GEOMETRIC L-PACKETS OF HOWE-UNRAMIFIED TORAL SUPERCUSPIDAL REPRESENTATIONS

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Abstract. We show that L-packets of toral supercuspidal representations arising from unramified maximal tori of p-adic groups are realized by Deligne–Lusztig varieties for parahoric subgroups. We prove this by exhibiting a direct comparison between the cohomology of these varieties and algebraic constructions of supercuspidal representations. Our approach is to establish that toral irreducible representations are uniquely determined by the values of their characters on a domain of sufficiently regular elements. This is an analogue of Harish-Chandra's characterization of real discrete series representations by their characters on regular elements of compact maximal tori, a characterization which Langlands relied on in his construction of L-packets of these representations. In parallel to the real case, we characterize the members of Kaletha's toral L-packets by their characters on sufficiently regular elements of elliptic maximal tori.

1. Introduction

This paper has several objectives, all of which are connected by the core motif that it is of significant interest to be able to recognize irreducible representations by the values of their characters on some domain. For real groups, it is the remarkable work of Harish-Chandra [HC65] that real discrete series representations are determined by their characters on the regular elements of compact maximal tori, a domain on which the character formula is very simple. This characterization was later used by Langlands [Lan89] to package these representations into L-packets. For p-adic groups, Kaletha [Kal19] recently proposed a construction of regular supercuspidal L-packets by reparametrizing (part of) Yu's construction of supercuspidal representations [Yu01] in terms of characters of certain elliptic maximal tori. As Kaletha mentions, ideally one should be able to characterize the members of these L-packets by their characters on some nice domain as in the real case; however, even the correct statement of the analogue of Harish-Chandra's result for general p-adic groups was essentially completely unknown. One of the main results of this paper (Theorem 9.1) is a resolution of this characterization problem for a class of regular supercuspidal representations corresponding to unramified elliptic maximal tori. This is a vast generalization of the unramified setting of Henniart's work on this problem for GL_n (see [Hen92] for arbitrary n and unramified extensions, [Hen93] for prime n and arbitrary extensions).

It is a folklore conjecture that every supercuspidal representation of a p-adic group is isomorphic to the compact induction of some finite-dimensional irreducible representation of a compact-modulo-center subgroup. In all known constructions of supercuspidal representations, modulo center, this compact subgroup can (essentially) be taken to be a so-called parahoric subgroup. Much of this paper is dedicated to establishing a characterization theorem at parahoric level; this characterization is significantly harder to establish than the

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above-mentioned result at the level of the p-adic group. We fix some notation: To any elliptic unramified maximal torus $\mathbf{S}$ of a connected reductive group $\mathbf{G}$ defined over a non-archimedean local field F with finite residue field $\mathbb{F}_q$, we may associate a unique point $\mathbf{x}$ in the reduced building $\mathcal{B}^{\mathrm{red}}(\mathbf{G}, F)$ together with a parahoric subgroup $G_{\mathbf{x},0} \subset G := \mathbf{G}(F)$. (Note in particular that we call G a p-adic group even though we do not require that F have characteristic zero.) We write $W_{G_{\mathbf{x},0}}(\mathbf{S})$ for the quotient of $N_{G_{\mathbf{x},0}}(\mathbf{S})$ (the normalizer group of $\mathbf{S}$ in $G_{\mathbf{x},0}$) by $S := \mathbf{S}(F)$. Our first theorem is that certain representations of $SG_{\mathbf{x},0}$ can be characterized by their trace on a special class of regular semisimple elements which we call *unramified very regular*[1].

Theorem A (Proposition 4.11 (existence), Theorem 5.17 (uniqueness))**.** *Let $\theta\colon S \to \mathbb{C}^\times$ be a toral character and assume that $p \gg 0$. There exists a unique irreducible representation τ of $SG_{\mathbf{x},0}$ such that its character Θ_τ at any unramified very regular element $\gamma \in S$ is*

$$\Theta_\tau(\gamma) = c \cdot \sum_{w \in W_{G_{\mathbf{x},0}}(\mathbf{S})} \theta({}^w\gamma)$$

for some constant $c \in \mathbb{C}^\times$ which does not depend on γ, where ${}^w\gamma := w\gamma w^{-1}$. Moreover, $c \in \{\pm 1\}$.

Completely separately and independently to the above algebraic developments, in recent years there has been a push towards constructing supercuspidal representations *geometrically* using constructions analogous to Deligne–Lusztig varieties for finite groups of Lie type. Central to this picture is Lusztig's work [Lus04] on such varieties for reductive groups over finite rings in equal characteristic, Stasinski's subsequent work [Sta09] for mixed characteristic, and the first author's joint work with Ivanov [CI21b] generalizing these works to arbitrary parahoric subgroups $G_{\mathbf{x},0}$ associated to unramified maximal tori. It is expected that Lusztig's conjecture on loop Deligne–Lusztig constructions for p-adic groups [Lus79] is very closely related to the parahoric picture, as demonstrated in [Boy12, CI21a] in the setting of GL_n and its inner forms. Among other things, the present paper resolves a basic and major gap in this geometric program: we prove that the irreducible representations of $SG_{\mathbf{x},0}$ arising from the cohomology of parahoric Deligne–Lusztig varieties indeed compactly induce to supercuspidal representations of G.

Following [CI21b], to every positive integer r, one can construct a smooth, separated, finite-type affine $\overline{\mathbb{F}}_q$-scheme X_r with a natural action by $G_{\mathbf{x},0} \times S_0$ where $S_0 = S \cap G_{\mathbf{x},0}$. This action can be extended by the center $Z_{\mathbf{G}}$ so that for any depth-r character $\theta\colon S \to \mathbb{C}^\times$, the corresponding isotropic subspace $H_c^*(X_r)[\theta] := H_c^*(X_r, \overline{\mathbb{Q}}_\ell)[\theta]$ of the cohomology $H_c^*(X_r, \overline{\mathbb{Q}}_\ell) := \sum_i (-1)^i H_c^i(X_r, \overline{\mathbb{Q}}_\ell)$ is in fact a (virtual) representation of $Z_{\mathbf{G}} G_{\mathbf{x},0} = SG_{\mathbf{x},0}$. (We note that because $\mathbf{S}$ is elliptic and unramified, we have $S = Z_{\mathbf{G}} S_0$.)

Theorem B (Theorems 7.2, 8.2, 8.3)**.** *Let $\theta\colon S \to \mathbb{C}^\times$ be 0-toral*[2] *of depth $r > 0$ and assume $p \gg 0$.*

(i) The compact induction $\text{c-Ind}_{SG_{\mathbf{x},0}}^G(H_c^*(X_r)[\theta])$ *is an irreducible supercuspidal representation of G.*

[1] Modulo the center of G, this notion agrees with Henniart's notion of very regular elements in the unramified elliptic torus of GL_n [Hen92] and with Chan–Ivanov's generalization to other unramified tori in p-adic groups in general [CI21b]. See Definition 4.2.

[2] This condition arises naturally for geometric reasons in [Lus04] and in subsequent work by others; in these past purely geometric investigations, *0-toral* is called *regular*. In this paper, *0-toral* is the same as in [FS21]. The reader is warned that *0-toral* is called *toral* in Reeder [Ree08] and DeBacker–Spice [DS18], and that in the present paper, *toral* is a much more general notion, see Definition 3.7.

(ii) The correspondence $(\mathbf{S}, \theta) \mapsto \text{c-Ind}_{SG_{\mathbf{x},0}}^{G}(H_c^*(X_r)[\theta])$ *preserves stability and* X_r *gives a geometric realization of 0-toral supercuspidal L-packets à la DeBacker–Spice* [DS18].

Allow us to immediately spoil the punch line relating this result and the discussion of characterizations of representations à la Harish-Chandra: Theorem B is an application of (a more precise version of) Theorem A.

We mention that when $r = 0$, the variety X_r is a classical Deligne–Lusztig variety and the conclusions of Theorem B are true for depth-0 characters θ in general position: (i) is due to Moy–Prasad [MP96], and (ii) is due to DeBacker–Reeder [DR09] and Kazhdan–Varshavsky [KV06].

For $r > 0$, it was proved in [CI21b] that $H_c^*(X_r, \overline{\mathbb{Q}}_\ell)[\theta]$ is an irreducible representation of $SG_{\mathbf{x},0}$. The additional assertion in Theorem B(i) that its compact induction to G is irreducible (and therefore supercuspidal) has been studied by various people in special cases—for inner forms of GL_n [CI21a, CI19] and for $\mathbf{x}$ hyperspecial, r odd [CS17]—by techniques totally different to ours (see Sections 7.3, 7.4 for further discussion). We remark that the assumption $p \gg 0$ appearing in Theorems A and B originates from two distinct sources: from constructions of supercuspidal representations (à la [Yu01, Kal19]) and from a technical part of the proof of our characterization theorem (Theorem A). The condition needed for our strategy is quite mild—we will prove in subsequent work that, at least for Coxeter tori, the latter assumption is weaker than the former assumption. In particular, our approach specialized to GL_n relaxes the $p > n$ assumption in [CI19] to $p > 2$ (see Section 7.3). On the other hand, there are two settings in which our Theorem B(i) falls short of existing results in the literature: for division algebras, this irreducibility was established for arbitrary θ with trivial Weyl stabilizer and all p in [Cha20, Theorem 7.1.2], and for general inner forms of GL_n, this irreducibility was established for 0-toral θ and all p in [CI21a, Theorem 12.1], both via purely geometric techniques (see Remark 7.10 for more details).

We actually prove something stronger than the supercuspidality assertion of Theorem B(i): we explicitly describe the supercuspidal $\text{c-Ind}_{SG_{\mathbf{x},0}}^{G}(H_c^*(X_r)[\theta])$ in terms of Yu's construction and Kaletha's reparametrization. This resolves a generalization of a question of Lusztig on comparing the representations in [Lus04] with non-cohomological constructions. Supercuspidal representations in the 0-toral setting had already been constructed and parametrized by Adler [Adl98]; our choice to write our paper within Yu's and Kaletha's framework is in anticipation of future work relaxing the genericity assumptions (toral, 0-toral) on θ.

Now let us explain the content of Theorem B(ii) in the context of past works. Following the construction of discrete series L-packets for real groups [Lan89] and of depth-0 L-packets of p-adic groups [DR09, KV06], one could extrapolate that for supercuspidal representations parametrized by characters θ of elliptic maximal tori $\mathbf{S}$, L-packets should be parametrized by stable conjugacy classes of $(\mathbf{S}, \theta)$. Using Adler's parametrization $(\mathbf{S}, \theta) \mapsto \pi_{(\mathbf{S},\theta)}$ of 0-toral supercuspidal representations, Reeder [Ree08] verified constancy of the formal degree on this packet of supercuspidals in the case that $\mathbf{S}$ is unramified. Later, DeBacker–Spice [DS18], still working in the setting that $\mathbf{S}$ is unramified, proved that this packet of supercuspidals fails (!) to satisfy *stability* (see Section 8 for more details); to make it stable, they prove that one must instead consider the twisted parametrization $(\mathbf{S}, \theta) \mapsto \pi_{(\mathbf{S},\theta\cdot\varepsilon[\theta])}$ by a quadratic character $\varepsilon[\theta]$ which depends on $(\mathbf{S}, \theta)$. This stability result has been generalized to 0-toral supercuspidals corresponding to tamely ramified $\mathbf{S}$ by Kaletha [Kal19], whose theory of regular supercuspidal representations develops a parametrization of a much larger class of supercuspidals in terms of $(\mathbf{S}, \theta)$ and also demands a generalized twisting character $\varepsilon[\theta]$.

The contribution of Theorem B to this picture is that $\text{c-Ind}_{SG_{\mathbf{x},0}}^{G}(H_c^*(X_r)[\theta]) \cong \pi_{(\mathbf{S},\theta\cdot\varepsilon[\theta])}$, so that in terms of the cohomologically arising parametrization of these supercuspidals, *no external twisting* is required to obtain a set of supercuspidals *satisfying stability* from a stable conjugacy class of $(\mathbf{S},\theta)$. We emphasize this point: the geometry seems to innately know about the automorphic side of the local Langlands correspondence.

1.1. **Outline of the paper.** A subtle point throughout this paper is taking stock of what assumptions one needs on p. For the most part, we have chosen to work in the greatest generality possible for each ingredient going into this paper, especially so as to illustrate the reasons various small primes are excluded. We collect a summary of these assumptions in Section 2.

In Section 3, we recall Yu's construction of supercuspidal representations and Kaletha's theory of regular supercuspidal representations. In particular, we recall how to associate to a tame elliptic regular pair $(\mathbf{S},\phi)$ a representation ${}^{\circ}\tau_d$ of $SG_{\mathbf{x},0}$ whose compact induction is the irreducible supercuspidal representation $\pi_{(\mathbf{S},\phi)}$[3]. We warn the reader that in the literature (for example, but not limited to, [Yu01, AS09]), it is more popular to work with a representation of the full stabilizer $G_{\mathbf{x}}$ of the point $\mathbf{x}$; it takes some care to work on $SG_{\mathbf{x},0} \subseteq G_{\mathbf{x}}$ instead, which we need to do for geometrically motivated reasons later. We additionally relax the ellipticity assumption on Kaletha's Howe factorization of tame elliptic pairs (Section 3.3) and use this to state a geometric conjecture (Conjecture 6.5) later in the paper.

Starting in Section 4, we assume that $\mathbf{S}$ is unramified. Sections 4 and 5 culminate in two characterization theorems for toral characters—one for representations of $SG_{\mathbf{x},0+}$ (Theorem 5.13) and one for representations of $SG_{\mathbf{x},0}$ (Theorem 5.17, presented as Theorem A in the Introduction). Of central importance in our analysis is a class of regular semisimple elements of G called *unramified very regular elements* (à la [CI21b]); we denote by S_{vreg} the set of unramified very regular elements contained in S. After reframing unramified very regular elements in the context of *normal r-approximations* (à la [AS08]), we prove the main result of Section 4: a simple and explicit character formula for the $SG_{\mathbf{x},0}$-representation ${}^{\circ}\tau_d$ on the unramified very regular locus of $SG_{\mathbf{x},0}$ (Proposition 4.11).

In Section 5, we only work with tame elliptic regular pairs $(\mathbf{S},\phi)$ where $\mathbf{S}$ is unramified and ϕ is toral. We prove our characterization theorems in this section. The linchpin that makes our approach possible is the seemingly innocent Lemma 5.12 (see Lemma 5.15 for the analogous argument for $SG_{\mathbf{x},0}$), whose content is the surprisingly simple and powerful trick that by using a telescoping series, one can show that character values on unramified very regular elements determine the representation on a pro-p subgroup. For all the arguments in this section, we need to assume that S_{vreg} generates S as a group (the assumption ($\mathfrak{vreg}$); see Section 5.2). We show in Section 5.1 that this is the case if a certain inequality ($\star$) related to a density of the unramified very regular elements holds, and that ($\star$) holds if $q \gg 0$. Moreover, the bound on q can be reduced to a calculation on reductive groups over finite fields because of a transferring trick (Lemma 5.6).

Our focus shifts in Section 6, where we recall what is known about the $\overline{\mathbb{F}}_q$-schemes X_r and their cohomology. In this section, we make no assumptions on p or on the ellipticity of $\mathbf{S}$. We recall the Drinfeld stratification [CI21c], which consists of subvarieties $X_r^{(\mathbf{L})} \subset X_r$ indexed by certain twisted Levi subgroups $\mathbf{L}$ containing $\mathbf{S}$. We conjecture (Conjecture 6.5)

[3]As mentioned above, we will show that the geometric representation arising from an unramified 0-toral pair $(\mathbf{S},\theta)$ coincides with the algebraic representation arising from the twisted pair $(\mathbf{S},\theta\cdot\varepsilon[\theta])$. This is the reason why a different symbol ϕ is used here; ϕ is intended to be taken to be $\theta\cdot\varepsilon[\theta]$ eventually.

that the Drinfeld stratification is the geometric version of the "stratification" on the set of regular supercuspidal representations given by the 0th piece $\mathbf{G}^0$ of the Howe factorization.

We prove our main results about the cohomology of X_r and its relation to supercuspidal representations in Section 7; note that supercuspidality corresponds to ellipticity of $\mathbf{S}$. In our framework, this comparison is simply an application of Section 5 to the cohomological representations discussed in Section 6. We compare $H_c^*(X_r)[\theta]$ to the $SG_{\mathbf{x},0}$-representation ${}^\circ\tau_d$ in Theorem 7.2 and prove Conjecture 6.5 in the setting $\mathbf{L} = \mathbf{S}$ in Theorem 7.6. We note that since we obtain these results as corollaries of the characterization theorems, there is an intriguing mystery surrounding the geometric representations $H_c^*(X_r)[\theta]$ for the small p excluded by Kaletha's theory of regular supercuspidal representations [Kal19, Kal21]. We make some comments about this in Section 7.4. In Section 7.3, we focus on the setting $\mathbf{G} = \mathrm{GL}_n$: we explicitly calculate the twisting character $\varepsilon[\theta]$ and show that a stronger version of the geometrically-proved supercuspidality results of [CI19] follows as a special case (Corollary 7.9) of our comparison theorem (Theorem 7.2). For convenience, we include a diagram depicting the main structure of the results needed to prove our comparison theorems (Theorems 7.2, 7.5):

$$\begin{array}{cccc}
SG_{\mathbf{x},0} & {}^\circ\tau_d & \xlongequal[\text{(Theorem 5.17)}]{\text{Theorem 7.2}} & |H_c^*(X_r, \overline{\mathbb{Q}}_\ell)[\theta]| \\
\cup & \uparrow & & \uparrow \text{Theorem 7.6 | (Conjecture 6.5)} \\
SG_{\mathbf{x},0+} & \mathrm{Ind}_{{}^\circ K^d}^{SG_{\mathbf{x},0+}} {}^\circ\rho_d' \otimes \phi_d & \xlongequal[\text{(Theorem 5.13)}]{\text{Theorem 7.5}} & |H_c^*(X_r \cap SG_{\mathbf{x},0+}, \overline{\mathbb{Q}}_\ell)[\theta]| \\
\cup & \uparrow & & \\
{}^\circ K^d & {}^\circ\rho_d' \otimes \phi_d & &
\end{array}$$

Here, the dashed vertical arrows indicate induction; these representations, all of which appear in Yu's construction, are recalled in Sections 3.1, 3.4. The horizontal equalities between the "algebraic" and "geometric" columns hold by the indicated theorems (Theorems 7.2 and 7.5), each of which is a direct application of the parenthetically indicated characterization theorems (Theorems 5.17 and 5.13).

In Section 8 we see the implications of our comparison theorem in the context of the local Langlands correspondence. We discuss Kaletha's construction of L-packets for 0-toral supercuspidal representations and use our comparison to deduce (Theorem 8.2) that L-packets of 0-toral supercuspidal representations associated to unramified $\mathbf{S}$ are realized by the natural correspondence arising via the cohomology of X_r. This yields Theorem 8.3, which is presented in the Introduction as Theorem B(ii).

Finally, in Section 9, we prove that toral supercuspidal representations associated to unramified $\mathbf{S}$ are determined by their characters on S_{vreg} (Theorem 9.1). The structure of this argument has a similar flavor to the parahoric-level characterization theorems of Section 5, but neither section implies the other logically. As mentioned in the Introduction, Theorem 9.1 is a p-adic analogue of Harish-Chandra's characterization of discrete series representations of real groups, and is the first of its kind at this level of generality. In particular, it allows one to characterize Kaletha's construction of these L-packets purely in terms of their character values on very regular elements of S.

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Contents

2. Notations and assumptions

Let F be a non-archimedean local field with finite residue field $\mathcal{O}_F/\mathfrak{p}_F \cong \mathbb{F}_q$ of prime characteristic p, where we write $\mathcal{O}_F$ and $\mathfrak{p}_F$ for the ring of its integers and the maximal ideal, respectively. We let F^{ur} denote the completion of the maximal unramified extension of F. We write Γ_F for the absolute Galois group of F.

For an algebraic variety $\mathbf{J}$ over F, we denote the set of its F-valued points by J. When $\mathbf{J}$ is an algebraic group, we write $\mathbf{Z_J}$ for its center.

Let us assume that $\mathbf{J}$ is a connected reductive group over F. We follow the notation around Bruhat–Tits theory used by [AS08, AS09, DS18]. (See, for example, [AS08, Section 3.1] for details.) Especially, $\mathcal{B}(\mathbf{J}, F)$ (resp. $\mathcal{B}^{\mathrm{red}}(\mathbf{J}, F)$) denotes the enlarged (resp. reduced) Bruhat–Tits building of $\mathbf{J}$ over F. For a point $\mathbf{x} \in \mathcal{B}(\mathbf{J}, F) = \mathcal{B}^{\mathrm{red}}(\mathbf{J}, F) \times X_*(\mathbf{Z_J})_{\mathbb{R}}$, we write $\bar{\mathbf{x}}$ for the image of $\mathbf{x}$ in $\mathcal{B}^{\mathrm{red}}(\mathbf{J}, F)$, and $J_{\bar{\mathbf{x}}}$ for the stabilizer of $\bar{\mathbf{x}}$ in J. We define $\widetilde{\mathbb{R}}$ to be the set $\mathbb{R} \sqcup \{r+ \mid r \in \mathbb{R}\} \sqcup \{\infty\}$ with its natural order. Then for any $r \in \widetilde{\mathbb{R}}$ we can consider the r-th Moy–Prasad filtration $J_{\mathbf{x},r}$ of J with respect to the point $\mathbf{x}$. For any $r, s \in \widetilde{\mathbb{R}}_{\geq 0}$ satisfying $r < s$, we write $J_{\mathbf{x},r:s}$ for the quotient $J_{\mathbf{x},r}/J_{\mathbf{x},s}$. Recall that $J_{\mathbf{x},0:0+}$ can be regarded as the set $\mathbb{J}(\mathbb{F}_q)$ of $\mathbb{F}_q$-valued points of a connected reductive group $\mathbb{J}$ defined over $\mathbb{F}_q$ (such a group $\mathbb{J}$ can be realized as the reductive quotient of the special fiber of the parahoric subgroup scheme attached to $\mathbf{x}$, see [MP96, Section 3.2]).

