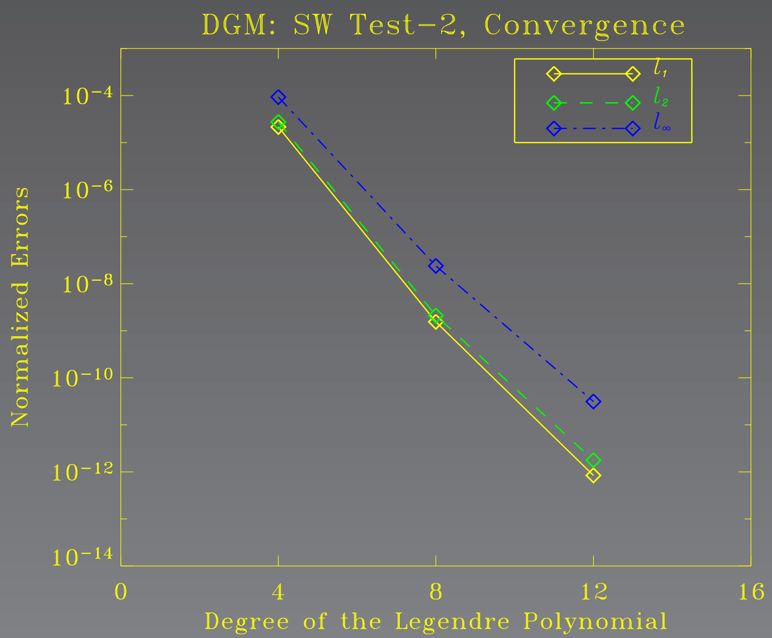


Idealized Cyclogenisis (Doswell 1985; Nair, Côté & Satniforth, 1999). Max error is $\mathcal{O}(10^{-6})$**.**