2.1. Assumptions on F and $\mathbf{G}$.

Let $\mathbf{G}$ be a tamely ramified connected reductive group over F, that is, $\mathbf{G}$ is a connected reductive group defined over F which splits over a tamely ramified extension of F. Unless otherwise stated, we will assume that p is odd, p is not bad for $\mathbf{G}$ (in the sense of Springer–Steinberg, see [SS70, Section 4]), and that $p \nmid |\pi_1(\mathbf{G}_{\mathrm{der}})|$ and $p \nmid |\pi_1(\widehat{\mathbf{G}}_{\mathrm{der}})|$. There are a few sections in the paper where we either relax or strengthen our assumptions on F; we specify these subsections here:

- Sections 3.1, 3.4 hold with the relaxed assumption that p is odd, but this is inconsequential for us.
- Section 3.3 holds without any assumption on p except for Lemma 3.8.
- In Section 4.2, we assume that the condition $(\mathbf{Gd}^G)$ of [AS08, Definition 6.3] is satisfied by certain unramified maximal tori of $\mathbf{G}$. This assumption is necessary for appealing to the theory of Adler–Spice on the character formula for tame supercuspidal representations. It is known that this assumption is satisfied if p does not divide the order of the absolute Weyl group of $\mathbf{G}$ (Remark 4.4).
- In Sections 5.2, 5.3, we additionally assume ($\mathfrak{vreg}$). This assumption is satisfied when an inequality ($\star$) related to the size of the residue field of F holds. This additional assumption is also needed in Sections 7, 8, 9, as these later sections rely on various arguments presented in Sections 5.2, 5.3.
- Section 6 recalls geometric constructions of representations of parahoric subgroups and holds with no assumptions on p. The discrepancy between this and the additional assumptions on p needed in other sections is an interesting point; we make some remarks about this in Section 7.4.

3. Tame supercuspidal representations

In this section, we first briefly review Yu's construction of tame supercuspidal representations (see [Yu01]; for an exposition, [AS09, Section 2]). Then, we summarize Kaletha's result ([Kal19, Section 3.4]) on a reparametrization of tame supercuspidal representations, which is based on the so-called Howe factorization for (certain) characters of elliptic maximal tori. We recall a part of Kaletha's Howe factorization process in Section 3.3; we will use this to state a geometric conjecture later (Conjecture 6.5). We finish with a discussion of passing from the full stabilizer $G_{\bar{\mathbf{x}}}$ to the smaller group $SG_{\mathbf{x},0}$ in Section 3.4, especially establishing some notation that will be used throughout the paper.

3.1. **Yu's construction of tame supercuspidal representations.** The constructions of this subsection hold with the relaxed assumption that p is odd. In [Yu01], Yu introduced the notion of a *cuspidal* $\mathbf{G}$*-datum* and to each such datum attached an irreducible supercuspidal representation of G. Recall that a cuspidal $\mathbf{G}$-datum is a quintuple

$$\Sigma = (\vec{\mathbf{G}}, \vec{\phi}, \vec{r}, \mathbf{x}, \rho_0')$$

consisting of the following objects:

- $\vec{\mathbf{G}}$ is a sequence $\mathbf{G}^0 \subsetneq \mathbf{G}^1 \subsetneq \cdots \subsetneq \mathbf{G}^d = \mathbf{G}$ of tame twisted Levi subgroups (i.e., each $\mathbf{G}^i$ is a subgroup of $\mathbf{G}$ which is defined over F and becomes a Levi subgroup of $\mathbf{G}$ over a tamely ramified extension of F) such that $\mathbf{Z}_{\mathbf{G}^0}/\mathbf{Z}_{\mathbf{G}}$ is anisotropic,
- $\mathbf{x}$ is a point of $\mathcal{B}(\mathbf{G}^0, F)$ whose image $\bar{\mathbf{x}}$ in $\mathcal{B}^{\mathrm{red}}(\mathbf{G}^0, F)$ is a vertex,
- $\vec{r}$ is a sequence $0 \leq r_0 < \cdots < r_{d-1} \leq r_d$ of real numbers such that $0 < r_0$ when $d > 0$,
- $\vec{\phi}$ is a sequence $(\phi_0, \ldots, \phi_d)$ of characters ϕ_i of G^i satisfying
  - for $0 \leq i < d$, ϕ_i is $\mathbf{G}^{i+1}$-generic of depth r_i at $\mathbf{x}$, and
  - for $i = d$,
$$\begin{cases} \mathrm{depth}_{\mathbf{x}}(\phi_d) = r_d & \text{if } r_{d-1} < r_d, \\ \phi_d = \mathbb{1} & \text{if } r_{d-1} = r_d, \end{cases}$$
- ρ_0' is an irreducible representation of $G^0_{\bar{\mathbf{x}}}$ whose restriction to $G^0_{\mathbf{x},0}$ contains the inflation of a cuspidal representation of the quotient $G^0_{\mathbf{x},0:0+}$.

We note that $\mathcal{B}^{\mathrm{red}}(\mathbf{G}^0, F)$ can be regarded as a subset of $\mathcal{B}^{\mathrm{red}}(\mathbf{G}^i, F)$ for any $0 \leq i \leq d$ thanks to the assumption $\mathbf{Z}_{\mathbf{G}^0}/\mathbf{Z}_{\mathbf{G}}$ (see [Yu01, Remark 3.4]). In particular, we may regard $\bar{\mathbf{x}} \in \mathcal{B}^{\mathrm{red}}(\mathbf{G}^0, F)$ as a point of $\mathcal{B}^{\mathrm{red}}(\mathbf{G}^i, F)$ for any $0 \leq i \leq d$.

Following [Yu01], the tame supercuspidal representation π_Σ associated to Σ is constructed as follows. We first put

$$(s_0, \ldots, s_d) := \Big(\frac{r_0}{2}, \ldots, \frac{r_d}{2}\Big)$$

and define the subgroups K^i, J^i, and J^i_+ of G for $1 \leq i \leq d$ by

$$K^i := G^0_{\bar{\mathbf{x}}}(G^0, \ldots, G^i)_{\mathbf{x},(0+,s_0,\ldots,s_{i-1})},$$

$$J^i := (G^{i-1}, G^i)_{\mathbf{x},(r_{i-1},s_{i-1})},$$

$$J^i_+ := (G^{i-1}, G^i)_{\mathbf{x},(r_{i-1},s_{i-1}+)},$$

where the right-hand sides denote the subgroups associated to pairs consisting of a tame twisted Levi sequence and an admissible sequence (see [Yu01, Sections 1 and 2]). Note that we have $K^{i+1} = K^i J^{i+1}$. For $i = 0$, we put

$$K^0 := G^0_{\bar{\mathbf{x}}}.$$

Then we construct a representation ρ'_{i+1} of K^{i+1} from ρ'_i of K^i inductively in the following manner. By investigating the quotient J^i/J^i_+ (which has a symplectic structure derived from the character ϕ_{i-1}), we obtain a finite Heisenberg group as a quotient of the group J^i. Then, as a consequence of the Stone–von Neumann theorem (together with the liftability of an associated projective representation to a linear representation), we obtain a Heisenberg–Weil representation $\tilde{\phi}_i$ of the semi-direct product $G^i_{\bar{\mathbf{x}}} \ltimes J^{i+1}$. The tensor representation

$$(\tilde{\phi}_i|_{K^i\ltimes J^{i+1}}) \otimes \big((\rho'_i \otimes \phi_i|_{K^i}) \ltimes \mathbb{1}\big)$$

of $K^i \ltimes J^{i+1}$ descends to $K^iJ^{i+1} = K^{i+1}$ (factors through the canonical map $K^i \ltimes J^{i+1} \twoheadrightarrow K^iJ^{i+1}$), and we define the representation ρ'_{i+1} of K^{i+1} to be the descended one. We define

$$\pi_\Sigma := \text{c-Ind}^G_{K^d}\, \rho'_d \otimes \phi_d.$$

This representation is irreducible [Yu01, Fin21a] and hence supercuspidal. The irreducible supercuspidal representations of G obtained from cuspidal $\mathbf{G}$-data in this way are called *tame supercuspidal representations*.

We also recall the definitions of a few more groups and representations which will be needed later (for describing the Adler–Spice character formula in Section 4):

$$K_{\sigma_i} := G^{i-1}_{\bar{\mathbf{x}}} G^i_{\mathbf{x},0+} \quad (K_{\sigma_0} := G^0_{\bar{\mathbf{x}}}),$$

$$\tilde{\rho}'_i := \mathrm{Ind}_{K^i}^{G^{i-1}_{\bar{\mathbf{x}}} G^i_{\mathbf{x},s_{i-1}}}\, \rho'_i,$$

$$\sigma_i := \mathrm{Ind}_{K^i}^{K_{\sigma_i}}\, \rho'_i \ (\cong \mathrm{Ind}^{K_{\sigma_i}}_{G^{i-1}_{\bar{\mathbf{x}}} G^i_{\mathbf{x},s_{i-1}}}\, \tilde{\rho}'_i),$$

$$\tau_i := \mathrm{Ind}_{K^i}^{G^i_{\bar{\mathbf{x}}}}\, \rho'_i \otimes \phi_i \ (\cong \mathrm{Ind}_{K_{\sigma_i}}^{G^i_{\bar{\mathbf{x}}}}\, \sigma_i \otimes \phi_i).$$

We finally recall the notion of a generic reduced cuspidal $\mathbf{G}$-datum due to Hakim–Murnaghan ([HM08]). By the theory of Moy–Prasad, the induced representation

$$\pi_{-1} := \text{c-Ind}^{G^0}_{K^0}\, \rho'_0$$

is an irreducible depth-zero supercuspidal representation of G^0 ([MP96, Proposition 6.6]). Conversely, any irreducible depth-zero supercuspidal representation π_{-1} of G^0 is obtained by the compact induction of a representation ρ'_0 satisfying the condition mentioned above in a unique (up to conjugation) way ([MP96, Proposition 6.8]). From this observation we conclude that the triple $(\vec{\mathbf{G}}, \pi_{-1}, \vec{\phi})$ is essentially equivalent to the original quintuple $(\vec{\mathbf{G}}, \vec{\phi}, \vec{r}, \mathbf{x}, \rho'_0)$. In [HM08], Hakim–Murnaghan called such triples *generic reduced cuspidal* $\mathbf{G}$*-data* and defined an equivalence relation called $\mathbf{G}$*-equivalence* on them. Here we do not recall the definition of the $\mathbf{G}$-equivalence (see [HM08, Definition 6.3]). The important nature of this equivalence relation is that it describes the "fibers" of Yu's construction: for two given generic reduced cuspidal $\mathbf{G}$-data Σ and Σ', the associated supercuspidal representations π_Σ and $\pi_{\Sigma'}$ are isomorphic if and only if two data Σ and Σ' are $\mathbf{G}$-equivalent. In other words, Yu's construction gives the following bijective map:

$$\begin{array}{ccc} & & \{\text{irred. s.c. rep'ns of } G\}/\!\sim \\ & & \cup \\ \{\text{gen. red. cusp. } \mathbf{G}\text{-data}\}/\mathbf{G}\text{-eq.} & \xrightarrow[\text{Yu's construction}]{1:1} & \{\text{tame s.c. rep'ns of } G\}/\!\sim \end{array}$$

3.2. **Kaletha's reparametrization of tame supercuspidal representations.** We first recall Kaletha's classification of regular depth-zero supercuspidal representations. Let $\mathbf{G}^0$ be a tamely ramified connected reductive group over F which belongs to a tame Levi sequence $\vec{\mathbf{G}} = (\mathbf{G}^0 \subsetneq \mathbf{G}^1 \subsetneq \cdots)$ as in the previous subsection. Let us suppose that we have an irreducible depth-zero supercuspidal representation π_{-1} of G^0. Then, by the theory of Moy–Prasad ([MP96]), there exists a point $\mathbf{x} \in \mathcal{B}(\mathbf{G}^0, F)$ such that its image $\bar{\mathbf{x}}$ in $\mathcal{B}^{\mathrm{red}}(\mathbf{G}^0, F)$ is a vertex and the restriction $\pi_{-1}|_{G^0_{\mathbf{x},0}}$ contains the inflation of an irreducible cuspidal representation κ of $G^0_{\mathbf{x},0:0+}$. Note that such a pair of $\mathbf{x}$ and κ is essentially unique (up to "association" in the sense of Moy–Prasad; see [MP96, Theorem 3.5]). We put $\mathbb{G}^0_{\mathbf{x}}$ to be the connected reductive group over $\mathbb{F}_q$ obtained by taking the reductive quotient of the special fiber of the parahoric subgroup scheme attached to $\mathbf{x}$. Then we have a natural identification $\mathbb{G}^0_{\mathbf{x}}(\mathbb{F}_q) \cong G^0_{\mathbf{x},0:0+}$. By Deligne–Lusztig theory, the cuspidality of κ implies that there exists a unique (up to $\mathbb{G}^0_{\mathbf{x}}(\mathbb{F}_q)$-conjugation) pair $(\mathbb{S}, \bar{\phi})$ of

- an elliptic maximal torus $\mathbb{S}$ of $\mathbb{G}^0_{\mathbf{x}}$ defined over $\mathbb{F}_q$, and
- a character $\bar{\phi}$ of $\mathbb{S}(\mathbb{F}_q)$

such that the associated Deligne–Lusztig representation $\pm R_{\mathbb{S}}^{\mathbb{G}^0_{\mathbf{x}}}(\bar{\phi})$ contains κ.

By [Kal19, Lemma 3.4.4] (due to [BT84, Proposition 5.1.10], cf. [DeB06, Section 2.2-2.3]), there exists a maximally unramified (in the sense of [Kal19, Definition 3.4.2]) elliptic maximal torus $\mathbf{S}$ of $\mathbf{G}^0$ defined over F whose connected Néron model has $\mathbb{S}$ as its special fiber. We let $N_{G^0}(\mathbf{S})$ be the normalizer group of $\mathbf{S}$ in G^0 and put

$$W_{G^0}(\mathbf{S}) := N_{G^0}(\mathbf{S})/S.$$

Following Kaletha [Kal19, Definitions 3.4.16 and 3.4.19], we say

- the character $\bar{\phi}$ is *regular* if the stabilizer of $\bar{\phi}$ in $W_{G^0}(\mathbf{S})$ is trivial, and
- the irreducible depth-zero supercuspidal representation π_{-1} is *regular* if $\bar{\phi}$ associated to π_{-1} in the above manner is regular

(note that the group $W_{G^0}(\mathbf{S})$ acts on the group of characters of $\mathbb{S}(\mathbb{F}_q)$ since we have $S_{0:0+} \cong \mathbb{S}(\mathbb{F}_q)$). In summary, we may associate a pair $(\mathbf{S}, \bar{\phi})$ to each irreducible depth zero supercuspidal representation π_{-1} of G^0 and define the notion of regularity for π by looking at the pair $(\mathbf{S}, \bar{\phi})$.

Remark 3.1. If $\bar{\phi}$ is regular, then it is in general position in the sense of Deligne–Lusztig [Kal19, Fact 3.4.18]. If $\bar{\phi}$ is in general position, $(-1)^{r(\mathbb{S})-r(\mathbb{G}^0_{\mathbf{x}})} R_{\mathbb{S}}^{\mathbb{G}^0_{\mathbf{x}}}(\bar{\phi})$ is an irreducible representation, where $r(\mathbb{S})$ and $r(\mathbb{G}^0_{\mathbf{x}})$ are the split ranks of $\mathbb{S}$ and $\mathbb{G}^0_{\mathbf{x}}$, respectively. Thus κ is necessarily equal to $(-1)^{r(\mathbb{S})-r(\mathbb{G}^0_{\mathbf{x}})} R_{\mathbb{S}}^{\mathbb{G}^0_{\mathbf{x}}}(\bar{\phi})$ itself. Furthermore, the orthogonality relation of Deligne–Lusztig ([DL76, Theorem 6.8]) assures that such a pair $(\mathbb{S}, \bar{\phi})$ is unique up to $\mathbb{G}^0_{\mathbf{x}}(\mathbb{F}_q)$-conjugacy. Hence $(\mathbf{S}, \bar{\phi})$ is unique up to $G^0_{\mathbf{x},0}$-conjugacy.

We next consider the "converse" of the above procedure. Let us suppose that we have a maximally unramified elliptic maximal torus $\mathbf{S}$ of $\mathbf{G}^0$ and a regular depth-zero character ϕ_{-1} of S, i.e., the character $\bar{\phi}$ of $S_{0:0+} = \mathbb{S}(\mathbb{F}_q)$ induced from ϕ_{-1} is regular in the above sense. Since the torus $\mathbf{S}$ is elliptic, it canonically defines a subset $\mathcal{A}^{\mathrm{red}}(\mathbf{S}, F)$ of $\mathcal{B}^{\mathrm{red}}(\mathbf{G}^0, F)$ consisting of only one point. More precisely, $\mathcal{A}^{\mathrm{red}}(\mathbf{S}, F)$ consists of the unique Frobenius-fixed point in the reduced apartment $\mathcal{A}^{\mathrm{red}}(\mathbf{S}_{F^{\mathrm{ur}}}, F^{\mathrm{ur}}) \subset \mathcal{B}^{\mathrm{red}}(\mathbf{G}^0, F^{\mathrm{ur}})$ of the maximal F^{ur}-split torus $\mathbf{S}_{F^{\mathrm{ur}}}$ of $\mathbf{G}_{F^{\mathrm{ur}}}$ (see the paragraph before [Kal19, Lemma 3.4.3]). We take a point $\mathbf{x} \in \mathcal{B}(\mathbf{G}^0, F)$ whose image $\bar{\mathbf{x}}$ in $\mathcal{B}^{\mathrm{red}}(\mathbf{G}^0, F)$ equals this unique point of $\mathcal{A}^{\mathrm{red}}(\mathbf{S}, F)$. Note that then S normalizes $G^0_{\mathbf{x},0}$ and we have $\mathbb{S} \subset \mathbb{G}^0_{\mathbf{x}}$. As explained in Remark 3.1, from the pair $(\mathbb{S}, \bar{\phi})$, we get an irreducible cuspidal representation

$$\kappa_{(\mathbf{S},\bar{\phi})} := (-1)^{r(\mathbb{S})-r(\mathbb{G}^0_{\mathbf{x}})} R_{\mathbb{S}}^{\mathbb{G}^0_{\mathbf{x}}}(\bar{\phi})$$

of $\mathbb{G}^0_{\mathbf{x}}(\mathbb{F}_q)$. In [Kal19, Section 3.4.4], Kaletha constructs an extension of (the inflation of) the representation $\kappa_{(\mathbf{S},\bar{\phi})}$ to $SG^0_{\mathbf{x},0}$ in a geometric way. Let $\kappa_{(\mathbf{S},\phi_{-1})}$ denote the extended representation of $SG^0_{\mathbf{x},0}$. Now Kaletha's classification theorem of regular depth zero supercuspidal representations is summarized as follows:

Proposition 3.2 ([Kal19, Lemma 3.4.20 and Proposition 3.4.27]). *The representation*

$$\pi^{\mathbf{G}^0}_{(\mathbf{S},\phi_{-1})} := \text{c-Ind}^{G^0}_{SG^0_{\mathbf{x},0}} \kappa_{(\mathbf{S},\phi_{-1})}$$

is an irreducible depth-zero regular supercuspidal representation of G^0. Conversely, every irreducible depth-zero regular supercuspidal representation of G^0 is obtained in this way. Furthermore, two such representations $\pi^{\mathbf{G}^0}_{(\mathbf{S},\phi_{-1})}$ and $\pi^{\mathbf{G}^0}_{(\mathbf{S}',\phi'_{-1})}$ are isomorphic if and only if the pairs $(\mathbf{S},\phi_{-1})$ and $(\mathbf{S}',\phi'_{-1})$ are G^0-conjugate.

We now return to tame supercuspidal representations. Let $\Sigma = (\vec{\mathbf{G}}, \pi_{-1}, \vec{\phi})$ be a generic reduced cuspidal $\mathbf{G}$-datum and π_Σ its associated supercuspidal representation of G. We call Σ *regular* if π_{-1} is regular. We call π_Σ a *regular supercuspidal representation* if Σ is regular. Let us suppose that Σ is regular. Then, thanks to Proposition 3.2, we have a pair $(\mathbf{S},\phi_{-1})$ consisting of a maximally unramified elliptic maximal torus $\mathbf{S}$ of $\mathbf{G}^0$ and a regular depth-zero character ϕ_{-1} of S satisfying $\pi_{-1} \cong \pi^{\mathbf{G}^0}_{(\mathbf{S},\phi_{-1})}$. We put $\mathbf{G}^{-1} := \mathbf{S}$ and define a character ϕ of S by

$$\phi := \prod_{i=-1}^{d} \phi_i|_S.$$

Kaletha's reparametrizing result is as follows:

Proposition 3.3 ([Kal19, Proposition 3.7.8]). *The map*

$$(\vec{\mathbf{G}}, \pi_{-1}, \vec{\phi}) \mapsto (\mathbf{S}, \phi)$$

defined in the above manner induces a bijection from the set of $\mathbf{G}$-equivalence classes of regular generic reduced cuspidal $\mathbf{G}$-data to the set of G-conjugacy classes of tame elliptic regular pairs in $\mathbf{G}$.

Remark 3.4. Note that we need our baseline assumptions on p (p odd, p not bad for $\mathbf{G}$, $p \nmid |\pi_1(\widehat{\mathbf{G}}_{\mathrm{der}})|$, and $p \nmid |\pi_1(\mathbf{G}_{\mathrm{der}})|$) for this proposition, especially in establishing the surjectivity part of the map (this is called the "Howe factorization" process, which will be recalled more precisely in the next subsection). One crucial step in proving [Kal19, Proposition 3.7.8] is to establish [Kal19, Lemma 3.6.8], whose proof utilizes a technical result of Yu concerning the genericity of characters [Yu01, Lemma 8.1]. The assumption required by [Yu01, Lemma 8.1] is that p is not a torsion prime for the root datum of the dual group $\mathbf{G}$, which is equivalent to that p is not bad for $\mathbf{G}$ and does not divide the order of $|\pi_1(\widehat{\mathbf{G}}_{\mathrm{der}})|$. Therefore we need to assume $p \nmid |\pi_1(\widehat{\mathbf{G}}_{\mathrm{der}})|$ in addition to the non-badness for the root datum of $\mathbf{G}$ imposed in the beginning of [Kal19, Section 3.6]. This subtlety is carefully explained in [Kal21]. See [Kal21, Section 4] for details.

Recall that a pair $(\mathbf{S},\phi)$ of a maximal torus $\mathbf{S}$ of $\mathbf{G}$ defined over F and a character $\phi\colon S \to \mathbb{C}^\times$ is called a *tame elliptic regular pair* if it satisfies the following conditions ([Kal19, Definition 3.7.5]):

- $\mathbf{S}$ is a tamely ramified elliptic maximal torus of $\mathbf{G}$,
- the action of the inertia subgroup I_F of $\mathrm{Gal}(\overline{F}/F)$ on the root subsystem

  $$R_{0+} := \{\alpha \in R(\mathbf{S},\mathbf{G}) \mid \phi|_{\mathrm{Nr}_{E/F}(\alpha^\vee(E^\times_{0+}))} \equiv \mathbb{1}\}$$

  of the root system $R(\mathbf{S},\mathbf{G})$ preserves a set of positive roots, where
  - E is the minimal extension of F splitting $\mathbf{S}$, and
  - $\mathrm{Nr}_{E/F}$ is the norm map $\mathbf{S}(E) \to \mathbf{S}(F)$,

- the restriction $\phi|_{S_0}$ has trivial stabilizer for the action of the group $W_{G^0}(\mathbf{S})$, where $\mathbf{G}^0 \subset \mathbf{G}$ is the reductive group with maximal torus $\mathbf{S}$ and root system R_{0+}.

According to Proposition 3.3, we conclude that (isomorphism classes of) regular supercuspidal representations bijectively correspond to (G-conjugacy classes of) tame elliptic regular pairs. Let $\pi_{(\mathbf{S},\phi)}$ denote the representation corresponding to a tame elliptic regular pair $(\mathbf{S}, \phi)$.

$$\begin{array}{ccc}
\{\text{gen. red. cusp. } \mathbf{G}\text{-data}\}/\mathbf{G}\text{-eq.} & \xrightarrow{1:1} & \{\text{tame s.c. rep'ns of } G\}/\sim \\
\cup & & \cup \\
\{\text{regular gen. red. cusp. } \mathbf{G}\text{-data}\}/\mathbf{G}\text{-eq.} & \xrightarrow{1:1} & \{\text{regular s.c. rep'ns of } G\}/\sim \\
\updownarrow {\scriptstyle 1:1\ (\text{Prop. } 3.3)} & \nearrow {\scriptstyle (\mathbf{S},\phi)\mapsto\pi_{(\mathbf{S},\phi)}} & \\
\{\text{tame elliptic regular pairs}\}/G\text{-conj.} & &
\end{array}$$

Remark 3.5. If we take a cuspidal $\mathbf{G}$-datum $(\vec{\mathbf{G}}, \vec{\phi}, \vec{r}, \mathbf{x}, \rho_0')$ corresponding to $(\vec{\mathbf{G}}, \pi_{-1}, \vec{\phi})$, then we have $\pi_{-1} \cong \text{c-Ind}_{K^0}^{G^0} \rho_0'$ (recall that $K^0 = G^0_{\bar{\mathbf{x}}}$). On the other hand, since we have $\pi_{-1} \cong \pi^{\mathbf{G}^0}_{(\mathbf{S},\phi_{-1})} = \text{c-Ind}_{SG^0_{\mathbf{x},0}}^{G^0} \kappa_{(\mathbf{S},\phi_{-1})}$, we may suppose that

$$\rho_0' \cong \text{Ind}_{SG^0_{\mathbf{x},0}}^{G^0_{\bar{\mathbf{x}}}} \kappa_{(\mathbf{S},\phi_{-1})}.$$

Finally we introduce the notion of Howe-unramifiedness as follows:

Definition 3.6. If a regular supercuspidal representation π is associated to a tame elliptic regular pair $(\mathbf{S}, \phi)$ with unramified $\mathbf{S}$, we say that π is *Howe-unramified*.

3.3. **Tame twisted Levi sequence associated to a character.** As the surjectivity part of Proposition 3.3 shows, we may associate to any tame elliptic regular pair $(\mathbf{S}, \phi)$ a sequence of tame twisted subgroups $\vec{\mathbf{G}} = (\mathbf{G}^{-1}, \ldots, \mathbf{G}^d)$ and a sequence of characters $(\phi_{-1}, \ldots, \phi_d)$. Indeed, in the proof of Proposition 3.3 ([Kal19, Proposition 3.7.8]), Kaletha gives a construction of such sequences explicitly. As explained in Section [Kal19, Section 3.6], this can be understood as a generalization of the *Howe factorization*, a factorization of characters of S defined in the GL_n setting in his construction of supercuspidal representations of GL_n. The Howe factorization comes with an associated sequence of subgroups which capture the relative genericity of the individual factors in the product. Although the characters in this factorization are *not* unique, the associated subgroups *are*. In this section, following [Kal19, Section 3.6], we recall how the Howe factorization attaches a tame twisted Levi sequence to a tame elliptic regular pair. In fact, we may work in a more general setting; let $\mathbf{S}$ be a tamely ramified maximal torus of $\mathbf{G}$ defined over F and θ a character of S. Hence here $(\mathbf{S}, \theta)$ is allowed to be a pair which is not tame elliptic regular.

For $r \in \widetilde{\mathbb{R}}_{>0}$, we define a subset R_r of $R(\mathbf{S}, \mathbf{G})$ as in [Kal19, 1107 page, (3.6.1)] by

$$R_r := \{\alpha \in R(\mathbf{S}, \mathbf{G}) \mid \theta|_{\mathrm{Nr}_{E/F}(\alpha^\vee(E_r^\times))} \equiv \mathbb{1}\},$$

where E is the splitting field of the torus $\mathbf{S}$. Note that R_{0+} introduced in the definition of a tame elliptic regular pair is nothing but a special case of R_r where r is taken to be $0+$. We let $r_{d-1} > \cdots > r_1 > r_0$ be the real numbers satisfying $R_r \subsetneq R_{r+}$ (in particular, r_{d-1} is the real number satisfying $R_{r_{d-1}} \subsetneq R_{r_{d-1}+} = R(\mathbf{S}, \mathbf{G})$). We put $r_d := \mathrm{depth}(\theta)$ and $r_{-1} := 0$ (note that $r_d \geq r_{d-1}$ since $R_{r_d+} = R(\mathbf{S}, \mathbf{G})$). Let $\mathbf{G}^0 \subsetneq \cdots \subsetneq \mathbf{G}^{d-1}$ be the tamely ramified reductive subgroups of $\mathbf{G}$ corresponding to the sequence $R_{r_0} \subsetneq \cdots \subsetneq R_{r_{d-1}}$ (i.e., each $\mathbf{G}^i$ is the reductive subgroup of $\mathbf{G}$ which contains $\mathbf{S}$ and has R_{r_i} as its roots). We put $\mathbf{G}^{-1} := \mathbf{S}$ and $\mathbf{G}^d := \mathbf{G}$ so that we have $\mathbf{G}^{-1} \subset \mathbf{G}^0 \subsetneq \cdots \subsetneq \mathbf{G}^{d-1} \subsetneq \mathbf{G}^d$. When we want to emphasize the dependence on $(\mathbf{S}, \theta)$, we write $\mathbf{G}^i(\mathbf{S}, \theta)$ for the i-th subgroup $\mathbf{G}^i$ associated to $(\mathbf{S}, \theta)$ via the Howe factorization.

We introduce the notion of torality and 0-torality as follows:

Definition 3.7. Let $\mathbf{S}$ be a tamely ramified maximal torus of $\mathbf{G}$ defined over F. Let θ be a character of S.

(1) We call θ a *toral* character if $\mathbf{G}^0(\mathbf{S}, \theta) = \mathbf{S}$.
(2) We call θ a 0-*toral* character if $d = 1$ and $\mathbf{G}^0(\mathbf{S}, \theta) = \mathbf{S}$ (i.e., $\vec{\mathbf{G}}$ consists only of $\mathbf{G}^0 = \mathbf{S}$ and $\mathbf{G}^1 = \mathbf{G}$).

We note that, under the assumption that p is not bad for $\mathbf{G}$, each $\mathbf{G}^i(\mathbf{S}, \theta)$ is a tame twisted Levi subgroup of $\mathbf{G}$ by [Kal19, Lemma 3.6.1].

We now return to the setting of elliptic $\mathbf{S}$. The following lemma will be useful for us, especially in Section 7. We warn the reader that this lemma does not hold for arbitrary p! We discuss subtleties regarding small residue characteristic in Section 7.4.

Lemma 3.8. *Let $(\mathbf{S}, \theta)$ be a tame elliptic regular pair. If θ is a toral character of S, then $\theta|_{S_{0+}}$ has trivial $W_G(\mathbf{S})$-stabilizer.*

Proof. By [Kal19, Proposition 3.6.7], θ has a Howe factorization. To be more precise, let $\vec{\mathbf{G}}$ denote the twisted Levi sequence $\vec{\mathbf{G}} = (\mathbf{G}^{-1}, \ldots, \mathbf{G}^d)$ attached to θ as above. Then, there exists a sequence $\vec{\theta} = (\theta_{-1}, \ldots, \theta_d)$ of characters θ_i of G^i such that $\theta = \prod_{i=-1}^{d} \theta_i|_S$ and we have a regular generic reduced cuspidal $\mathbf{G}$-datum $((\mathbf{G}^0, \ldots, \mathbf{G}^d), \pi_{-1}, (\theta_0, \ldots, \theta_d))$, where $\pi_{-1} \cong \pi^{\mathbf{G}^0}_{(\mathbf{S}, \theta_{-1})}$. By [Kal19, Lemma 3.6.5 (2)], we have $\mathrm{Stab}_{W_G(\mathbf{S})}(\theta|_{S_{0+}}) = \mathrm{Stab}_{W_{G^0}(\mathbf{S})}(\theta_{-1}|_{S_{0+}}) = W_{G^0}(\mathbf{S})$, where the last equality holds since $\theta_{-1}|_{S_{0+}}$ is trivial. The conclusion clearly follows as θ being toral (i.e., $\mathbf{G}^0 = \mathbf{S}$) implies that $W_{\mathbf{G}^0}(\mathbf{S})$ is trivial. ☐

3.4. Point stabilizer vs. parahoric subgroup.

Let $(\mathbf{S}, \phi)$ be a tame elliptic regular pair. As explained in Section 3.2, this pair gives rise to a $\mathbf{G}$-equivalence class of cuspidal $\mathbf{G}$-data; let $\Sigma = (\vec{\mathbf{G}}, \vec{\phi}, \vec{r}, \mathbf{x}, \rho'_0)$ be a representative of this equivalence class, where $\rho'_0 \cong \mathrm{Ind}_{SG^0_{\mathbf{x},0}}^{G^0_{\bar{\mathbf{x}}}} \kappa_{(\mathbf{S}, \phi_{-1})}$. Recall that, in Section 3.1, we considered the groups

$$G^i_{\bar{\mathbf{x}}} \supset K_{\sigma_i} \supset G^{i-1}_{\bar{\mathbf{x}}} G^i_{\mathbf{x}, s_{i-1}} \supset K^i$$

and defined the representations τ_i, σ_i, and $\tilde{\rho}'_i$ by inducing ρ'_i:

$$\tau_i := \mathrm{Ind}_{K_{\sigma_i}}^{G^i_{\bar{\mathbf{x}}}} \sigma_i \otimes \phi_i, \quad \sigma_i := \mathrm{Ind}_{G^{i-1}_{\bar{\mathbf{x}}} G^i_{\mathbf{x}, s_{i-1}}}^{K_{\sigma_i}} \tilde{\rho}'_i, \quad \tilde{\rho}'_i := \mathrm{Ind}_{K^i}^{G^{i-1}_{\bar{\mathbf{x}}} G^i_{\mathbf{x}, s_{i-1}}} \rho'_i.$$

For our convenience, we also introduce the following slightly smaller groups by replacing the role of $G^i_{\bar{\mathbf{x}}}$ with $SG^i_{\mathbf{x},0}$:

$$
\begin{array}{ccc}
G^i_{\bar{\mathbf{x}}} & \supset & SG^i_{\mathbf{x},0} \\
\cup & & \cup \\
K_{\sigma_i} := G^{i-1}_{\bar{\mathbf{x}}} G^i_{\mathbf{x},0+} & \supset & {}^\circ K_{\sigma_i} := SG^{i-1}_{\mathbf{x},0} G^i_{\mathbf{x},0+} \\
\cup & & \cup \\
G^{i-1}_{\bar{\mathbf{x}}} G^i_{\mathbf{x},s_{i-1}} & \supset & SG^{i-1}_{\mathbf{x},0} G^i_{\mathbf{x},s_{i-1}} \\
\cup & & \cup \\
K^i := G^0_{\bar{\mathbf{x}}}(G^0, \ldots, G^i)_{\mathbf{x},(0+,s_0,\ldots,s_{i-1})} & \supset & {}^\circ K^i := SG^0_{\mathbf{x},0}(G^0, \ldots, G^i)_{\mathbf{x},(0+,s_0,\ldots,s_{i-1})}
\end{array}
$$

We define a representation ${}^\circ\rho'_0$ of $SG^0_{\mathbf{x},0}$ by

$$
{}^\circ\rho'_0 := \kappa_{(\mathbf{S},\phi_{-1})}
$$

and construct representations ${}^\circ\rho'_i$ of ${}^\circ K^i$ in the same manner as before starting from ${}^\circ\rho'_0$ instead of ρ'_0.

Let us recall the following general lemma:

Lemma 3.9 (projection formula)**.** *Let G be a locally profinite group and H a closed subgroup. For any smooth representation π of G and smooth representation τ of H, we have*

$$
\mathrm{Ind}_H^G(\pi|_H \otimes \tau) \cong \pi \otimes (\mathrm{Ind}_H^G \tau)
$$

Proof. Let $\mathrm{Rep}(G)$ (resp. $\mathrm{Rep}(H)$) denote the abelian category of smooth representations of G (resp. H). We fix a smooth representation π of G. By Frobenius reciprocity and the Hom-$\otimes$ adjointness, we have a canonical isomorphism

$$
\mathrm{Hom}_G\bigl(\rho, \mathrm{Ind}_H^G(\pi|_H \otimes \tau)\bigr) \cong \mathrm{Hom}_H(\rho|_H, \pi|_H \otimes \tau) \cong \mathrm{Hom}_H\bigl((\rho \otimes \pi^\vee)|_H, \tau\bigr)
$$

for any $\rho \in \mathrm{Rep}(G)$ and $\tau \in \mathrm{Rep}(H)$, which is functorial in both variables. On the other hand, again by the Hom-$\otimes$ adjointness and Frobenius reciprocity, we also have a canonical and functorial isomorphism

$$
\mathrm{Hom}_G\bigl(\rho, \pi \otimes (\mathrm{Ind}_H^G \tau)\bigr) \cong \mathrm{Hom}_G(\rho \otimes \pi^\vee, \mathrm{Ind}_H^G \tau) \cong \mathrm{Hom}_H\bigl((\rho \otimes \pi^\vee)|_H, \tau\bigr).
$$

This implies that both the functors $\mathrm{Ind}_H^G \circ (\pi|_H \otimes (-)) \colon \mathrm{Rep}(H) \to \mathrm{Rep}(G)$ and $(\pi \otimes (-)) \circ \mathrm{Ind}_H^G \colon \mathrm{Rep}(H) \to \mathrm{Rep}(G)$ are the right adjoint to the functor $\mathrm{Res}_H^G \circ ((-) \otimes \pi^\vee) \colon \mathrm{Rep}(G) \to \mathrm{Rep}(H)$. Thus we conclude that these two functors are isomorphic by the uniqueness of the adjoint functor. In particular, we get $\mathrm{Ind}_H^G(\pi|_H \otimes \tau) \cong \pi \otimes (\mathrm{Ind}_H^G \tau)$ for any $\tau \in \mathrm{Rep}(H)$. □

The relationship between ρ'_i and ${}^\circ\rho'_i$ is described as follows.

Lemma 3.10. *The K^i-representation ρ'_i is isomorphic to the induction of the ${}^\circ K^i$-representation ${}^\circ\rho'_i$:*

$$
\rho'_i \cong \mathrm{Ind}_{{}^\circ K^i}^{K^i} {}^\circ\rho'_i.
$$

Proof. It follows from the definition of ${}^\circ\rho'_0$ that we have $\rho'_0 \cong \mathrm{Ind}_{SG^0_{\mathbf{x},0}}^{G^0_{\bar{\mathbf{x}}}} \kappa_{(\mathbf{S},\phi_{-1})}$; hence the assertion for $i = 0$ follows. Let us check the assertion for $i+1$ by assuming its validity for i. Recall that the representation ρ'_{i+1} is the push-forward of

$$
(\tilde{\phi}_i|_{K^i \ltimes J^{i+1}}) \otimes \bigl((\rho'_i \otimes \phi_i|_{K^i}) \ltimes \mathbb{1}\bigr)
$$

via the canonical map $K^i \ltimes J^{i+1} \twoheadrightarrow K^i J^{i+1} = K^{i+1}$. Similarly, the representation ${}^\circ\rho'_{i+1}$ is the push-forward of

$$
(\tilde{\phi}_i|_{{}^\circ K^i \ltimes J^{i+1}}) \otimes \bigl(({}^\circ\rho'_i \otimes \phi_i|_{{}^\circ K^i}) \ltimes \mathbb{1}\bigr)
$$

via the canonical map ${}^{\circ}K^{i} \ltimes J^{i+1} \twoheadrightarrow {}^{\circ}K^{i}J^{i+1} = {}^{\circ}K^{i+1}$. Noting that $(\rho'_i \otimes \phi_i|_{K^i}) \ltimes \mathbb{1} = (\rho'_i \ltimes \mathbb{1}) \otimes (\phi_i|_{K^i} \ltimes \mathbb{1})$, it is enough to check that we have

$$\begin{aligned}\mathrm{Ind}_{{}^{\circ}K^i \ltimes J^{i+1}}^{K^i \ltimes J^{i+1}} \Big((\tilde{\phi}_i|_{{}^{\circ}K^i \ltimes J^{i+1}}) \otimes \big(({}^{\circ}\rho'_i \otimes \phi_i|_{{}^{\circ}K^i}) \ltimes \mathbb{1} \big) \Big) \\ \cong (\tilde{\phi}_i|_{K^i \ltimes J^{i+1}}) \otimes \big((\rho'_i \ltimes \mathbb{1}) \otimes (\phi_i|_{K^i} \ltimes \mathbb{1}) \big).\end{aligned}$$

Obviously $\tilde{\phi}_i|_{{}^{\circ}K^i \ltimes J^{i+1}}$ is the restriction of $\tilde{\phi}_i|_{K^i \ltimes J^{i+1}}$ and $\phi_i|_{{}^{\circ}K^i}$ is the restriction of $\phi_i|_{K^i}$. Hence, by Lemma 3.9,

$$\begin{aligned}\mathrm{Ind}_{{}^{\circ}K^i \ltimes J^{i+1}}^{K^i \ltimes J^{i+1}} \Big((\tilde{\phi}_i|_{{}^{\circ}K^i \ltimes J^{i+1}}) \otimes \big(({}^{\circ}\rho'_i \otimes \phi_i|_{{}^{\circ}K^i}) \ltimes \mathbb{1} \big) \Big) \\ \cong (\tilde{\phi}_i|_{K^i \ltimes J^{i+1}}) \otimes \Big(\big(\mathrm{Ind}_{{}^{\circ}K^i \ltimes J^{i+1}}^{K^i \ltimes J^{i+1}} ({}^{\circ}\rho'_i \ltimes \mathbb{1}) \big) \otimes (\phi_i|_{K^i} \ltimes \mathbb{1}) \Big).\end{aligned}$$

As we are supposing that $\rho'_i \cong \mathrm{Ind}_{{}^{\circ}K^i}^{K^i} {}^{\circ}\rho'_i$, we get $\mathrm{Ind}_{{}^{\circ}K^i \ltimes J^{i+1}}^{K^i \ltimes J^{i+1}} ({}^{\circ}\rho'_i \ltimes \mathbb{1}) \cong \rho'_i \ltimes \mathbb{1}$. □

We define representations ${}^{\circ}\tau_i$, ${}^{\circ}\sigma_i$, and ${}^{\circ}\tilde{\rho}'_i$ by inducing ${}^{\circ}\rho'_i$:

$${}^{\circ}\tau_i := \mathrm{Ind}_{{}^{\circ}K_{\sigma_i}}^{SG^i_{\mathbf{x},0}} {}^{\circ}\sigma_i \otimes \phi_i, \quad {}^{\circ}\sigma_i := \mathrm{Ind}_{SG^{i-1}_{\mathbf{x},0} G^i_{\mathbf{x},s_{i-1}}}^{{}^{\circ}K_{\sigma_i}} {}^{\circ}\tilde{\rho}'_i, \quad {}^{\circ}\tilde{\rho}'_i := \mathrm{Ind}_{{}^{\circ}K^i}^{SG^{i-1}_{\mathbf{x},0} G^i_{\mathbf{x},s_{i-1}}} {}^{\circ}\rho'_i.$$

Then Lemma 3.10 implies that

$$\tau_i \cong \mathrm{Ind}_{SG^i_{\mathbf{x},0}}^{G^i_{\bar{\mathbf{x}}}} {}^{\circ}\tau_i, \quad \sigma_i \cong \mathrm{Ind}_{{}^{\circ}K_{\sigma_i}}^{K_{\sigma_i}} {}^{\circ}\sigma_i, \quad \tilde{\rho}'_i \cong \mathrm{Ind}_{SG^{i-1}_{\mathbf{x},0} G^i_{\mathbf{x},s_{i-1}}}^{G^{i-1}_{\bar{\mathbf{x}}} G^i_{\mathbf{x},s_{i-1}}} {}^{\circ}\tilde{\rho}'_i.$$

$$\begin{array}{ccc} G^i_{\bar{\mathbf{x}}} & \supset & SG^i_{\mathbf{x},0} \\ \cup & & \cup \\ K_{\sigma_i} & \supset & {}^{\circ}K_{\sigma_i} \\ \cup & & \cup \\ G^{i-1}_{\bar{\mathbf{x}}} G^i_{\mathbf{x},s_{i-1}} & \supset & SG^{i-1}_{\mathbf{x},0} G^i_{\mathbf{x},s_{i-1}} \\ \cup & & \cup \\ K^i & \supset & {}^{\circ}K^i \end{array} \qquad \begin{array}{ccc} \tau_i & \leftarrow\!-\!-\!-\!- & {}^{\circ}\tau_i \\ \uparrow & & \uparrow \\ \sigma_i \otimes \phi_i & \leftarrow\!-\!- & {}^{\circ}\sigma_i \otimes \phi_i \\ \uparrow & & \uparrow \\ \tilde{\rho}'_i \otimes \phi_i & \leftarrow\!-\!- & {}^{\circ}\tilde{\rho}'_i \otimes \phi_i \\ \uparrow & & \uparrow \\ \rho'_i \otimes \phi_i & \leftarrow\!-\!- & {}^{\circ}\rho'_i \otimes \phi_i \end{array}$$

4. Adler–DeBacker–Spice's character formula

The purpose of this section is to give a character formula for a crucial intermediate representation used in Yu's construction of supercuspidal representations, following the theory of Adler–DeBacker–Spice ([AS09, DS18]). We first note that several technical assumptions on p are required so that their theory works well (see [AS08, Section 2]), but it is enough to assume only the oddness and the non-badness of p whenever $\mathbf{G}$ is tamely ramified (see [Kal19, Section 4.1]).