DG-SW: Test Case-2

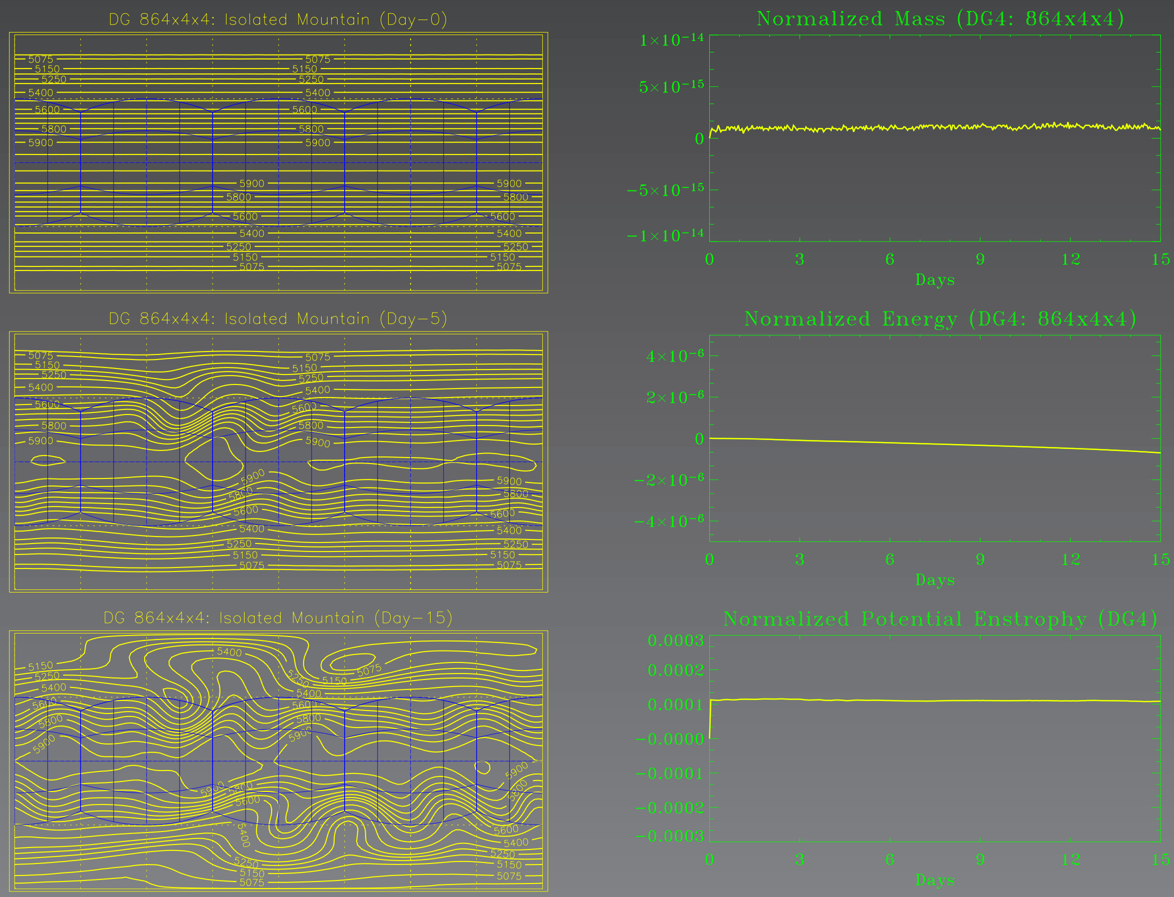






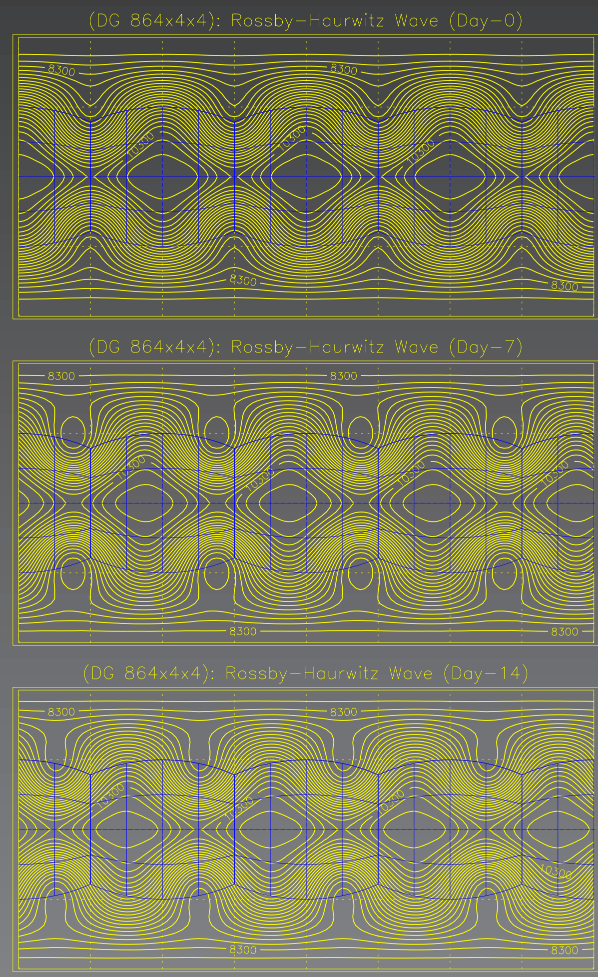
Steady state geostrophic flow $(\alpha = \pi/4)$. Max height error is $\mathcal{O}(10^{-6})$ m.

DG-SW: Test Case-5



Zonal flow over a mountain: $(864 \times 4 \times 4)$ **grid, after 5 and 15 days of integration**

DG-SW Test: Rossby-Haurwitz Wave



$(864 \times 4 \times 4)$ **Grid.**

Summary

- **Discontinuous Galerkin Method (DGM) based flux form shallow water model has been developed on the cubed-sphere (*Nair, Thomas & Loft 2004 MWR, submitted*).**

- **The standard relative error metrics are significantly smaller for the equiangular as opposed to the equidistant projection.**

- **Numerical results either comparable or better than a standard spectral element method.**

- **DG scheme exhibits exponential convergence for SW test case-2**

- **DG solutions of the SW test cases are much better than those of a spectral model *(Jacob-Chien et al. 1995)* for a given spatial resolution.**

- **For high-order spatial discretization, the solution do not exhibit spurious oscillation for the flow over a mountain test case.**

- **DG model conserves mass to machine precision. Conservation of total energy and enstrophy is better preserved than the finite-volume models (*Lin & Rood 1997; Thuburn 1996*).**

- **Future work: Time integration scheme, limiters. Parallel implementaion of the DG model in the NCAR SE modeling framework.**

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