We focus on the setting of *Howe-unramified* regular supercuspidal representations. To this end, let $(\mathbf{S}, \phi)$ be a tame elliptic regular pair such that

$$\mathbf{S} \text{ is an unramified elliptic maximal torus of } \mathbf{G}.$$

Let $(\vec{\mathbf{G}}, \pi_{-1}, \vec{\phi})$ be a regular generic reduced cuspidal $\mathbf{G}$-datum corresponding to $(\mathbf{S}, \phi)$ as in Section 3.2 and recall the various intermediate representations described in Section 3.4. In Section 4.2 we give a simple character formula (Proposition 4.11) for the $SG_{\mathbf{x},0}$-representation ${}^{\circ}\tau_d$ on the locus of *unramified very regular elements* (Definition 4.2, following [CI21b]).

4.1. **DeBacker–Spice's invariants.** In this section, we recall several invariants introduced by DeBacker–Spice in [DS18, Section 4.3] to describe the characters of tame supercuspidal representations.

Let $\mathbf{J}$ be a connected reductive group over F. We take a maximal torus $\mathbf{T_J}$ of $\mathbf{J}$ defined over F. Then we get the set $R(\mathbf{J}, \mathbf{T_J})$ of absolute roots of $\mathbf{T_J}$ in $\mathbf{J}$ which has an action of the absolute Galois group Γ_F of F. For each $\alpha \in R(\mathbf{J}, \mathbf{T_J})$, we put Γ_α (resp. $\Gamma_{\pm\alpha}$) to be the stabilizer of α (resp. $\{\pm\alpha\}$) in Γ_F. Let F_α (resp. $F_{\pm\alpha}$) be the subfield of $\overline{F}$ fixed by Γ_α (resp. $\Gamma_{\pm\alpha}$):

$$F \subset F_{\pm\alpha} \subset F_\alpha \quad \longleftrightarrow \quad \Gamma_F \supset \Gamma_{\pm\alpha} \supset \Gamma_\alpha.$$

- When $F_\alpha = F_{\pm\alpha}$, we say α is an *asymmetric* root.
- When $F_\alpha \supsetneq F_{\pm\alpha}$, we say α is a *symmetric* root. Note that, in this case, the extension $F_\alpha/F_{\pm\alpha}$ is necessarily quadratic. Furthermore,
  - when $F_\alpha/F_{\pm\alpha}$ is unramified, we say α is *symmetric unramified*, and
  - when $F_\alpha/F_{\pm\alpha}$ is ramified, we say α is *symmetric ramified*.

Note that a root α is symmetric if and only if the Γ_F-orbit of α contains $-\alpha$. We write $R(\mathbf{J}, \mathbf{T_J})^{\mathrm{sym}}$, $R(\mathbf{J}, \mathbf{T_J})_{\mathrm{sym}}$, $R(\mathbf{J}, \mathbf{T_J})_{\mathrm{sym,ur}}$, and $R(\mathbf{J}, \mathbf{T_J})_{\mathrm{sym,ram}}$ for the set of asymmetric roots, symmetric roots, symmetric unramified roots, and symmetric ramified roots, respectively.

According to [DS18, Definition 3.6], for each $\alpha \in R(\mathbf{J}, \mathbf{T_J})$ and a point $\mathbf{y} \in \mathcal{B}^{\mathrm{red}}(\mathbf{J}, F)$, we define a set $\mathrm{ord}_{\mathbf{y}}\, \alpha$ of real numbers by

$$\mathrm{ord}_{\mathbf{y}}\, \alpha := \{i \in \mathbb{R} \mid \mathfrak{g}_\alpha(F_\alpha)_{\mathbf{y},i:i+} \neq 0\},$$

where $\mathfrak{g}_\alpha$ is the root space of α in the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$ and $\mathfrak{g}_\alpha(F_\alpha)_{\mathbf{y},i} := \mathfrak{g}(F_\alpha)_{\mathbf{y},i} \cap \mathfrak{g}_\alpha(F_\alpha)$. Then, for a positive real number $r \in \mathbb{R}_{>0}$, we define a subset $R^{\mathbf{J}}_{\mathbf{y},r/2}$ of $R(\mathbf{J}, \mathbf{T_J})$ by

$$R^{\mathbf{J}}_{\mathbf{y},r/2} := \{\alpha \in R(\mathbf{J}, \mathbf{T_J}) \mid r/2 \in \mathrm{ord}_{\mathbf{y}}\, \alpha\}.$$

For an element δ of $T_{\mathbf{J}} = \mathbf{T_J}(F)$, we put

$$R^{\mathbf{J}}_{\mathbf{y},(r-\mathrm{ord}_\delta)/2} := \{\alpha \in R(\mathbf{J}, \mathbf{T_J}) \mid \alpha(\delta) \neq 1,\, (r - \mathrm{ord}_\delta\, \alpha)/2 \in \mathrm{ord}_{\mathbf{y}}\, \alpha\},$$

where $\mathrm{ord}_\delta\, \alpha := \mathrm{ord}(\alpha(\delta) - 1)$.

Now let us recall the definition of the invariants $\varepsilon^{\mathrm{ram}}$, $\varepsilon_{\mathrm{sym,ram}}$, and $\tilde{e}$ ([DS18, Definition 4.14]). Let δ be an element of the maximal bounded subgroup of $T_{\mathbf{J}}$.

- We first define a sign $\varepsilon_\alpha(\delta)$ for an asymmetric or symmetric unramified root α as follows:
  - For an asymmetric root $\alpha \in R(\mathbf{J}, \mathbf{T_J})^{\mathrm{sym}}$, we put

    $$\varepsilon_\alpha(\delta) := \left(\frac{\overline{\alpha(\delta)}}{k_{F_\alpha}^\times}\right),$$

    where $\left(\frac{-}{k_{F_\alpha}^\times}\right)$ is the unique nontrivial quadratic character of the multiplicative group $k_{F_\alpha}^\times$ of the residue field k_{F_α} of F_α.
  - For a symmetric unramified root $\alpha \in R(\mathbf{J}, \mathbf{T_J})_{\mathrm{sym,ur}}$, we put

    $$\varepsilon_\alpha(\delta) := \left(\frac{\overline{\alpha(\delta)}}{k_{F_\alpha}^1}\right),$$

    where $\left(\frac{-}{k_{F_\alpha}^1}\right)$ is the unique nontrivial quadratic character of the kernel $k_{F_\alpha}^1$ of the norm map $\mathrm{Nr}_{k_{F_\alpha}/k_{F_{\pm\alpha}}} \colon k_{F_\alpha}^\times \twoheadrightarrow k_{F_{\pm\alpha}}^\times$.

Here, in both cases, $\alpha(\delta)$ belongs to $\mathcal{O}_{F_\alpha}^\times$ since δ belongs to the maximal bounded subgroup of $T_{\mathbf{J}}$. Thus we can take its reduction $\overline{\alpha(\delta)} \in k_{F_\alpha}^\times$. Note that, in the latter case, the definition makes sense since $\alpha(\delta)$ always belongs to the kernel of the norm map $\mathrm{Nr}_{F_\alpha/F_{\pm\alpha}}\colon F_\alpha^\times \to F_{\pm\alpha}^\times$, hence $\overline{\alpha(\delta)} \in k_{F_\alpha}^1$.

Then we define

$$\varepsilon^{\mathrm{sym}}(\mathbf{J}, \mathbf{T}_{\mathbf{J}}, r, \delta) := \prod_{\alpha \in \Gamma_F \times \{\pm 1\} \backslash (R^{\mathbf{J}}_{\mathbf{y}, r/2} \cap R(\mathbf{J}, \mathbf{T}_{\mathbf{J}})^{\mathrm{sym}})} \varepsilon_\alpha(\delta),$$

$$\varepsilon_{\mathrm{sym,ur}}(\mathbf{J}, \mathbf{T}_{\mathbf{J}}, r, \delta) := \prod_{\alpha \in \Gamma_F \backslash (R^{\mathbf{J}}_{\mathbf{y}, r/2} \cap R(\mathbf{J}, \mathbf{T}_{\mathbf{J}})_{\mathrm{sym,ur}})} \varepsilon_\alpha(\delta),$$

$$\varepsilon^{\mathrm{ram}}(\mathbf{J}, \mathbf{T}_{\mathbf{J}}, r, \delta) := \varepsilon^{\mathrm{sym}}(\mathbf{J}, \mathbf{T}_{\mathbf{J}}, r, \delta) \cdot \varepsilon_{\mathrm{sym,ur}}(\mathbf{J}, \mathbf{T}_{\mathbf{J}}, r, \delta).$$

Here the index set of the first product denotes the set of orbits under the action of $\Gamma_F \times \{\pm 1\}$ (the action of $-1 \in \{\pm 1\}$ is given by $\alpha \mapsto -\alpha$).

- We put $\varepsilon_{\mathrm{sym,ram}}(\mathbf{J}, \mathbf{T}_{\mathbf{J}}, r, \delta)$ to be the product of

$$(-1)^{\mathrm{rank}_{F_{\pm\alpha}}(\mathbf{J}_{\pm\alpha})-1}(-\mathfrak{G})^{f_\alpha}\left(\frac{t_\alpha}{k_{F_\alpha}^\times}\right)\mathrm{sgn}_{F_{\pm\alpha}}(\mathbf{J}_{\pm\alpha})$$

  over $\alpha \in \Gamma_F \backslash (R^{\mathbf{J}}_{\mathbf{y},(r-\mathrm{ord}_\delta)/2} \cap R(\mathbf{J}, \mathbf{T}_{\mathbf{J}})_{\mathrm{sym,ram}})$. We do not recall the definitions of various symbols appearing here (see [DS18, Definition 4.14] for the details). We only remark that the index set is empty when $\mathbf{T}_{\mathbf{J}}$ is unramified. In particular, $\varepsilon_{\mathrm{sym,ram}}(\mathbf{J}, \mathbf{T}_{\mathbf{J}}, r, \delta)$ is always trivial as long as $\mathbf{T}_{\mathbf{J}}$ is unramified.
- We put

$$\tilde{e}(\mathbf{J}, \mathbf{T}_{\mathbf{J}}, r, \delta) := (-1)^{|\Gamma_F \backslash R^{\mathbf{J}}_{\mathbf{y},(r-\mathrm{ord}_\delta)/2}|}.$$

Remark 4.1. (i) When $\mathbf{T}_{\mathbf{J}}$ is elliptic in $\mathbf{J}$, for any $\delta \in T_{\mathbf{J}}$, the image $\alpha(\delta)$ of δ under the root $\alpha\colon \mathbf{T}_{\mathbf{J}}(F_\alpha) \to F_\alpha^\times$ belongs to $\mathcal{O}_{F_\alpha}^\times$. Hence the above definition makes sense for any $\delta \in T_{\mathbf{J}}$ and $\varepsilon^{\mathrm{ram}}(\mathbf{J}, \mathbf{T}_{\mathbf{J}}, r, \delta)$ defines a character on $T_{\mathbf{J}}$.
(ii) By definition, all of $\varepsilon^{\mathrm{ram}}(\mathbf{J}, \mathbf{T}_{\mathbf{J}}, r, \delta)$, $\varepsilon_{\mathrm{sym,ram}}(\mathbf{J}, \mathbf{T}_{\mathbf{J}}, r, \delta)$, and $\tilde{e}(\mathbf{J}, \mathbf{T}_{\mathbf{J}}, r, \delta)$ are invariant under J-conjugation ([DS18, Remark 4.18]).

4.2. **Character values at unramified very regular elements.** Let $(\mathbf{S}, \phi)$ be a tame elliptic regular pair. We take a regular generic reduced cuspidal $\mathbf{G}$-datum

$$(\vec{\mathbf{G}}, \pi_{-1}, \vec{\phi}) := \bigl(\mathbf{G}^0 \subsetneq \mathbf{G}^1 \subsetneq \cdots \subsetneq \mathbf{G}^d, \pi_{-1} \cong \pi^{\mathbf{G}^0}_{(\mathbf{S}, \phi_{-1})}, (\phi_0, \ldots, \phi_d)\bigr)$$

corresponding to $(\mathbf{S}, \phi)$ as in Section 3. In the following, we assume that

$$\mathbf{S} \text{ is an unramified elliptic maximal torus of } \mathbf{G}.$$

Let us list a few consequences of this assumption:

- Recall that, in general, $\mathbf{S}$ is a maximally unramified maximal torus of $\mathbf{G}^0$. Thus the unramifiedness of $\mathbf{S}$ implies the same of $\mathbf{G}^0$ (and the converse also holds). Note that then $\mathbf{G}^1, \ldots, \mathbf{G}^d$ are also unramified.
- Let $\mathbf{x}$ be a point of $\mathcal{B}(\mathbf{G}^0, F)$ whose image $\bar{\mathbf{x}}$ in $\mathcal{B}^{\mathrm{red}}(\mathbf{G}^0, F)$ is associated with $\mathbf{S}$. Then we have

$$G^0_{\bar{\mathbf{x}}} \supset SG^0_{\mathbf{x},0} = Z_{\mathbf{G}^0} G^0_{\mathbf{x},0} = Z_{\mathbf{G}} G^0_{\mathbf{x},0}.$$

  (This follows from the property $S = S_0 Z_{\mathbf{G}}$, see [Kal11, Lemma 7.1.1]).

- Thanks to the equality $SG^0_{\mathbf{x},0} = Z_{\mathbf{G}}G^0_{\mathbf{x},0}$, Kaletha's extension $\kappa_{(\mathbf{S},\phi_{-1})}$ of the representation $\kappa_{(\mathbf{S},\bar{\phi})}$ of $G^0_{\mathbf{x},0}$ to $SG^0_{\mathbf{x},0}$ is simply described as follows: for $g = z \cdot g_0 \in Z_{\mathbf{G}}G^0_{\mathbf{x},0}$, we have

$$\kappa_{(\mathbf{S},\phi_{-1})}(g) = \phi_{-1}(z) \cdot \kappa_{(\mathbf{S},\bar{\phi})}(g_0).$$

Following [CI21b], we introduce the notion of the *unramified very regularity* for elements of $SG_{\mathbf{x},0}$ as follows:

Definition 4.2. We say that an element $\gamma \in SG_{\mathbf{x},0}\,(= Z_{\mathbf{G}}G_{\mathbf{x},0})$ is *unramified very regular (with respect to* $\mathbf{S}$*)* if it satisfies the following conditions:

(1) γ is regular semisimple in $\mathbf{G}$,
(2) the connected centralizer $\mathbf{T}_\gamma := C_{\mathbf{G}}(\gamma)^\circ$ is an unramified maximal torus of $\mathbf{G}$ such that the set $\mathcal{A}(\mathbf{T}_\gamma, F)$ of Frobenius-fixed points of the apartment $\mathcal{A}(\mathbf{T}_{\gamma,F^{\mathrm{ur}}}, F^{\mathrm{ur}})$ associated with $\mathbf{T}_{\gamma,F^{\mathrm{ur}}}$ contains the point $\mathbf{x}$, and
(3) $\alpha(\gamma) \not\equiv 1 \pmod{\mathfrak{p}_F}$ for any root α of $\mathbf{T}_\gamma$ in $\mathbf{G}$.

The purpose of this section is to give an explicit formula for the character $\Theta_{{}^\circ\tau_d}$ of ${}^\circ\tau_d$ at unramified very regular elements of $SG_{\mathbf{x},0}$ (Proposition 4.11). Our formula is just an easy consequence of the theory of Adler–DeBacker–Spice (established in [AS09] and [DS18]) and not new at all. However we have to be careful of the following points:

- for our purpose, we want to compute the character of ${}^\circ\tau_d$, not τ_d as in [AS09], and
- in the full character formula [AS09, Theorem 7.1], the index set is expressed via a set of conjugates of γ, not a set of elements conjugating γ as in Proposition 4.11.

For these reasons, we cannot deduce the formula of Proposition 4.11 from the results of [AS09] immediately. What we will do in the following is just to repeat the proofs of [AS09, Theorems 6.4 and 7.1] while paying attention to these points. However, we note that most of delicate computations carried out in [AS09] can be skipped by focusing only on unramified very regular elements.

We start from checking that every unramified very regular element of $SG_{\mathbf{x},0}$ satisfies the following basic properties necessary for the computation of Adler–Spice:

Lemma 4.3. *Let γ be an unramified very regular element of $SG_{\mathbf{x},0}$. We assume the condition* ($\mathbf{Gd}^G$) *of* [AS08, Definition 6.3] *is satisfied by the maximal torus $\mathbf{T}_\gamma$ of $\mathbf{G}$. Then, for any $r \in \mathbb{R}_{\geq 0}$,*

(i) γ has a normal r-approximation in the sense of [AS08, Definition 6.8]*, and*
(ii) the point $\mathbf{x}$ belongs to the set $\mathcal{B}_r(\gamma)$ defined in [AS08, Definition 9.5].

Proof. Since we have $SG_{\mathbf{x},0} = Z_{\mathbf{G}}G_{\mathbf{x},0}$ by the unramifiedness of $\mathbf{S}$, it is enough to treat the case where $\gamma \in G_{\mathbf{x},0}$.

When $r = 0$, by definition, any element of $G_{\bar{\mathbf{x}}}$ has a normal 0-approximation ([AS08, Definition 6.8]). Furthermore, since we have

$$\mathcal{B}_0(\gamma) := \{\mathbf{y} \in \mathcal{B}(\mathbf{G}, F) \mid \gamma \cdot \bar{\mathbf{y}} = \bar{\mathbf{y}}\}$$

([AS08, Definition 9.5]), the assertion (ii) is obvious.

In the following, we consider the case where $r > 0$. We first check the assertion (i). Since γ belongs to $G_{\mathbf{x},0}$, in particular it is bounded. Moreover, by the unramified very regularity, γ belongs to an unramified torus $\mathbf{T}_\gamma$, which satisfies the condition ($\mathbf{Gd}^G$) by the assumption. Hence γ has a normal r-approximation by [AS08, Lemma 8.1]. More precisely, by applying [AS08, Lemma 8.1] to $\mathbf{G}' = \mathbf{G}$, $d = 0$, $r = r$, and $\mathbf{S} = \mathbf{T}_\gamma$, we can find a normal r-approximation of γ. Note that we do not need to check the assumption that $\widetilde{G}_{0+}$ is contained in the image of G_{0+} of [AS08, Lemma 8.1] by applying Spice's topological Jordan decomposition ([Spi08]) to γ itself instead of $\overline{\gamma}$ in the proof of [AS08, Lemma 8.1].

We take a normal r-approximation $(\gamma_i)_{0\leq i<r}$ of γ and put

$$\gamma_{<r} := \prod_{0\leq i<r} \gamma_i \quad \text{and} \quad \gamma_{\geq r} := \gamma \cdot \gamma_{<r}^{-1}.$$

We next check that the point $\mathbf{x}$ belongs to the set $\mathcal{B}_r(\gamma)$. For this, we recall that the set $\mathcal{B}_r(\gamma)$ is defined by

$$\mathcal{B}_r(\gamma) := \{\mathbf{y} \in \mathcal{B}(C^{(r)}_{\mathbf{G}}(\gamma), F) \mid Z^{(r)}_G(\gamma)\gamma \cap G_{\mathbf{y},r} \neq \varnothing\}.$$

Here the group $C^{(r)}_{\mathbf{G}}(\gamma)$ is defined by

$$C^{(r)}_{\mathbf{G}}(\gamma) := \Bigl(\bigcap_{0\leq i<r} C_{\mathbf{G}}(\gamma_i)\Bigr)^{\circ}$$

and $Z^{(r)}_G(\gamma)$ denotes the group consisting of the F-valued points of the center of $C^{(r)}_{\mathbf{G}}(\gamma)$. We note that the depth-zero part γ_0 of γ is regular semisimple in $\mathbf{G}$ by the unramified very regularity of γ. Indeed, by the definition of a normal approximation, we have

$$\alpha(\gamma) = \alpha(\gamma_{<r} \cdot \gamma_{\geq r}) = \prod_{0\leq i<r} \alpha(\gamma_i) \cdot \alpha(\gamma_{\geq r}) \equiv \alpha(\gamma_0) \pmod{\mathfrak{p}_F}$$

for any root α of $\mathbf{T}_\gamma$ in $\mathbf{G}$. Since $\alpha(\gamma) \not\equiv 1 \pmod{\mathfrak{p}_F}$, also $\alpha(\gamma_0) \not\equiv 1 \pmod{\mathfrak{p}_F}$, hence in particular γ_0 is regular semisimple. Hence the connected centralizer $C_{\mathbf{G}}(\gamma_0)^\circ$ of γ_0 in $\mathbf{G}$ is a maximal torus. By noting the commutativity of γ with γ_0, we see that $C_{\mathbf{G}}(\gamma_0)^\circ$ is nothing but $\mathbf{T}_\gamma$. Thus the group $C^{(r)}_{\mathbf{G}}(\gamma)$ is contained in $C_{\mathbf{G}}(\gamma_0)^\circ = \mathbf{T}_\gamma$. On the other hand, again by the definition of a normal r-approximation, there exists a tame torus $\mathbf{T}$ such that, for every $0 \leq i < r$, γ_i is contained in $\mathbf{T}$. As γ_0 is regular semisimple, this torus $\mathbf{T}$ is necessarily equal to $\mathbf{T}_\gamma$. By this observation, we know that each group $C_{\mathbf{G}}(\gamma_i)$ contains $\mathbf{T}_\gamma$. In conclusion, we have

$$C^{(r)}_{\mathbf{G}}(\gamma) = \mathbf{T}_\gamma \quad \text{and} \quad Z^{(r)}_G(\gamma) = T_\gamma.$$

By the unramified very regularity of γ, the building $\mathcal{B}(C^{(r)}_{\mathbf{G}}(\gamma), F)$ contains the point $\mathbf{x}$. Moreover, obviously the intersection $Z^{(r)}_G(\gamma)\gamma \cap G_{\mathbf{x},r}$ appearing in the definition of $\mathcal{B}_r(\gamma)$ is not empty (both $Z^{(r)}_G(\gamma)\gamma = T_\gamma\gamma = T_\gamma$ and $G_{\mathbf{x},r}$ contain 1). □

In the following, we assume that

> for any unramified very regular element $\gamma \in SG_{\mathbf{x},0}$, the torus $\mathbf{T}_\gamma$ satisfies the condition $(\mathbf{Gd}^G)$.

Remark 4.4. (1) According to [Fin21b, Theorem 3.6], when p does not divide the order of the absolute Weyl group of $\mathbf{G}$, the condition $(\mathbf{Gd}^G)$ is satisfied by any maximal torus of $\mathbf{G}$. (In fact, this condition on p can be slightly weakened more, see also [Fin21b, Remarks 3.4 and 3.7].)

(2) When $\mathbf{G} = \mathrm{GL}_n$, the assumption $(\mathbf{Gd}^G)$ is satisfied by every unramified maximal torus $\mathbf{S}$ without any condition on p. Indeed, such $\mathbf{S}$ is isomorphic to $\prod_{i=1}^{l} \mathrm{Res}_{E_i/F}\,\mathbb{G}_{\mathrm{m}}$ for unramified extensions E_i/F satisfying $\sum_{i=1}^{l}[E_i : F] = n$. Hence S_r (resp. $S_{r:r+}$)

is given by $\prod_{i=1}^{l} E_{i,r}^{\times}$ (resp. $\prod_{i=1}^{l} E_{i,r:r+}^{\times}$). The set of roots of $\mathbf{S}$ in $\mathbf{G}$ is indexed by the set

$$\{(\sigma_i, \sigma_j) \mid \sigma_i \in \mathrm{Gal}(E_i/F),\, \sigma_j \in \mathrm{Gal}(E_j/F) \text{ such that } \sigma_i \neq \sigma_j\}.$$

For any element $\gamma = (\gamma_i)_i$ of $S = \prod_{i=1}^{l} E_i^{\times}$, its image under the root corresponding to (σ_i, σ_j) is given by $\sigma_i(\gamma_i)/\sigma_j(\gamma_j)$. We fix a uniformizer ϖ_F of F (hence of E_i) and regard $k_{E_i}^{\times}$ as a subset of $E_i^{\times}$ via the Teichmüller lift. Then, by the above description of the set of roots of $\mathbf{S}$ in $\mathbf{G}$, we can easily check that $(1+\varpi_F^r\zeta_i)_i \in S_r$ is a good element of depth r for any $(\zeta_i)_i \in \prod_{i=1}^{l} k_{E_i}$. Furthermore, any coset in $S_{r:r+}$ contains such $(1+\varpi_F^r\zeta_i)_i \in S_r$ for some $(\zeta_i)_i \in \prod_{i=1}^{l} k_{E_i}$.

Lemma 4.5 ([Kal19, Corollary 3.4.26])**.** *Let γ_0 be a regular semisimple element of $SG^0_{\mathbf{x},0}$ whose image in G^0_{ad} is topologically semisimple. Then the character $\Theta_{\kappa_{(\mathbf{S},\phi_{-1})}}(\gamma_0)$ of $\kappa_{(\mathbf{S},\phi_{-1})}$ at γ_0 is zero unless γ_0 is $SG^0_{\mathbf{x},0}$-conjugate to an element of S. When γ_0 is an element of S, we have*

$$\Theta_{\kappa_{(\mathbf{S},\phi_{-1})}}(\gamma_0) = (-1)^{r(\mathbf{G}^0)-r(\mathbf{S})} \sum_{w \in W_{G^0_{\mathbf{x},0}}(\mathbf{S})} \phi_{-1}({}^{w}\gamma_0),$$

where

- *$r(\mathbf{G}^0)$ and $r(\mathbf{S})$ are the split ranks of $\mathbf{G}^0$ and $\mathbf{S}$, respectively,*
- *$W_{G^0_{\mathbf{x},0}}(\mathbf{S}) = N_{G^0_{\mathbf{x},0}}(\mathbf{S})/S_0$, and*
- *${}^{w}\gamma_0 = w\gamma_0 w^{-1}$ denotes the w-conjugate of γ_0.*

Remark 4.6. As discussed in [Kal19, Proof of Propositions 3.4.23 and 3.4.24], $r(\mathbf{G}^0)$ (resp. $r(\mathbf{S})$) is equal to the split rank $r(\mathbb{G}^0_{\mathbf{x}})$ of $\mathbb{G}^0_{\mathbf{x}}$ (resp. $r(\mathbb{S})$ of $\mathbb{S}$).

Lemma 4.7. *Let $\gamma \in {}^{\circ}K^d = SG^0_{\mathbf{x},0}(G^0, \ldots, G^d)_{\mathbf{x},(0+,s_0,\ldots,s_{d-1})}$ be an unramified very regular element. Then γ is ${}^{\circ}K^d$-conjugate to an element of $SG^0_{\mathbf{x},0}$.*

Proof. Let γ be an unramified very regular element of ${}^{\circ}K^d$. By Lemma 4.3, we can take a normal s_{d-1}-approximation to γ and then $\mathbf{x}$ belongs to $\mathcal{B}_{s_{d-1}}(\gamma)$. Then, by applying [AS08, Proposition 9.14] to $r = s_{d-1}$ and $(\mathbf{G}', \mathbf{G}) = (\mathbf{G}^{d-1}, \mathbf{G}^d)$, there exists an element $k \in [\gamma; \mathbf{x}, s_{d-1}]_{G^d}$ satisfying ${}^{k}Z_G^{(s_{d-1})}(\gamma) = kZ_G^{(s_{d-1})}(\gamma)k^{-1} \subset G^{d-1}$.

Here we recall that, for a connected reductive group $\mathbf{J}$ over F and an element $\delta \in J$ having a normal t-approximation ($t \in \mathbb{R}_{\geq 0}$) with respect to $\mathbf{y} \in \mathcal{B}(\mathbf{J}, F)$, the group $[\delta; \mathbf{y}, t]_J$ is defined to be the group $\vec{J}_{\mathbf{y},\vec{s}}$ ([AS08, Definition 5.14]) for

- $\vec{\mathbf{J}} := (C_{\mathbf{J}}^{(t-i)}(\delta))_{0<i\leq t}$, and
- $\vec{s} := (i)_{0<i\leq t}$

(see [AS08, Definition 6.6] or [AS09, Section 1.4]). Since our γ is unramified very regular, we have

$$C_{\mathbf{G}^d}^{(s_{d-1}-i)}(\gamma) = \begin{cases} \mathbf{T}_\gamma & 0 < i < s_{d-1}, \\ \mathbf{G}^d & i = s_{d-1}, \end{cases}$$

where $\mathbf{T}_\gamma$ is the connected centralizer of γ in $\mathbf{G}^d$. Hence we have

$$[\gamma; \mathbf{x}, s_{d-1}]_{G^d} = T_{\gamma,0+}G^d_{\mathbf{x},s_{d-1}}.$$

By also noting that $\gamma \in Z_G^{(s_{d-1})}(\gamma) = T_\gamma$, we conclude that there exists an element $k' \in G_{\mathbf{x},s_{d-1}}$ satisfying ${}^{k'}\gamma \in G^{d-1}$. Therefore, after possibly conjugating by an element of ${}^\circ K^d$, we may assume that γ itself lies in G^{d-1} (note that $G_{\mathbf{x},s_{d-1}} \subset {}^\circ K^d$). Then, by a descent property of the subgroups associated to concave functions, we have

$$\begin{aligned}\gamma \in {}^\circ K^d \cap G^{d-1} &= SG^0_{\mathbf{x},0}(G^0,\ldots,G^d)_{\mathbf{x},(0+,s_0,\ldots,s_{d-1})} \cap G^{d-1}\\ &= SG^0_{\mathbf{x},0}(G^0,\ldots,G^{d-1})_{\mathbf{x},(0+,s_0,\ldots,s_{d-2})} = {}^\circ K^{d-1}\end{aligned}$$

(similarly to the proof of [AS09, Lemma 2.4], this is justified by using [AS08, Lemmas 5.29 and 5.33]). By repeating this argument inductively, we finally conclude that some ${}^\circ K^d$-conjugate of γ belongs to $SG^0_{\mathbf{x},0}$. □

The following sign character is of central importance.

Definition 4.8. Define the character $\varepsilon^{\mathrm{ram}}[\phi]\colon S \to \mathbb{C}^\times$ to be

$$\varepsilon^{\mathrm{ram}}[\phi](\gamma) := \prod_{i=0}^{d-1} \varepsilon^{\mathrm{ram}}(\mathbf{G}^{i+1}/\mathbf{G}^i, r_i, \gamma),$$

where

$$\varepsilon^{\mathrm{ram}}(\mathbf{G}^{i+1}/\mathbf{G}^i, r_i, \gamma) := \frac{\varepsilon^{\mathrm{ram}}(\mathbf{G}^{i+1}, \mathbf{S}, r_i, \gamma)}{\varepsilon^{\mathrm{ram}}(\mathbf{G}^{i}, \mathbf{S}, r_i, \gamma)}.$$

Proposition 4.9. *Let γ be an unramified very regular element of ${}^\circ K^d$.*

(1) If γ is not ${}^\circ K^d$-conjugate to an element of S, then we have $\Theta_{{}^\circ\rho'_d\otimes\phi_d}(\gamma) = 0$.

(2) If γ is an element of S, then we have

$$\Theta_{{}^\circ\rho'_d\otimes\phi_d}(\gamma) = (-1)^{r(\mathbf{G}^0)-r(\mathbf{S})+r(\mathbf{S},\phi)} \sum_{w\in W_{G^0_{\mathbf{x},0}}(\mathbf{S})} \varepsilon^{\mathrm{ram}}[\phi]({}^w\gamma)\phi({}^w\gamma),$$

where

- *$r(\mathbf{G}^0)$ and $r(\mathbf{S})$ are the split ranks of $\mathbf{G}^0$ and $\mathbf{S}$,*
- *$r(\mathbf{S},\phi) := \sum_{i=0}^{d-1} |\Gamma_F\backslash(R^{\mathbf{G}^{i+1}}_{\mathbf{x},r_i/2} \smallsetminus R^{\mathbf{G}^{i}}_{\mathbf{x},r_i/2})|$,*
- *$W_{G^0_{\mathbf{x},0}}(\mathbf{S}) = N_{G^0_{\mathbf{x},0}}(\mathbf{S})/S_0$.*

Proof. Since the character $\Theta_{{}^\circ\rho'_d\otimes\phi_d}$ is invariant under ${}^\circ K^d$-conjugation, we may suppose that γ belongs to $SG^0_{\mathbf{x},0}$ by Lemma 4.7. By definition, ${}^\circ\rho'_d$ is the representation of ${}^\circ K^d = {}^\circ K^{d-1}J^d$ descended from the ${}^\circ K^{d-1} \ltimes J^d$-representation $(\tilde{\phi}_{d-1}|_{{}^\circ K^{d-1}\ltimes J^d}) \otimes (({}^\circ\rho'_{d-1}\otimes\phi_{d-1}|_{{}^\circ K^{d-1}}) \ltimes \mathbb{1})$. Hence

$$\Theta_{{}^\circ\rho'_d}(\gamma) = \Theta_{\tilde{\phi}_{d-1}}(\gamma\ltimes 1)\cdot\Theta_{{}^\circ\rho'_{d-1}}(\gamma)\cdot\phi_{d-1}(\gamma).$$

For the same reason, we have

$$\Theta_{{}^\circ\rho'_{d-1}}(\gamma) = \Theta_{\tilde{\phi}_{d-2}}(\gamma\ltimes 1)\cdot\Theta_{{}^\circ\rho'_{d-2}}(\gamma)\cdot\phi_{d-2}(\gamma).$$

By repeating this computation inductively, we get

$$\Theta_{{}^\circ\rho'_d\otimes\phi_d}(\gamma) = \Theta_{{}^\circ\rho'_0}(\gamma)\prod_{i=0}^{d-1}\Theta_{\tilde{\phi}_i}(\gamma\ltimes 1)\prod_{i=0}^{d}\phi_i(\gamma).$$

We recall that ${}^\circ\rho'_0$ is given by $\kappa_{(\mathbf{S},\phi_{-1})}$, hence we have

$$\Theta_{{}^\circ\rho'_0}(\gamma) = \Theta_{\kappa_{(\mathbf{S},\phi_{-1})}}(\gamma_0).$$

If γ_0 is not $SG^0_{\mathbf{x},0}$-conjugate to an element of S, then we have $\Theta_{\kappa_{(\mathbf{S},\phi_{-1})}}(\gamma_0) = 0$ by Lemma 4.5. Hence we have $\Theta_{{}^\circ\rho'_d\otimes\phi_d}(\gamma) = 0$ in this case. By noting that γ_0 is $SG^0_{\mathbf{x},0}$-conjugate to an element of S if and only if so is γ, we get the assertion (1).

We next compute $\Theta_{{}^\circ\rho'_d\otimes\phi_d}(\gamma)$ by assuming that γ belongs to S. Note that then we have $\mathbf{T}_\gamma = \mathbf{S}$. In this case, by the argument in the previous paragraph and Lemma 4.5, we have

$$\begin{aligned}\Theta_{{}^\circ\rho'_d\otimes\phi_d}(\gamma) &= (-1)^{r(\mathbf{G}^0)-r(\mathbf{S})} \sum_{w\in W_{G^0_{\mathbf{x},0}}(\mathbf{S})} \phi_{-1}({}^w\gamma_0) \prod_{i=0}^{d-1} \Theta_{\tilde{\phi}_i}(\gamma \ltimes 1) \prod_{i=0}^{d} \phi_i(\gamma) \\ &= (-1)^{r(\mathbf{G}^0)-r(\mathbf{S})} \prod_{i=0}^{d-1} \Theta_{\tilde{\phi}_i}(\gamma \ltimes 1) \sum_{w\in W_{G^0_{\mathbf{x},0}}(\mathbf{S})} \phi({}^w\gamma)\end{aligned}$$

(recall that $\phi = \prod_{i=-1}^{d} \phi_i$ and ϕ_{-1} is of depth zero).

By [AS09, Proposition 3.8], we have

$$\Theta_{\tilde{\phi}_i}(\gamma \ltimes 1) = |(C^{(0+)}_{G^i}(\gamma), C^{(0+)}_{G^{i+1}}(\gamma))_{\mathbf{x},(r_i,s_i):(r_i,s_i+)}|^{\frac{1}{2}} \cdot \varepsilon(\phi_i,\gamma)$$

with the notations used in [AS09, Proposition 3.8]. Since γ is unramified very regular, its depth zero part γ_0 is regular semisimple in $\mathbf{G}$ so that $C^{(0+)}_{G^i}(\gamma) = C^{(0+)}_{G^{i+1}}(\gamma) = T_\gamma$. Hence the factor $|(C^{(0+)}_{G^i}(\gamma), C^{(0+)}_{G^{i+1}}(\gamma))_{\mathbf{x},(r_i,s_i):(r_i,s_i+)}|^{\frac{1}{2}}$ is trivial.

To understand the factors $\varepsilon(\phi_i,\gamma)$ for $0 \le i < d$, we use [DS18, Proposition 4.21], which implies that the product

$$\mathfrak{G}(\phi_i,\gamma)\varepsilon(\phi_i,\gamma)$$

is equal to

$$\varepsilon_{\mathrm{sym,ram}}(\pi',\gamma)\cdot\varepsilon^{\mathrm{ram}}(\pi',\gamma)\cdot\tilde{e}(\pi',\gamma),$$

where $\mathfrak{G}(\phi_i,\gamma)$ is the constant defined in [AS09, Definition 5.24]. Here we temporarily follow the notation of [DS18]; especially, we are applying results in [DS18] by taking $(\mathbf{G},\mathbf{G}')$ to be $(\mathbf{G}^{i+1},\mathbf{G}^i)$. See [DS18, Definition 4.14] for the definitions of three terms in the above product. Again noting that the centralizer group $C^{(r_i)}_{\mathbf{G}}(\gamma)$ is equal to $\mathbf{T}_\gamma$ by the unramified very regularity of γ, hence equal to $\mathbf{S}$, we get

$$\varepsilon_{\mathrm{sym,ram}}(\pi',\gamma)\cdot\varepsilon^{\mathrm{ram}}(\pi',\gamma)\cdot\tilde{e}(\pi',\gamma) =$$

$$\varepsilon_{\mathrm{sym,ram}}(\mathbf{G}^{i+1}/\mathbf{G}^i,r_i,\gamma)\cdot\varepsilon^{\mathrm{ram}}(\mathbf{G}^{i+1}/\mathbf{G}^i,r_i,\gamma)\cdot\tilde{e}(\mathbf{G}^{i+1}/\mathbf{G}^i,r_i,\gamma),$$

where

$$\varepsilon_{\mathrm{sym,ram}}(\mathbf{G}^{i+1}/\mathbf{G}^i,r_i,\gamma) := \frac{\varepsilon_{\mathrm{sym,ram}}(\mathbf{G}^{i+1},\mathbf{S},r_i,\gamma)}{\varepsilon_{\mathrm{sym,ram}}(\mathbf{G}^{i},\mathbf{S},r_i,\gamma)}$$

(similarly for $\varepsilon^{\mathrm{ram}}$ and $\tilde{e}$).

According to the description in [AS09, Proposition 5.2.13], the invariant $\mathfrak{G}(\phi_i,\gamma)$ is equal to 1 when γ is unramified very regular because the sets $\dot{\Upsilon}_{\mathrm{sym}}(\phi_i,\gamma)$, $\Upsilon_{\mathrm{sym,ram}}(\phi_i,\gamma)$ are empty (see [AS09, Notation 5.2.11]). On the other hand, by Remark 4.1,

- all three terms $\varepsilon_{\mathrm{sym,ram}}(\mathbf{G}^{i+1}/\mathbf{G}^i,r_i,\gamma)$, $\varepsilon^{\mathrm{ram}}(\mathbf{G}^{i+1}/\mathbf{G}^i,r_i,\gamma)$, and $\tilde{e}(\mathbf{G}^{i+1}/\mathbf{G}^i,r_i,\gamma)$, are invariant under G^0-conjugation, and
- the first term $\varepsilon_{\mathrm{sym,ram}}$ is trivial since $\mathbf{S}$ is unramified.

Thus we get

$$\Theta_{{}^\circ\rho'_d\otimes\phi_d}(\gamma) = (-1)^{r(\mathbf{G}^0)-r(\mathbf{S})}\prod_{i=0}^{d-1}\tilde{e}(\mathbf{G}^{i+1}/\mathbf{G}^i, r_i, \gamma)\cdot\varepsilon^{\mathrm{ram}}[\phi](\gamma)\sum_{w\in W_{G^0_{\mathbf{x},0}}(\mathbf{S})}\phi({}^w\gamma)$$

$$= (-1)^{r(\mathbf{G}^0)-r(\mathbf{S})}\prod_{i=0}^{d-1}\tilde{e}(\mathbf{G}^{i+1}/\mathbf{G}^i, r_i, \gamma)\sum_{w\in W_{G^0_{\mathbf{x},0}}(\mathbf{S})}\varepsilon^{\mathrm{ram}}[\phi]({}^w\gamma)\phi({}^w\gamma)$$

We finally investigate the product $\prod_{i=0}^{d-1}\tilde{e}(\mathbf{G}^{i+1}/\mathbf{G}^i, r_i, \gamma)$. Each $\tilde{e}(\mathbf{G}^{i+1}/\mathbf{G}^i, r_i, \gamma)$ is given by the quotient $\tilde{e}(\mathbf{G}^{i+1}, \mathbf{S}, r_i, \gamma)/\tilde{e}(\mathbf{G}^i, \mathbf{S}, r_i, \gamma)$ and we have

$$\tilde{e}(\mathbf{G}^{i+1}, \mathbf{S}, r_i, \gamma) = (-1)^{|\Gamma_F\backslash R^{\mathbf{G}^{i+1}}_{\mathbf{x},(r_i-\mathrm{ord}_\gamma)}|}$$

and

$$\tilde{e}(\mathbf{G}^i, \mathbf{S}, r_i, \gamma) = (-1)^{|\Gamma_F\backslash R^{\mathbf{G}^i}_{\mathbf{x},(r_i-\mathrm{ord}_\gamma)}|}.$$

By the unramified very regularity of γ, for any root $\alpha\in R(\mathbf{G}^i, \mathbf{S})$, we have

$$\alpha(\gamma)\neq 1 \quad\text{and}\quad \mathrm{ord}_\gamma\,\alpha = \mathrm{ord}(\alpha(\gamma)-1) = 0.$$

Thus we have

$$\begin{aligned} R^{\mathbf{G}^{i+1}}_{\mathbf{x},(r_i-\mathrm{ord}_\gamma)/2} &:= \{\alpha\in R(\mathbf{G}^{i+1}, \mathbf{S}) \mid \alpha(\gamma)\neq 1,\ (r_i-\mathrm{ord}_\gamma\,\alpha)/2\in\mathrm{ord}_{\mathbf{x}}\,\alpha\} \\ &= \{\alpha\in R(\mathbf{G}^{i+1}, \mathbf{S}) \mid r_i/2\in\mathrm{ord}_{\mathbf{x}}\,\alpha\} \\ &=: R^{\mathbf{G}^{i+1}}_{\mathbf{x},r_i/2}. \end{aligned}$$

Similarly, we have $R^{\mathbf{G}^i}_{\mathbf{x},(r_i-\mathrm{ord}_\gamma)/2} = R^{\mathbf{G}^i}_{\mathbf{x},r_i/2}$.

Therefore, using notation defined in the assertion, we get

$$\Theta_{{}^\circ\rho'_d\otimes\phi_d}(\gamma) = (-1)^{r(\mathbf{G}^0)-r(\mathbf{S})+r(\mathbf{S},\phi)}\sum_{w\in W_{G^0_{\mathbf{x},0}}(\mathbf{S})}\varepsilon^{\mathrm{ram}}[\phi]({}^w\gamma)\phi({}^w\gamma). \qquad \square$$

Lemma 4.10. *We have the following equalities:*

(1) ${}^\circ K^d\cap N_{G_{\mathbf{x},0}}(\mathbf{S}) = N_{G^0_{\mathbf{x},0}}(\mathbf{S})$,

(2) ${}^\circ K^d\cap N_{G_{\bar{\mathbf{x}}}}(\mathbf{S}) = N_{SG^0_{\mathbf{x},0}}(\mathbf{S})$.

Proof. The first equality can be deduced from the second one. Indeed, by assuming (2), we get

$${}^\circ K^d\cap N_{G_{\mathbf{x},0}}(\mathbf{S}) = {}^\circ K^d\cap N_{G_{\bar{\mathbf{x}}}}(\mathbf{S})\cap G_{\mathbf{x},0} \overset{(2)}{=} N_{SG^0_{\mathbf{x},0}}(\mathbf{S})\cap G_{\mathbf{x},0} = N_{G^0_{\mathbf{x},0}}(\mathbf{S}).$$

Let us show the equality in (2). Since the inclusion ${}^\circ K^d\cap N_{G_{\bar{\mathbf{x}}}}(\mathbf{S})\supset N_{SG^0_{\mathbf{x},0}}(\mathbf{S})$ is trivial, it suffices to check the converse inclusion. Let $g\in{}^\circ K^d\cap N_{G_{\bar{\mathbf{x}}}}(\mathbf{S})$. As g belongs to ${}^\circ K^d = SG^0_{\mathbf{x},0}(G^0,\ldots,G^d)_{\mathbf{x},(0+,s_0,\ldots,s_{d-1})}$, we may write $g = g^0k$ with elements $g^0\in SG^0_{\mathbf{x},0}$ and $k\in(G^0,\ldots,G^d)_{\mathbf{x},(0+,s_0,\ldots,s_{d-1})}$. Additionally, since g normalizes S by assumption, we have ${}^kS\subset G^0$. Then, by using [AS08, Lemma 9.10] with $(\mathbf{G}',\mathbf{G}) := (\mathbf{G}^{d-1},\mathbf{G}^d)$, we get $k\in G^{d-1}_{\mathbf{x},0+}S_{0+} = G^{d-1}_{\mathbf{x},0+}$. Hence $k\in G^{d-1}_{\mathbf{x},0+}\cap(G^0,\ldots,G^d)_{\mathbf{x},(0+,s_0,\ldots,s_{d-1})} = (G^0,\ldots,G^{d-1})_{\mathbf{x},(0+,s_0,\ldots,s_{d-2})}$. By repeatedly applying [AS08, Lemma 9.10] with $(\mathbf{G}',\mathbf{G}) := (\mathbf{G}^{d-2},\mathbf{G}^{d-1}),\ldots,(\mathbf{G}^0,\mathbf{G}^1)$, we finally get $k\in G^0_{\mathbf{x},0+}$. Thus we have $g = g^0k\in SG^0_{\mathbf{x},0}$, which implies that $g\in N_{SG^0_{\mathbf{x},0}}(\mathbf{S})$. $\square$

Proposition 4.11. *Let γ be an unramified very regular element of $SG_{\mathbf{x},0}$.*

(1) If γ is not $SG_{\mathbf{x},0}$-conjugate to an element of S, then we have $\Theta_{{}^\circ\tau_d}(\gamma)=0$.
(2) If γ is an element of S, we have

$$\Theta_{{}^\circ\tau_d}(\gamma)=(-1)^{r(\mathbf{G}^0)-r(\mathbf{S})+r(\mathbf{S},\phi)}\sum_{w\in W_{G_{\mathbf{x},0}}(\mathbf{S})}\varepsilon^{\mathrm{ram}}[\phi]({}^w\gamma)\phi({}^w\gamma),$$

where $W_{G_{\mathbf{x},0}}(\mathbf{S}):=N_{G_{\mathbf{x},0}}(\mathbf{S})/S_0$.

Proof. Let γ be an unramified very regular element of $SG_{\mathbf{x},0}$. Then, since we have ${}^\circ\tau_d\cong\mathrm{Ind}_{{}^\circ K^d}^{SG_{\mathbf{x},0}}({}^\circ\rho'_d\otimes\phi_d)$ by definition, the Frobenius formula for induced representations implies that

$$\Theta_{{}^\circ\tau_d}(\gamma)=\sum_{\substack{g\in{}^\circ K^d\backslash SG_{\mathbf{x},0}\\ {}^g\gamma\in{}^\circ K^d}}\Theta_{{}^\circ\rho'_d\otimes\phi_d}({}^g\gamma).$$

Hence, if γ is not $SG_{\mathbf{x},0}$-conjugate to an element of ${}^\circ K^d$, then the character is equal to zero. Moreover, by Proposition 4.9, if ${}^g\gamma$ is not ${}^\circ K^d$-conjugate to an element of S, then we have $\Theta_{{}^\circ\rho'_d\otimes\phi_d}({}^g\gamma)=0$. Therefore we get the assertion (1).

We now assume that γ belongs to S. Then, again by the same argument as in the previous paragraph, we get

$$\Theta_{{}^\circ\tau_d}(\gamma)=\sum_{\substack{g\in{}^\circ K^d\backslash SG_{\mathbf{x},0}\\ {}^g\gamma\in S}}\Theta_{{}^\circ\rho'_d\otimes\phi_d}({}^g\gamma),$$

where the sum is over elements of ${}^\circ K^d\backslash SG_{\mathbf{x},0}$ containing a representative g satisfying ${}^g\gamma\in S$.

We investigate the index set of this formula. First, as ${}^\circ K^d$ contains S, we may suppose that g belongs to $G_{\mathbf{x},0}$. Since γ is a regular semisimple element belonging to S, if $g\in G_{\mathbf{x},0}$ satisfies ${}^g\gamma\in S$, then g belongs to $N_{G_{\mathbf{x},0}}(\mathbf{S})$. Conversely, any element g of $N_{G_{\mathbf{x},0}}(\mathbf{S})$ satisfies ${}^g\gamma\in S$. In other words, the index set can be rewritten as ${}^\circ K^d\cap N_{G_{\mathbf{x},0}}(\mathbf{S})\backslash N_{G_{\mathbf{x},0}}(\mathbf{S})$, which furthermore equals $N_{G^0_{\mathbf{x},0}}(\mathbf{S})\backslash N_{G_{\mathbf{x},0}}(\mathbf{S})$ by Lemma 4.10 (1).

Then Proposition 4.9 implies that

$$\begin{aligned}\Theta_{{}^\circ\tau_d}(\gamma)&=\sum_{g\in N_{G^0_{\mathbf{x},0}}(\mathbf{S})\backslash N_{G_{\mathbf{x},0}}(\mathbf{S})}(-1)^{r(\mathbf{G}^0)-r(\mathbf{S})+r(\mathbf{S},\phi)}\sum_{w\in W_{G^0_{\mathbf{x},0}}(\mathbf{S})}\varepsilon^{\mathrm{ram}}[\phi]({}^{wg}\gamma)\phi({}^{wg}\gamma)\\&=(-1)^{r(\mathbf{G}^0)-r(\mathbf{S})+r(\mathbf{S},\phi)}\sum_{w\in W_{G_{\mathbf{x},0}}(\mathbf{S})}\varepsilon^{\mathrm{ram}}[\phi]({}^w\gamma)\phi({}^w\gamma).\end{aligned}$$

□

5. Parahoric representations characterized by S_{vreg}

Let $(\mathbf{S},\phi)$ be an elliptic regular pair and let $(\vec{\mathbf{G}},\pi_{-1},\vec{\phi})$ be a regular generic reduced cuspidal $\mathbf{G}$-datum corresponding to $(\mathbf{S},\phi)$ as in Section 3.2. Recall from Section 3.4 that from $(\vec{\mathbf{G}},\pi_{-1},\vec{\phi})$, Yu constructs various intermediate representations; in this section we will be especially interested in ${}^\circ\rho'_d$ and ${}^\circ\tau_d$.

In Section 4.2, we worked with elliptic regular pairs $(\mathbf{S},\phi)$ where $\mathbf{S}$ is unramified. In this section (and in fact in the rest of the paper, excluding Section 6), we additionally assume:

$$\phi \text{ is toral, i.e., } \mathbf{G}^0(\mathbf{S},\phi)=\mathbf{S}.$$

In this case, for any i, the group ${}^\circ K^i=SG^0_{\mathbf{x},0}(G^0,\ldots,G^i)_{\mathbf{x},(0+,s_0,\ldots,s_{i-1})}$ defined in Section 3.4 is equal to the (*a priori* slightly larger) group $K^i=G^0_{\bar{\mathbf{x}}}(G^0,\ldots,G^i)_{\mathbf{x},(0+,s_0,\ldots,s_{i-1})}$, which furthermore equals $Z_{\mathbf{G}}S_0(G^0,\ldots,G^i)_{\mathbf{x},(0+,s_0,\ldots,s_{i-1})}$. Accordingly, for any i, we have

${}^{\circ}\rho'_i = \rho'_i$. We also remark that whether ϕ is toral or not depends only on $\phi|_{S_{0+}}$. In particular, ϕ is toral if and only if $\phi \cdot \varepsilon^{\mathrm{ram}}[\phi]$ is toral since $\varepsilon^{\mathrm{ram}}[\phi]|_{S_{0+}}$ is trivial.

5.1. **Unramified very regular elements.** Let S_{vreg} denote the set of unramified very regular elements of $SG_{\mathbf{x},0}$ contained in S.

Lemma 5.1. *For any $s \in S_{\mathrm{vreg}}$ and any $g_+ \in G_{\mathbf{x},0+}$, the product $\gamma := s \cdot g_+$ is unramified very regular and $\mathbf{T}_\gamma$ is $G_{\mathbf{x},0+}$-conjugate to $\mathbf{S}$.*

Proof. We first take a topological Jordan decomposition (or, equivalently, a normal (0+)-approximation) $s = s_0 \cdot s_+$ with topologically semisimple part s_0 and topologically unipotent part s_+. Then the product decomposition $\gamma = s_0 \cdot (s_+ g_+)$ gives a (0+)-approximation to γ; more precisely, the pair $(\underline{\gamma}, \mathbf{x})$ of the good sequence $\underline{\gamma} = (\gamma_i)_{0 \leq i < 0+}$ consisting of only one element $\gamma_0 := s_0$ and the point $\mathbf{x}$ is a (0+)-approximation to γ. By [AS08, Lemma 9.2], there exists an element $k \in G_{\mathbf{x},0+}$ such that ${}^k\underline{\gamma} = ({}^k\gamma_i)_{0 \leq i < 0+}$ is a normal (0+)-approximation to γ. In other words, we have the following:

- Since ${}^k\underline{\gamma} = ({}^k\gamma_i)_{0 \leq i < 0+}$ is a (0+)-approximation to γ, we have $\gamma \in {}^k\gamma_0 G_{\mathbf{y},0+}$ for some point $\mathbf{y}$ of $\mathcal{B}(C^{(0+)}_{\mathbf{G}}({}^k\underline{\gamma}), F)$. Here note that $C^{(0+)}_{\mathbf{G}}({}^k\underline{\gamma}) = C_{\mathbf{G}}({}^k\gamma_0)^\circ = {}^k\mathbf{S}$ by the regularity of $\gamma_0 = s_0$, which follows from the unramified very regularity of s. Thus, as $k \in G_{\mathbf{x},0+}$ stabilizes $\mathbf{x}$, we have
$$\mathcal{B}(C^{(0+)}_{\mathbf{G}}({}^k\underline{\gamma}), F) = \mathcal{A}({}^k\mathbf{S}, F) = k \cdot \mathcal{A}(\mathbf{S}, F) \ni k \cdot \mathbf{x} = \mathbf{x}.$$
Since $\mathcal{A}^{\mathrm{red}}({}^k\mathbf{S}, F)$ consists of only one point, we have $\mathcal{A}^{\mathrm{red}}({}^k\mathbf{S}, F) = \{\bar{\mathbf{x}}\} = \{\bar{\mathbf{y}}\}$. Let us write $\gamma = {}^k\gamma_0 \cdot \gamma_+$ with $\gamma_+ \in G_{\mathbf{y},0+} = G_{\mathbf{x},0+}$.
- Since ${}^k\underline{\gamma}$ is normal, γ lies in $C^{(0+)}_{\mathbf{G}}({}^k\underline{\gamma})(F) = {}^kS$. Hence so does γ_+.

We now check the unramified very regularity of γ using the decomposition $\gamma = {}^k\gamma_0 \cdot \gamma_+$. Since γ belongs to ${}^kS$, γ is semisimple. Moreover, we claim that γ satisfies $\alpha(\gamma) \not\equiv 1 \pmod{\mathfrak{p}_F}$ for any $\alpha \in R({}^k\mathbf{S}, \mathbf{G})$. Indeed, for any root $\alpha \in R({}^k\mathbf{S}, \mathbf{G})$, we have
$$\alpha(\gamma) = \alpha({}^k\gamma_0) \cdot \alpha(\gamma_+).$$
While the unramified very regularity of s implies that $\alpha({}^k\gamma_0) \not\equiv 1 \pmod{\mathfrak{p}_F}$, we have $\alpha(\gamma_+) \equiv 1 \pmod{\mathfrak{p}_F}$ by the topological unipotency of γ_+. In particular, we have $\alpha(\gamma) \not\equiv 1 \pmod{\mathfrak{p}_F}$. Obviously, this implies $\alpha(\gamma) \neq 1$ for all $\alpha \in R({}^k\mathbf{S}, \mathbf{G})$, and so we see that γ is regular. Finally, since the connected centralizer $\mathbf{T}_\gamma$ is ${}^k\mathbf{S}$, we have $\mathcal{A}(\mathbf{T}_\gamma, F) = \mathcal{A}({}^k\mathbf{S}, F) \ni \mathbf{x}$, which now completes the proof that γ is unramified very regular and also that $\mathbf{T}_\gamma$ is $G_{\mathbf{x},0+}$-conjugate to $\mathbf{S}$ (since $k \in G_{\mathbf{x},0+}$). □

Lemma 5.2. *We have $S_{\mathrm{vreg}} G_{\mathbf{x},0+} = \{\gamma \in SG_{\mathbf{x},0+} \mid \gamma$ is unramified very regular$\}$. Moreover, every unramified very regular element of $SG_{\mathbf{x},0+}$ is $G_{\mathbf{x},0+}$-conjugate to an element of S_{vreg}.*

Proof. By Lemma 5.1, every element of $S_{\mathrm{vreg}} G_{\mathbf{x},0+}$ is unramified very regular. To see the reverse inclusion, let $\gamma \in SG_{\mathbf{x},0+}$ be unramified very regular. We may write $\gamma = s \cdot g_+$ for some $s \in S$ and $g_+ \in G_{\mathbf{x},0+}$. By Lemma 5.1, $s \in S_{\mathrm{vreg}}$, so $\gamma \in S_{\mathrm{vreg}} G_{\mathbf{x},0+}$.

Since every element of $S_{\mathrm{vreg}} G_{\mathbf{x},0+}$ is $G_{\mathbf{x},0+}$-conjugate to an element of S_{vreg} by Lemma 5.1, the final assertion now holds. □

We put $S_{0,\mathrm{vreg}} := S_{\mathrm{vreg}} \cap S_0$. Note that, by the definition of unramified very regular elements, we can easily see that $S_{\mathrm{vreg}} = Z_{\mathbf{G}} S_{0,\mathrm{vreg}}$ (recall that $S = Z_{\mathbf{G}} S_0$).

Definition 5.3. We define the subset $\mathbb{S}(\mathbb{F}_q)_{\mathrm{vreg}}$ of $\mathbb{S}(\mathbb{F}_q)$ to be the image of $S_{0,\mathrm{vreg}}$ under the reduction map $S_0 \twoheadrightarrow S_{0:0+} \cong \mathbb{S}(\mathbb{F}_q)$. Let $\mathbb{S}(\mathbb{F}_q)_{\mathrm{nvreg}}$ denote its complement in $\mathbb{S}(\mathbb{F}_q)$, i.e., $\mathbb{S}(\mathbb{F}_q)_{\mathrm{nvreg}} := \mathbb{S}(\mathbb{F}_q) \smallsetminus \mathbb{S}(\mathbb{F}_q)_{\mathrm{vreg}}$.

Lemma 5.4. *When $|\mathbb{S}(\mathbb{F}_q)|/|\mathbb{S}(\mathbb{F}_q)_{\mathrm{nvreg}}| > 2$, for any element $s \in \mathbb{S}(\mathbb{F}_q)$, there exist elements $t_1, t_2 \in \mathbb{S}(\mathbb{F}_q)_{\mathrm{vreg}}$ such that $s = t_1t_2$.*

Proof. For a given $s \in \mathbb{S}(\mathbb{F}_q)$, we consider the subset $s \cdot \mathbb{S}(\mathbb{F}_q)_{\mathrm{vreg}}$ of $\mathbb{S}(\mathbb{F}_q)$. If we have $|s \cdot \mathbb{S}(\mathbb{F}_q)_{\mathrm{vreg}}| > |\mathbb{S}(\mathbb{F}_q)_{\mathrm{nvreg}}|$, then $s \cdot \mathbb{S}(\mathbb{F}_q)_{\mathrm{vreg}}$ is not contained in $\mathbb{S}(\mathbb{F}_q)_{\mathrm{nvreg}}$. In other words, $s \cdot \mathbb{S}(\mathbb{F}_q)_{\mathrm{vreg}}$ intersects $\mathbb{S}(\mathbb{F}_q)_{\mathrm{vreg}}$. Thus there exist elements $t_1, t_2 \in \mathbb{S}(\mathbb{F}_q)_{\mathrm{vreg}}$ such that $st_1 = t_2$. By noting that the set $S_{0,\mathrm{vreg}}$ is stable under the inversion, t_1^{-1} lies in $\mathbb{S}(\mathbb{F}_q)_{\mathrm{vreg}}$ and we get the assertion. The inequality $|s \cdot \mathbb{S}(\mathbb{F}_q)_{\mathrm{vreg}}| > |\mathbb{S}(\mathbb{F}_q)_{\mathrm{nvreg}}|$ is equivalent to $|\mathbb{S}(\mathbb{F}_q)|/|\mathbb{S}(\mathbb{F}_q)_{\mathrm{nvreg}}| > 2$. □

Corollary 5.5. *When $|\mathbb{S}(\mathbb{F}_q)|/|\mathbb{S}(\mathbb{F}_q)_{\mathrm{nvreg}}| > 2$, for any element $s \in S_0$, there exist elements $t_1, t_2 \in S_{0,\mathrm{vreg}}$ such that $s = t_1t_2$. In particular, $S_{0,\mathrm{vreg}}$ generates S_0 as a group.*

Proof. Let s be an element of S_0. By the assumption and Lemma 5.4, we can take elements t_1 and t_2 of $S_{0,\mathrm{vreg}}$ such that t_1t_2 and s have the same image in $S_{0:0+} \cong \mathbb{S}(\mathbb{F}_q)$. In other words, there exists an element $t_+ \in S_{0+}$ satisfying $t_1t_2t_+ = s$. By Lemma 5.1, t_2t_+ is an unramified very regular element of S. Since t_2t_+ also belongs to S_0, this shows that s can be written as a product of two elements of $S_{0,\mathrm{vreg}}$. □

Let us show that the inequality

$$(\star) \qquad \frac{|\mathbb{S}(\mathbb{F}_q)|}{|\mathbb{S}(\mathbb{F}_q)_{\mathrm{nvreg}}|} > 2$$

is satisfied when q is sufficiently large.

Lemma 5.6. *The unramified elliptic maximal torus $\mathbf{S}$ of $\mathbf{G}$ transfers to an unramified elliptic maximal torus $\mathbf{S}^*$ of the quasi-split inner form $\mathbf{G}^*$ of $\mathbf{G}$ such that the associated point $\bar{\mathbf{x}}^*$ of the building $\mathcal{B}^{\mathrm{red}}(\mathbf{G}^*, F)$ corresponds to a Chevalley valuation of $\mathbf{G}^*$.*

Proof. The precise meaning of the "transfer" is as follows: there exists an inner twist $\psi\colon \mathbf{G} \to \mathbf{G}^*$ such that its restriction to $\mathbf{S}$ induces an isomorphism $\psi|_{\mathbf{S}}\colon \mathbf{S} \to \mathbf{S}^*$ defined over F. The existence of a transfer $\mathbf{S}^*$ of $\mathbf{S}$ is a standard fact guaranteed by the ellipticity of $\mathbf{S}$ or the quasi-splitness of $\mathbf{G}^*$; see, for example, [Kal19, Section 3.2] for references about this fact. Note that $\mathbf{S}^*$ is unramified and elliptic since $\psi|_{\mathbf{S}}$ is defined over F and maps $\mathbf{Z}_{\mathbf{G}}$ to $\mathbf{Z}_{\mathbf{G}^*}$.

Hence it suffices to show that such an $\mathbf{S}^*$ can be taken so that the associated point $\bar{\mathbf{x}}^* \in \mathcal{B}^{\mathrm{red}}(\mathbf{G}^*, F)$ corresponds to a Chevalley valuation. For this, we utilize the results of Kaletha in [Kal19, Section 3.4.1]. Since $\mathbf{G}^*$ is quasi-split, there exists a point $\bar{\mathbf{x}}_1^* \in \mathcal{B}^{\mathrm{red}}(\mathbf{G}^*, F)$ which corresponds to a Chevalley valuation; such a point $\bar{\mathbf{x}}_1^*$ is superspecial in the sense of Kaletha (see [Kal19, Remark 3.4.9]). By applying [Kal19, Lemma 3.4.12 (1)] to $\bar{\mathbf{x}}_1^*$ and $\mathbf{S}^* \subset \mathbf{G}^*$, we can find a maximal torus $\mathbf{S}_1^*$ of $\mathbf{G}^*$ stably conjugate to $\mathbf{S}^*$ with associated point $\bar{\mathbf{x}}_1^*$. By replacing $\mathbf{S}^*$ with $\mathbf{S}_1^*$ (hence $\bar{\mathbf{x}}^*$ with $\bar{\mathbf{x}}_1^*$), we obtain the desired assertion. □

Lemma 5.7. *Let $\psi\colon \mathbf{G} \to \mathbf{G}^*$ be an inner twist whose restriction to $\mathbf{S}$ gives an isomorphism $\mathbf{S} \cong \mathbf{S}^*$ defined over F. Then, under the isomorphism $\mathbb{S}(\mathbb{F}_q) \to \mathbb{S}^*(\mathbb{F}_q)$ induced by ψ, $\mathbb{S}(\mathbb{F}_q)_{\mathrm{vreg}}$ is identified with $\mathbb{S}^*(\mathbb{F}_q)_{\mathrm{vreg}}$.*

Proof. As $\psi|_{\mathbf{S}}$ is defined over F, we have identifications $S \cong S^*$ and $\mathbb{S}(\mathbb{F}_q) \cong \mathbb{S}^*(\mathbb{F}_q)$ which are consistent with reduction morphisms:

$$\begin{array}{ccc} S_0 & \xrightarrow[\cong]{\psi} & S_0^* \\ \downarrow & & \downarrow \\ \mathbb{S}(\mathbb{F}_q) & \xrightarrow[\cong]{\psi} & \mathbb{S}^*(\mathbb{F}_q) \end{array}$$

Since ψ induces a bijection $R(\mathbf{S}, \mathbf{G}) \to R(\mathbf{S}^*, \mathbf{G}^*)\colon \alpha \mapsto \alpha \circ \psi|_{\mathbf{S}}^{-1}$, ψ gives an identification of S_{vreg} with S^*_{vreg}, and also $S_{0,\mathrm{vreg}}$ with $S^*_{0,\mathrm{vreg}}$, by the definition of unramified very regularity. As $\mathbb{S}(\mathbb{F}_q)_{\mathrm{vreg}}$ (resp. $\mathbb{S}^*(\mathbb{F}_q)_{\mathrm{vreg}}$) is defined to be the image of $S_{0,\mathrm{vreg}}$ (resp. $S^*_{0,\mathrm{vreg}}$) under the reduction morphism, ψ gives an identification of $\mathbb{S}(\mathbb{F}_q)_{\mathrm{vreg}}$ with $\mathbb{S}^*(\mathbb{F}_q)_{\mathrm{vreg}}$. □

Proposition 5.8. *There exists a constant C depending only on the absolute rank of $\mathbf{G}$ such that the inequality ($\star$) is satisfied when $q > C$.*

Proof. We take an inner twist $\psi\colon \mathbf{G} \to \mathbf{G}^*$ transferring $\mathbf{S}$ to $\mathbf{S}^*$ as in Lemma 5.6. Then, by Lemma 5.7, we have

$$\frac{|\mathbb{S}(\mathbb{F}_q)|}{|\mathbb{S}(\mathbb{F}_q)_{\mathrm{nvreg}}|} = \frac{|\mathbb{S}^*(\mathbb{F}_q)|}{|\mathbb{S}^*(\mathbb{F}_q)_{\mathrm{nvreg}}|}.$$

Hence it is enough to show the assertion for $\mathbf{S}^* \subset \mathbf{G}^*$ whose associated point $\bar{\mathbf{x}}^* \in \mathcal{B}^{\mathrm{red}}(\mathbf{G}^*, F)$ corresponds to a Chevalley valuation.

Let $\mathbb{G}^*_{\mathbf{x}^*}$ be the reductive quotient of the special fiber of the parahoric subgroup scheme of $\mathbf{G}^*$ with respect to $\bar{\mathbf{x}}^*$, and regard $\mathbb{S}^*$ as a maximal torus of $\mathbb{G}^*_{\mathbf{x}^*}$. Since the point $\bar{\mathbf{x}}^*$ corresponds to a Chevalley valuation, the reduction map induces a bijection from $R(\mathbf{S}^*, \mathbf{G}^*)$ to $R(\mathbb{S}^*, \mathbb{G}^*_{\mathbf{x}^*})$. This implies that an element $\gamma \in S_0^*$ is unramified very regular if and only if its reduction $\overline{\gamma} \in \mathbb{S}^*(\mathbb{F}_q)$ is regular semisimple in $\mathbb{G}^*_{\mathbf{x}^*}$. Therefore the set $\mathbb{S}^*(\mathbb{F}_q)_{\mathrm{vreg}}$ is nothing but the set $\mathbb{S}^*_{\mathrm{reg}}(\mathbb{F}_q)$ of $\mathbb{F}_q$-valued points of the regular semisimple locus $\mathbb{S}^*_{\mathrm{reg}}$ of $\mathbb{S}^*$ in $\mathbb{G}^*_{\mathbf{x}^*}$. Similarly, $\mathbb{S}^*(\mathbb{F}_q)_{\mathrm{nvreg}}$ $(:= \mathbb{S}^*(\mathbb{F}_q) \smallsetminus \mathbb{S}^*(\mathbb{F}_q)_{\mathrm{vreg}})$ is nothing but $\mathbb{S}^*_{\mathrm{nreg}}(\mathbb{F}_q)$ $(:= \mathbb{S}^*(\mathbb{F}_q) \smallsetminus \mathbb{S}^*_{\mathrm{reg}}(\mathbb{F}_q))$.

Now the statement follows from a well-known fact of finite reductive groups. For example, by [GM20, Lemma 2.3.11], for any connected reductive group $\mathbb{G}$ over $\mathbb{F}_q$ and its $\mathbb{F}_q$-rational maximal torus $\mathbb{T}$, there exists a positive number $C_R > 0$ depending only on the absolute root system R of $\mathbb{G}$ such that we have

$$\frac{|\mathbb{T}_{\mathrm{reg}}(\mathbb{F}_q)|}{|\mathbb{T}(\mathbb{F}_q)|} \geq 1 - \frac{C_R}{q},$$

or equivalently,

$$\frac{|\mathbb{T}(\mathbb{F}_q)|}{|\mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q)|} \geq \frac{q}{C_R}.$$

(Although it is stated that the constant C_R depends only on the root datum of $\mathbb{G}$ in [GM20, Lemma 2.3.11], we can check that in fact such C_R depends only on the root system of $\mathbb{G}$.) Note that, if we let l be the absolute rank of $\mathbf{G}$, then $\mathbb{G}^*_{\mathbf{x}^*}$ is a connected reductive group over $\mathbb{F}_q$ whose rank is at most l. Thus, by putting

$$C' := \max\{C_R \mid R \text{ is a root system whose rank is at most } l\}$$

(note that the set of root systems with rank at most l is finite), we get

$$\frac{|\mathbb{S}^*(\mathbb{F}_q)|}{|\mathbb{S}^*_{\mathrm{nreg}}(\mathbb{F}_q)|} \geq \frac{q}{C'}.$$

Thus $C := 2C'$ satisfies the desired condition. □

Remark 5.9. The subtlety handled in the above lemmas culminating in Proposition 5.8 is exactly that in general $\mathbb{S}(\mathbb{F}_q)_{\mathrm{vreg}} \subsetneq \mathbb{S}_{\mathrm{reg}}(\mathbb{F}_q)$. For example, in the extreme case that $G_{\mathbf{x},0}$ is an Iwahori subgroup, $\mathbb{S}_{\mathrm{reg}}(\mathbb{F}_q) = \mathbb{S}(\mathbb{F}_q)$.

Remark 5.10. There is a natural question of how large q must be in order for $(\star)$ to be satisfied. The estimate in Proposition in 5.8 is very crude since we deduced it only from the absolute rank of $\mathbf{G}$. As long as $\mathbf{G}$ and $\mathbf{S}$ are given explicitly, we can determine the associated groups $\mathbb{G}^*_{\mathbf{x}^*}$ and $\mathbb{S}^*$ explicitly; hence it is possible to compute the ratio $|\mathbb{S}(\mathbb{F}_q)|/|\mathbb{S}(\mathbb{F}_q)_{\mathrm{nvreg}}|$ precisely. At least for Coxeter tori of split simple groups, the resulting bound is very mild (forthcoming work). For example, if $\mathbf{S}$ is the Coxeter torus of $\mathbf{G} = G_2$, then $|\mathbb{S}(\mathbb{F}_q)_{\mathrm{nreg}}|$ is either 1 or 3, depending on q, and $|\mathbb{S}(\mathbb{F}_q)| \geq 7$ for $q > 3$. Hence only $q = 2$ does not satisfy $(\star)$, so in fact $(\star)$ is a *weaker* condition than the conditions on p required by the theory of Kaletha and Yu reviewed in Section 3. The GL_n case (which was known already to Henniart [Hen92]) is explained in Section 7.3.

5.2. **Representations of $SG_{\mathbf{x},0+}$ characterized by their trace on S_{vreg}.** In this subsection, we prove (Theorem 5.13) that some irreducible virtual representations of $SG_{\mathbf{x},0+}$ are characterized by character values on unramified very regular elements S_{vreg} of S. In this subsection, in addition to our basic assumptions introduced in Section 2.1, we will also assume that

$$(\mathfrak{vreg}) \qquad S_{0,\mathrm{vreg}} \text{ generates } S_0 \text{ as a group.}$$

As discussed in Section 5.1, this assumption is satisfied when the inequality $(\star)$, which is true for $q \gg 0$, holds.

This subsection pertains to smooth irreducible representations τ_+ of $SG_{\mathbf{x},0+}$ such that for a nonzero constant $c \in \mathbb{C}$,

$$(\dagger_+) \qquad \Theta_{\tau_+}(\gamma) = c \cdot \theta(\gamma), \qquad \text{for all } \gamma \in S_{\mathrm{vreg}}$$

for a character θ of S with $\mathbf{G}^0(\mathbf{S}, \theta) = \mathbf{S}$. This determines Θ_{τ_+} on the unramified very regular elements of $SG_{\mathbf{x},0+}$ since every unramified regular element of $SG_{\mathbf{x},0+}$ is $G_{\mathbf{x},0+}$-conjugate to an element of S_{vreg} by Lemma 5.2.

Recall that at the beginning of this section, we fixed a regular generic reduced cuspidal $\mathbf{G}$-datum $(\vec{\mathbf{G}}, \pi_{-1}, \vec{\phi})$ corresponding to a tame elliptic regular pair $(\mathbf{S}, \phi)$ whose ϕ is toral. Also recall that this gives rise to the following chain of open compact subgroups (note that $\mathbf{G}^0 = \mathbf{S}$ by the torality assumption on ϕ):

$$\begin{aligned} S_0G_{\mathbf{x},0+} &\supset S_0G_{\mathbf{x},0+} \cap K^d = S_0(G^0, \ldots, G^d)_{\mathbf{x},(0+,s_0,\ldots,s_{d-1})} \\ &\supset G_{\mathbf{x},0+} \cap K^d = (G^0, \ldots, G^d)_{\mathbf{x},(0+,s_0,\ldots,s_{d-1})} \end{aligned}$$

Let r_d denote the depth of the character ϕ.

Lemma 5.11. *For any $t \in \widetilde{\mathbb{R}}_{>0}$ satisfying $t > r_d$, we put $(S_{0,\mathrm{vreg}}G_{\mathbf{x},0+} \cap K^d)'_t$ to be the set*

$$\{\gamma \in S_{0,\mathrm{vreg}}G_{\mathbf{x},0+} \cap K^d \mid {}^k\gamma' \in S_0 \text{ for some } k \in K^d \text{ and } \gamma' \in \gamma G_{\mathbf{x},t}\}.$$

Then the association $(\gamma, k) \mapsto {}^k\gamma$ gives a bijection

$$(S_{0,\mathrm{vreg}}/S_t) \times (K^d/SG_{\mathbf{x},t}) \xrightarrow{1:1} (S_{0,\mathrm{vreg}}G_{\mathbf{x},0+} \cap K^d)'_t/G_{\mathbf{x},t},$$

where $S_t := S_0 \cap G_{\mathbf{x},t}$.

Proof. Let us first check that the association $(\gamma, k) \mapsto {}^k\gamma$ gives a well-defined map $S_{0,\mathrm{vreg}} \times K^d \to (S_{0,\mathrm{vreg}}G_{\mathbf{x},0+} \cap K^d)'_t$. For any $\gamma \in S_{0,\mathrm{vreg}}$ and $k \in K^d$, ${}^k\gamma$ belongs to K^d. As $S_0G_{\mathbf{x},0+}$ is normal in K^d (this follows from the assumption that $\mathbf{G}^0 = \mathbf{S}$), ${}^k\gamma$ also belongs to $S_0G_{\mathbf{x},0+}$.

Since ${}^k\gamma$ is unramified very regular, Lemma 5.2 implies that ${}^k\gamma$ lies in $S_{\mathrm{vreg}}G_{\mathbf{x},0+}$, hence in $S_{\mathrm{vreg}}G_{\mathbf{x},0+} \cap S_0G_{\mathbf{x},0+} = S_{0,\mathrm{vreg}}G_{\mathbf{x},0+}$. Thus we get ${}^k\gamma \in (S_{0,\mathrm{vreg}}G_{\mathbf{x},0+} \cap K^d)'_t$.

Let us next show the surjectivity of the map

$$(1) \qquad S_{0,\mathrm{vreg}} \times K^d \to (S_{0,\mathrm{vreg}}G_{\mathbf{x},0+} \cap K^d)'_t/G_{\mathbf{x},t} \colon (\gamma, k) \mapsto {}^k\gamma \cdot G_{\mathbf{x},t}.$$

If $\gamma \in (S_{0,\mathrm{vreg}}G_{\mathbf{x},0+} \cap K^d)'_t$, then we can find $k \in K^d$ and $\gamma' \in \gamma G_{\mathbf{x},t}$ such that ${}^k\gamma' \in S_0$. Since γ is unramified very regular by Lemma 5.2, we have ${}^k\gamma' \in S_{0,\mathrm{vreg}}$ again by Lemma 5.2. Thus the coset $\gamma \cdot G_{\mathbf{x},t}$ is the image of $({}^k\gamma', k^{-1}) \in S_{0,\mathrm{vreg}} \times K^d$ under the map (1).

We consider the fibers of the map (1). Suppose that the images of $(\gamma_1, k_1) \in S_{0,\mathrm{vreg}} \times K^d$ and $(\gamma_2, k_2) \in S_{0,\mathrm{vreg}} \times K^d$ under this map coincide, i.e., ${}^{k_1}\gamma_1 \cdot G_{\mathbf{x},t} = {}^{k_2}\gamma_2 \cdot G_{\mathbf{x},t}$, or equivalently, ${}^{k_1^{-1}k_2}\gamma_2 \in \gamma_1 G_{\mathbf{x},t}$. By using [AS08, Proposition 9.14] with $\mathbf{G}' := \mathbf{S}$ (note that ${}^{k_1^{-1}k_2}\gamma_2 \in \gamma_1 G_{\mathbf{x},t} \subset SG_{\mathbf{x},t}$), we see that ${}^k(Z_G^{(t)}({}^{k_1^{-1}k_2}\gamma_2)) \subset S$ for some $k \in [{}^{k_1^{-1}k_2}\gamma_2; \mathbf{x}, t]$ with the notation as in [AS08, Proposition 9.14]. As in the proof of Lemma 4.7, we can check that $Z_G^{(t)}({}^{k_1^{-1}k_2}\gamma_2) = {}^{k_1^{-1}k_2}S$ and $[{}^{k_1^{-1}k_2}\gamma_2; \mathbf{x}, t] = {}^{k_1^{-1}k_2}S_{0+} \cdot G_{\mathbf{x},t}\,(\subset K^d)$ by the unramified very regularity of γ_2. Let us write $k = gs$ with $s \in {}^{k_1^{-1}k_2}S$ and $g \in G_{\mathbf{x},t}$ (note that ${}^{k_1^{-1}k_2}S_{0+} \cdot G_{\mathbf{x},t} = G_{\mathbf{x},t} \cdot {}^{k_1^{-1}k_2}S_{0+}$). Then, since ${}^{gs}({}^{k_1^{-1}k_2}S) = {}^g({}^{k_1^{-1}k_2}S)$, we may assume that $k \in G_{\mathbf{x},t}$ by replacing $k = gs$ with g.

Note that $kk_1^{-1}k_2$ normalizes S. Since the assumption ($\mathfrak{vreg}$) assures that there exists at least one unramified very regular element in S, this implies that $kk_1^{-1}k_2$ normalizes $\mathbf{S}$. Hence, by Lemma 4.10 (1), we have $kk_1^{-1}k_2 \in S$. In other words, $k_1 \equiv k_2$ in $K^d/SG_{\mathbf{x},t}$. We write $k_1^{-1}k_2 = g's'$ with $s' \in S$ and $g' \in G_{\mathbf{x},t}$. Then, as we have ${}^{k_1^{-1}k_2}\gamma_2 \in \gamma_1 G_{\mathbf{x},t}$, we get ${}^{g'}\gamma_2 \equiv \gamma_1$ in $K^d/G_{\mathbf{x},t}$. Hence we have $g' \cdot \gamma_2 g'^{-1}\gamma_2^{-1} \equiv \gamma_1\gamma_2^{-1}$ in $K^d/G_{\mathbf{x},t}$. Since both g' and $\gamma_2 g'^{-1}\gamma_2^{-1}$ belong to $G_{\mathbf{x},t}$, this implies that $\gamma_1 \equiv \gamma_2$ in $K^d/G_{\mathbf{x},t}$, hence in S_0/S_t.

Conversely, any (γ_1, k_1) and (γ_2, k_2) satisfying $\gamma_1 \equiv \gamma_2$ in S_0/S_t and $k_1 \equiv k_2$ in $K^d/SG_{\mathbf{x},t}$ map to the same $G_{\mathbf{x},t}$-coset. Therefore the above map (1) induces a bijection

$$(S_{0,\mathrm{vreg}}/S_t) \times (K^d/SG_{\mathbf{x},t}) \xrightarrow{1:1} (S_{0,\mathrm{vreg}}G_{\mathbf{x},0+} \cap K^d)'_t/G_{\mathbf{x},t}. \qquad \square$$

Lemma 5.12. *Let τ_+ be a smooth irreducible representation of $SG_{\mathbf{x},0+}$ satisfying ($\dagger_+$) for the character $\theta = \phi \cdot \varepsilon^{\mathrm{ram}}[\phi]$ of S. Then*

$$\mathrm{Hom}_{G_{\mathbf{x},0+} \cap K^d}(\rho'_d \otimes \phi_d, \tau_+) \neq 0.$$

Proof. Because of the smoothness assumption, the action of $S_0G_{\mathbf{x},0+}$ on τ_+ factors through a finite quotient $S_0G_{\mathbf{x},0+}/G_{\mathbf{x},t}$ for sufficiently large $t \in \mathbb{R}_{>0}$. Fix such a t which is greater than the depth of the toral character ϕ. Note that then also $\rho'_d \otimes \phi_d$ is trivial on $G_{\mathbf{x},t}$.

We first prove that

$$(2) \qquad \sum_{\gamma \in (S_{0,\mathrm{vreg}}G_{\mathbf{x},0+} \cap K^d)/G_{\mathbf{x},t}} \Theta_{\rho'_d \otimes \phi_d}(\gamma) \cdot \overline{\Theta_{\tau_+}(\gamma)} \neq 0.$$

By Proposition 4.9, $\Theta_{\rho'_d \otimes \phi_d}(\gamma) = 0$ if $\gamma \in S_{0,\mathrm{vreg}}G_{\mathbf{x},0+} \cap K^d$ is not K^d-conjugate to an element of S. Thus, by putting $(S_{0,\mathrm{vreg}}G_{\mathbf{x},0+} \cap K^d)'_t$ to be the set as in Lemma 5.11, we get

$$\sum_{\gamma \in (S_{0,\mathrm{vreg}}G_{\mathbf{x},0+} \cap K^d)/G_{\mathbf{x},t}} \Theta_{\rho'_d \otimes \phi_d}(\gamma) \cdot \overline{\Theta_{\tau_+}(\gamma)} = \sum_{\gamma \in (S_{0,\mathrm{vreg}}G_{\mathbf{x},0+} \cap K^d)'_t/G_{\mathbf{x},t}} \Theta_{\rho'_d \otimes \phi_d}(\gamma) \cdot \overline{\Theta_{\tau_+}(\gamma)}.$$

By Lemma 5.11, the right-hand side equals

$$\sum_{\gamma \in S_{0,\mathrm{vreg}}/S_t} \sum_{k \in K^d/SG_{\mathbf{x},t}} \Theta_{\rho'_d \otimes \phi_d}({}^k\gamma) \cdot \overline{\Theta_{\tau_+}({}^k\gamma)}.$$

Since both of $\Theta_{\rho'_d\otimes\phi_d}$ and $\overline{\Theta_{\tau_+}}$ are invariant under K^d-conjugation (note that K^d is contained in $SG_{\mathbf{x},0+}$, where τ_+ is defined), we get

$$\sum_{\gamma\in S_{0,\mathrm{vreg}}/S_t}\sum_{k\in K^d/SG_{\mathbf{x},t}}\Theta_{\rho'_d\otimes\phi_d}({}^k\gamma)\cdot\overline{\Theta_{\tau_+}({}^k\gamma)}=|K^d/SG_{\mathbf{x},t}|\sum_{\gamma\in S_{0,\mathrm{vreg}}/S_t}\Theta_{\rho'_d\otimes\phi_d}(\gamma)\cdot\overline{\Theta_{\tau_+}(\gamma)}.$$

By Proposition 4.9 and our assumption on τ_+, for any $\gamma\in S_{0,\mathrm{vreg}}$, we have

$$\Theta_{\rho'_d\otimes\phi_d}(\gamma)\cdot\overline{\Theta_{\tau_+}(\gamma)}=(-1)^{r(\mathbf{S},\phi)}\cdot\bar{c}\cdot\theta(\gamma)\overline{\theta(\gamma)}$$

(note that here the sum over $w\in W_{G^0_{\mathbf{x},0}}(\mathbf{S})$ as in Prop 4.9 does not appear since $\mathbf{G}^0=\mathbf{S}$ by the torality assumption on ϕ). Thus we get

$$\sum_{\gamma\in S_{0,\mathrm{vreg}}/S_t}\Theta_{\rho'_d\otimes\phi_d}(\gamma)\cdot\overline{\Theta_{\tau_+}(\gamma)}=(-1)^{r(\mathbf{S},\phi)}\bar{c}\sum_{\gamma\in S_{0,\mathrm{vreg}}/S_t}\theta(\gamma)\overline{\theta(\gamma)}.$$

Since $\theta(\gamma)\overline{\theta(\gamma)}$ is a positive number, this sum is not zero (recall that $S_{0,\mathrm{vreg}}\neq\varnothing$). Hence we get the assertion (2).

We next show

$$\sum_{\gamma\in(G_{\mathbf{x},0+}\cap K^d)/G_{\mathbf{x},t}}\Theta_{\rho'_d\otimes\phi_d}(\gamma)\cdot\overline{\Theta_{\tau_+}(\gamma)}\neq 0$$

by using (2), which we proved just now. For any subset $A\subset R(\mathbf{S},\mathbf{G})$, we consider the subgroup

$$S_A:=\{\delta\in S_0\mid\alpha(\delta)\equiv 1\pmod{\mathfrak{p}_F}\text{ for all }\alpha\in A\}$$

of S. Observe that $S_0=S_\varnothing$ and that $S_{0,\mathrm{vreg}}=S_0\smallsetminus\bigcup_{\varnothing\neq A\subset R(\mathbf{S},\mathbf{G})}S_A$. By the principle of inclusion and exclusion, the left-hand side of (2) is equal to

$$\sum_{A\subset R(\mathbf{S},\mathbf{G})}(-1)^{|A|}\sum_{\gamma\in(S_AG_{\mathbf{x},0+}\cap K^d)/G_{\mathbf{x},t}}\Theta_{\rho'_d\otimes\phi_d}(\gamma)\cdot\overline{\Theta_{\tau_+}(\gamma)}.$$

It follows that the sum $\sum_{\gamma\in(S_AG_{\mathbf{x},0+}\cap K^d)/G_{\mathbf{x},t}}\Theta_{\rho'_d\otimes\phi_d}(\gamma)\cdot\overline{\Theta_{\tau_+}(\gamma)}$ is not zero for some $A\subset R(\mathbf{S},\mathbf{G})$. In other words, $\dim\mathrm{Hom}_{S_AG_{\mathbf{x},0+}\cap K^d}(\rho'_d\otimes\phi_d,\tau_+)\neq 0$ for some subset $A\subset R(\mathbf{S},\mathbf{G})$. Since $S_AG_{\mathbf{x},0+}\supset G_{\mathbf{x},0+}$, the desired conclusion follows. □

Theorem 5.13. *Let τ_+ be a smooth irreducible representation of $SG_{\mathbf{x},0+}$ satisfying $(\dagger_+)$ for the character $\theta=\phi\cdot\varepsilon^{\mathrm{ram}}[\phi]$ of S. Then $c=(-1)^{r(\mathbf{S},\phi)}$ and $\tau_+\cong\mathrm{Ind}_{K^d}^{SG_{\mathbf{x},0+}}(\rho'_d\otimes\phi_d)$.*

Proof. We first note that the action of $Z_{\mathbf{G}}$ on $\mathrm{Ind}_{K^d}^{SG_{\mathbf{x},0+}}(\rho'_d\otimes\phi_d)$ is given by $\phi|_{Z_{\mathbf{G}}}$ by [Kal19, Fact 3.7.11]. As $\varepsilon^{\mathrm{ram}}[\phi]|_{Z_{\mathbf{G}}}$ is trivial by definition (Definition 4.8), we have $\theta|_{Z_{\mathbf{G}}}=\phi|_{Z_{\mathbf{G}}}$. On the other hand, also the action of $Z_{\mathbf{G}}$ on τ_+ is given by $\theta|_{Z_{\mathbf{G}}}$. Indeed, if we take an unramified very regular element $\gamma\in S_{\mathrm{vreg}}$ ($S_{\mathrm{vreg}}\neq\varnothing$ by the assumption ($\mathfrak{vreg}$)), then $z\gamma$ is unramified very regular for any $z\in Z_{\mathbf{G}}$. Thus the condition $(\dagger_+)$ implies that

$$\Theta_{\tau_+}(z\gamma)=c\cdot\theta(z\gamma)=c\cdot\theta(z)\cdot\theta(\gamma)=\theta(z)\cdot\Theta_{\tau_+}(\gamma).$$

Therefore, to get the assertion, it is enough to show that $c=(-1)^{r(\mathbf{S},\phi)}$ and $\tau_+|_{S_0G_{\mathbf{x},0+}}\cong\mathrm{Ind}_{K^d}^{SG_{\mathbf{x},0+}}(\rho'_d\otimes\phi_d)|_{S_0G_{\mathbf{x},0+}}$ since we have $SG_{\mathbf{x},0+}=Z_{\mathbf{G}}S_0G_{\mathbf{x},0+}$. Also note that $\mathrm{Ind}_{K^d}^{SG_{\mathbf{x},0+}}(\rho'_d\otimes\phi_d)|_{S_0G_{\mathbf{x},0+}}$ is nothing but $\mathrm{Ind}_{S_0G_{\mathbf{x},0+}\cap K^d}^{S_0G_{\mathbf{x},0+}}(\rho'_d\otimes\phi_d)$.

By Lemma 5.12 and Frobenius reciprocity for $G_{\mathbf{x},0+}\cap K^d\subset S_0G_{\mathbf{x},0+}\cap K^d$, we have

$$\mathrm{Hom}_{S_0G_{\mathbf{x},0+}\cap K^d}\bigl(\mathrm{Ind}_{G_{\mathbf{x},0+}\cap K^d}^{S_0G_{\mathbf{x},0+}\cap K^d}(\rho'_d\otimes\phi_d),\tau_+\bigr)\neq 0.\tag{3}$$

Since $\rho'_d \otimes \phi_d$ is a representation of K^d, which contains $S_0G_{\mathbf{x},0+} \cap K^d$, Lemma 3.9 gives

$$\mathrm{Ind}_{G_{\mathbf{x},0+}\cap K^d}^{S_0G_{\mathbf{x},0+}\cap K^d}(\rho'_d \otimes \phi_d) \cong (\rho'_d \otimes \phi_d) \otimes \bigl(\mathrm{Ind}_{G_{\mathbf{x},0+}\cap K^d}^{S_0G_{\mathbf{x},0+}\cap K^d} \mathbb{1}\bigr).$$

As we have $(S_0G_{\mathbf{x},0+} \cap K^d)/(G_{\mathbf{x},0+} \cap K^d) \cong S_{0:0+}$, we have

$$\mathrm{Ind}_{G_{\mathbf{x},0+}\cap K^d}^{S_0G_{\mathbf{x},0+}\cap K^d} \mathbb{1} \cong \bigoplus_{\theta_0\colon S_{0:0+}\to\mathbb{C}^\times} \theta_0,$$

hence

$$\mathrm{Ind}_{G_{\mathbf{x},0+}\cap K^d}^{S_0G_{\mathbf{x},0+}\cap K^d}(\rho'_d \otimes \phi_d) \cong \bigoplus_{\theta_0\colon S_{0:0+}\to\mathbb{C}^\times} (\rho'_d \otimes \phi_d) \otimes \theta_0.$$

Therefore (3) implies that we have

$$\mathrm{Hom}_{S_0G_{\mathbf{x},0+}\cap K^d}\bigl((\rho'_d \otimes \phi_d) \otimes \theta_0, \tau_+\bigr) \neq 0$$

for at least one $\theta_0\colon S_{0:0+} \to \mathbb{C}^\times$.

Next, by Frobenius reciprocity for $S_0G_{\mathbf{x},0+} \cap K^d \subset S_0G_{\mathbf{x},0+}$, we get

$$\mathrm{Hom}_{S_0G_{\mathbf{x},0+}}\bigl(\mathrm{Ind}_{S_0G_{\mathbf{x},0+}\cap K^d}^{S_0G_{\mathbf{x},0+}}((\rho'_d \otimes \phi_d) \otimes \theta_0), \tau_+\bigr) \neq 0. \tag{4}$$

Note that we have $S_0G_{\mathbf{x},0+}/G_{\mathbf{x},0+} \cong (S_0G_{\mathbf{x},0+} \cap K^d)/(G_{\mathbf{x},0+} \cap K^d) \cong S_{0:0+}$. In particular, θ_0 can be regarded as a character of $S_0G_{\mathbf{x},0+}$ (which is trivial on $G_{\mathbf{x},0+}$). Thus Lemma 3.9 gives

$$\mathrm{Ind}_{S_0G_{\mathbf{x},0+}\cap K^d}^{S_0G_{\mathbf{x},0+}}((\rho'_d \otimes \phi_d) \otimes \theta_0) \cong \bigl(\mathrm{Ind}_{S_0G_{\mathbf{x},0+}\cap K^d}^{S_0G_{\mathbf{x},0+}}(\rho'_d \otimes \phi_d)\bigr) \otimes \theta_0.$$

Since the compact induction of $\rho'_d \otimes \phi_d$ from to K^d to G is irreducible by Yu's theory, so is the induced representation $\mathrm{Ind}_{K^d}^{SG_{\mathbf{x},0+}}(\rho'_d \otimes \phi_d)$. By noting that $K^d = Z_{\mathbf{G}} \cdot (S_0G_{\mathbf{x},0+} \cap K^d)$ and $SG_{\mathbf{x},0+} = Z_{\mathbf{G}} \cdot S_0G_{\mathbf{x},0+}$, we see that $\mathrm{Ind}_{S_0G_{\mathbf{x},0+}\cap K^d}^{S_0G_{\mathbf{x},0+}}(\rho'_d \otimes \phi_d)$ is nothing but the restriction of $\mathrm{Ind}_{K^d}^{SG_{\mathbf{x},0+}}(\rho'_d \otimes \phi_d)$ to $S_0G_{\mathbf{x},0+}$ and irreducible. On the other hand, τ_+ is also irreducible as a representation of $S_0G_{\mathbf{x},0+}$. Thus, by (4), we get

$$\tau_+ \cong \bigl(\mathrm{Ind}_{S_0G_{\mathbf{x},0+}\cap K^d}^{S_0G_{\mathbf{x},0+}}(\rho'_d \otimes \phi_d)\bigr) \otimes \theta_0.$$

Now our task is to show that $\theta_0 = \mathbb{1}$. By a similar, but simpler, argument to the proof of Proposition 4.11, for all $\gamma \in S_{0,\mathrm{vreg}}$ we have

$$\begin{aligned}\Theta_{(\mathrm{Ind}_{S_0G_{\mathbf{x},0+}\cap K^d}^{S_0G_{\mathbf{x},0+}}(\rho'_d\otimes\phi_d))\otimes\theta_0}(\gamma) &= \Theta_{\mathrm{Ind}_{S_0G_{\mathbf{x},0+}\cap K^d}^{S_0G_{\mathbf{x},0+}}(\rho'_d\otimes\phi_d)}(\gamma) \cdot \theta_0(\gamma)\\ &= (-1)^{r(\mathbf{S},\phi)}\theta(\gamma) \cdot \theta_0(\gamma).\end{aligned}$$

Thus, by the assumption on τ_+, for all $\gamma \in S_{0,\mathrm{vreg}}$ we have

$$c \cdot \theta(\gamma) = (-1)^{r(\mathbf{S},\phi)}\theta(\gamma) \cdot \theta_0(\gamma).$$

Hence θ_0 must satisfy $\theta_0|_{S_{0,\mathrm{vreg}}} = c \cdot (-1)^{r(\mathbf{S},\phi)}$. By Corollary 5.5, we can find elements $\gamma_1, \gamma_2 \in S_{0,\mathrm{vreg}}$ such that the product $\gamma_1\gamma_2 \in S_0$ is again unramified very regular. Then we have

$$\theta_0(\gamma_1) = \theta_0(\gamma_2) = \theta_0(\gamma_1\gamma_2) = c \cdot (-1)^{r(\mathbf{S},\phi)}.$$

By noting that $\theta_0(\gamma_1\gamma_2) = \theta_0(\gamma_1)\theta_0(\gamma_2)$ and that $c \neq 0$ (this is part of the assumption ($\dagger_+$)), we see that $c \cdot (-1)^{r(\mathbf{S},\phi)} = 1$. Then the equality $\theta_0|_{S_{0,\mathrm{vreg}}} = \mathbb{1}$ implies $\theta_0 = \mathbb{1}$ since $S_{0,\mathrm{vreg}}$ generates S_0 by the assumption ($\mathfrak{vreg}$). □

Corollary 5.14. *Let τ_+ be a smooth irreducible representation of $SG_{\mathbf{x},0+}$ satisfying ($\dagger_+$) for the character $\theta = \phi \cdot \varepsilon^{\mathrm{ram}}[\phi]$ of S. Then $\mathrm{Ind}_{SG_{\mathbf{x},0+}}^{SG_{\mathbf{x},0}}(\tau_+)$ is an irreducible representation of $SG_{\mathbf{x},0}$.*

Proof. This follows from Theorem 5.13 together with the fact that $\mathrm{Ind}_{K^d}^{SG_{\mathbf{x},0}}(\rho'_d \otimes \phi_d)$ is irreducible. □

5.3. **Representations of $SG_{\mathbf{x},0}$ characterized by their trace on S_{vreg}.** In this section, we prove (Theorem 5.17) that some irreducible representations of $SG_{\mathbf{x},0}$ are characterized by character values on unramified very regular elements S_{vreg} of S. As in the previous subsection, we will assume ($\mathfrak{vreg}$) here. The reader should think of this section in parallel to Section 5.2, though we will need an additional argument to establish the analogues of Lemma 5.12 and Theorem 5.13.

In this section, we will consider smooth irreducible representations τ of $SG_{\mathbf{x},0}$ such that for some nonzero constant $c \in \mathbb{C}$,

$$(\dagger) \qquad \Theta_\tau(\gamma) = c \sum_{w \in W_{G_{\mathbf{x},0}}(\mathbf{S})} \theta^w(\gamma), \qquad \text{for all } \gamma \in S_{\mathrm{vreg}},$$

where θ^w is the character of S defined by $\theta^w(\gamma) = \theta({}^w\gamma) = \theta(w\gamma w^{-1})$.

Lemma 5.15. *Let π be a smooth irreducible representation of $SG_{\mathbf{x},0}$ satisfying ($\dagger$) for the character $\theta = \phi \cdot \varepsilon^{\mathrm{ram}}[\phi]$ of S. Then*

$$\mathrm{Hom}_{G_{\mathbf{x},0+} \cap K^d}(\rho'_d \otimes \phi_d, \tau) \neq 0.$$

Proof. This is very similar to the proof of Lemma 5.12 but with one additional argument. As in that proof, by choosing a sufficiently large $r \in \mathbb{R}_{>0}$, it is enough to show that we have

$$\sum_{\gamma \in (S_{0,\mathrm{vreg}} G_{\mathbf{x},0+} \cap K^d)/G_{\mathbf{x},t}} \Theta_{\rho'_d \otimes \phi_d}(\gamma) \cdot \overline{\Theta_\tau(\gamma)} \neq 0.$$

By the same argument as in the proof of Lemma 5.12,

$$\sum_{\gamma \in (S_{0,\mathrm{vreg}} G_{\mathbf{x},0+} \cap K^d)/G_{\mathbf{x},t}} \Theta_{\rho'_d \otimes \phi_d}(\gamma) \cdot \overline{\Theta_\tau(\gamma)} = |K^d / SG_{\mathbf{x},t}| \sum_{\gamma \in S_{0,\mathrm{vreg}}/S_t} \Theta_{\rho'_d \otimes \phi_d}(\gamma) \cdot \overline{\Theta_\tau(\gamma)}.$$

By Proposition 4.9 and the assumption on τ, we have

$$\sum_{\gamma \in S_{0,\mathrm{vreg}}/S_t} \Theta_{\rho'_d \otimes \phi_d}(\gamma) \cdot \overline{\Theta_\tau(\gamma)} = (-1)^{r(\mathbf{S},\phi)} \bar{c} \sum_{w \in W_{G_{\mathbf{x},0}(\mathbf{S})}} \sum_{\gamma \in S_{0,\mathrm{vreg}}/S_t} \theta(\gamma) \overline{\theta^w(\gamma)}.$$

Recall the summand $\sum_{\gamma \in S_{0,\mathrm{vreg}}} \theta(\gamma)\overline{\theta(\gamma)}$ corresponding to $w = 1$ is not zero by the positivity of $\theta(\gamma)\overline{\theta(\gamma)}$ and the non-emptiness of $S_{0,\mathrm{vreg}}$. Hence it suffices to check that for $w \in W_{G_{\mathbf{x},0}}(\mathbf{S}) \smallsetminus \{1\}$, the corresponding sum $\sum_{\gamma \in S_{0,\mathrm{vreg}}} \theta(\gamma)\overline{\theta^w(\gamma)}$ vanishes. By fixing a set of representatives $\{\tilde{\gamma}\}$ of the quotient $S_{0,\mathrm{vreg}}/S_{0+} \cong \mathbb{S}(\mathbb{F}_q)_{\mathrm{vreg}}$, we get

$$\begin{aligned} \sum_{\gamma \in S_{0,\mathrm{vreg}}/S_t} \theta(\gamma) \cdot \overline{\theta^w(\gamma)} &= \sum_{\gamma \in \mathbb{S}(\mathbb{F}_q)_{\mathrm{vreg}}} \sum_{\gamma_+ \in S_{0+:t}} \theta(\tilde{\gamma}\gamma_+) \cdot \overline{\theta^w(\tilde{\gamma}\gamma_+)} \\ &= \sum_{\gamma \in \mathbb{S}(\mathbb{F}_q)_{\mathrm{vreg}}} \theta(\tilde{\gamma}) \cdot \overline{\theta^w(\tilde{\gamma})} \sum_{\gamma_+ \in S_{0+:t}} \theta(\gamma_+) \cdot \overline{\theta^w(\gamma_+)}. \end{aligned}$$

Since $\theta|_{S_{0+}}$ has trivial $W_{G_{\mathbf{x},0}}(\mathbf{S})$-stabilizer by Lemma 3.8, it follows that $\sum_{\gamma_+ \in S_{0+:t}} \theta(\gamma_+) \cdot \overline{\theta^w(\gamma_+)} = 0$ whenever $w \neq 1$. This completes the proof. □

When we prove the analogue of Theorem 5.13 in the setting of $SG_{\mathbf{x},0}$-representations, we will need the following lemma. In the GL_n setting, this result is due to Henniart [Hen93], and the proof of the general setting given here is a direct generalization of Henniart's proof.

Lemma 5.16. *Let W be a subgroup of $W_G(\mathbf{S})$. Let $\theta, \theta' \colon S_0 \to \mathbb{C}^\times$ be two smooth characters and assume that $\theta|_{S_{0+}}$ has trivial W-stabilizer. If for some nonzero constant $c \in \mathbb{C}$,*

$$\sum_{w \in W} \theta^w(\gamma) = c \cdot \sum_{w \in W} \theta'^w(\gamma)$$

for all $\gamma \in S_{0,\mathrm{vreg}}$, then $c = 1$ and $\theta' = \theta^w$ for some unique $w \in W$.

Proof. We follow the same strategy as Henniart in [Hen93, Section 5.3]. Henniart works in the setting of $G = \mathrm{GL}_\ell$ for a prime ℓ different from p, but the proof generalizes with no problems. We present it here.

We first show that the conclusion must hold on S_{0+}. Fix $\gamma \in S_{0,\mathrm{vreg}}$. Then $\gamma S_{0+} \subset S_{0,\mathrm{vreg}}$. From this, the assumption of the lemma implies that we have a linear dependence between the $2|W|$ (not necessarily distinct) characters $\theta^w|_{S_{0+}}$ and $\theta'^w|_{S_{0+}}$ for $w \in W$:

$$\sum_{w \in W} \theta^w(\gamma) \cdot (\theta^w | S_{0+}) = c \cdot \sum_{w \in W} \theta'^w(\gamma) \cdot (\theta'^w|_{S_{0+}}).$$

Thus, by linear independence of characters (of $S|_{0+}$) and the triviality of the stabilizer of $\theta|_{S_{0+}}$ in W we see that there exists $w \in W$ satisfying $\theta|_{S_{0+}} = \theta'^{w^{-1}}|_{S_{0+}}$, or equivalently, $\theta'|_{S_{0+}} = \theta^w|_{S_{0+}}$. Since the equality $\theta'|_{S_{0+}} = \theta^w|_{S_{0+}}$ implies that the stabilizer of $\theta'|_{S_{0+}}$ in W is trivial, we also know that an element $w \in W$ satisfying $\theta'|_{S_{0+}} = \theta^w|_{S_{0+}}$ is in fact unique.

Then, again by linear independence of characters, the above identity implies that $\theta^w(\gamma) = c \cdot \theta'(\gamma)$ for any $\gamma \in S_{0,\mathrm{vreg}}$. By the same argument as at the end of the proof of Theorem 5.13, the assumption ($\mathfrak{vreg}$) implies that if a character of S_0 restricts identically to c on $S_{0,\mathrm{vreg}}$, then $c = 1$, and the conclusion of the lemma now follows. □

We now prove the analogue of Theorem 5.13.

Theorem 5.17. *Let τ be a smooth irreducible representation of $SG_{\mathbf{x},0}$ satisfying (†) for the character $\theta = \phi \cdot \varepsilon^{\mathrm{ram}}[\phi]$ of S. Then $c = (-1)^{r(\mathbf{S},\phi)}$ and $\tau \cong {}^\circ\tau_d$.*

Proof. Recall that ${}^\circ\tau_d$ is defined to be $\mathrm{Ind}_{K^d}^{SG_{\mathbf{x},0}}(\rho'_d \otimes \phi_d)$ (in our situation, we have ${}^\circ K^d = K^d$ and ${}^\circ\rho'_d = \rho'_d$). By comparing the contribution of $Z_{\mathbf{G}}$ as in the proof of Theorem 5.13, it is enough to show that $c = (-1)^{r(\mathbf{S},\phi)}$ and $\tau \cong \mathrm{Ind}_{S_0 G_{\mathbf{x},0+} \cap K^d}^{G_{\mathbf{x},0}}(\rho'_d \otimes \phi_d)$ as a representation of $G_{\mathbf{x},0}$.

Lemma 5.15 and Frobenius reciprocity imply that

$$\mathrm{Hom}_{G_{\mathbf{x},0}}\bigl(\mathrm{Ind}_{G_{\mathbf{x},0+} \cap K^d}^{G_{\mathbf{x},0}}(\rho'_d \otimes \phi_d), \tau\bigr) \neq 0.$$

Recall from the proof of Theorem 5.13 that

$$\mathrm{Ind}_{G_{\mathbf{x},0+} \cap K^d}^{S_0 G_{\mathbf{x},0+} \cap K^d}(\rho'_d \otimes \phi_d) \cong \bigoplus_{\theta_0 \colon S_{0:0+} \to \mathbb{C}^\times} (\rho'_d \otimes \phi_d) \otimes \theta_0.$$

Thus we have

$$\mathrm{Ind}_{G_{\mathbf{x},0+} \cap K^d}^{G_{\mathbf{x},0}}(\rho'_d \otimes \phi_d) \cong \bigoplus_{\theta_0 \colon S_{0:0+} \to \mathbb{C}^\times} \mathrm{Ind}_{S_0 G_{\mathbf{x},0+} \cap K^d}^{G_{\mathbf{x},0}}\bigl((\rho'_d \otimes \phi_d) \otimes \theta_0\bigr).$$

Hence, there exists at least one character $\theta_0\colon S_{0:0+}\to\mathbb{C}^\times$ such that

$$\mathrm{Hom}_{G_{\mathbf{x},0}}\Big(\mathrm{Ind}_{S_0G_{\mathbf{x},0+}\cap K^d}^{G_{\mathbf{x},0}}\big((\rho'_d\otimes\phi_d)\otimes\theta_0\big),\tau\Big)\neq 0. \tag{5}$$

Let us investigate the representation $\mathrm{Ind}_{S_0G_{\mathbf{x},0+}\cap K^d}^{G_{\mathbf{x},0}}((\rho'_d\otimes\phi_d)\otimes\theta_0)$. Recall that $\rho'_d\otimes\phi_d$ is the representation of K^d obtained from a tame elliptic regular pair $(\mathbf{S},\phi)$. If we take an extension $\tilde{\theta}_0$ of θ_0 from S_0 to S, then the pair $(\mathbf{S},\phi\otimes\tilde{\theta}_0)$ is again tame elliptic regular. Furthermore, we can easily check that the Yu's construction (reviewed in Section 3.1) attaches to $(\mathbf{S},\phi\otimes\tilde{\theta}_0)$ the representation $(\rho'_d\otimes\phi_d)\otimes\tilde{\theta}_0$ of K^d, where $\tilde{\theta}_0$ is regarded as a character of K^d through the map $K^d\twoheadrightarrow K^d/(K^d\cap G_{\mathbf{x},0+})\cong S/S_{0+}$. Therefore the compact induction of $(\rho'_d\otimes\phi_d)\otimes\tilde{\theta}_0$ from K^d to G is an irreducible (supercuspidal) representation. In particular, also $\mathrm{Ind}_{K^d}^{SG_{\mathbf{x},0}}((\rho'_d\otimes\phi_d)\otimes\tilde{\theta}_0)$ is irreducible, hence so is its restriction to $G_{\mathbf{x},0}$ (note that $SG_{\mathbf{x},0}=Z_{\mathbf{G}}G_{\mathbf{x},0}$). As we have $\mathrm{Ind}_{K^d}^{SG_{\mathbf{x},0}}((\rho'_d\otimes\phi_d)\otimes\tilde{\theta}_0)|_{G_{\mathbf{x},0}}\cong\mathrm{Ind}_{S_0G_{\mathbf{x},0+}\cap K^d}^{G_{\mathbf{x},0}}((\rho'_d\otimes\phi_d)\otimes\theta_0)$, we eventually get the irreducibility of $\mathrm{Ind}_{S_0G_{\mathbf{x},0+}\cap K^d}^{G_{\mathbf{x},0}}((\rho'_d\otimes\phi_d)\otimes\theta_0)$. Then, by combining this with the irreducibility of τ, (5) implies that

$$\mathrm{Ind}_{S_0G_{\mathbf{x},0+}\cap K^d}^{G_{\mathbf{x},0}}\big((\rho'_d\otimes\phi_d)\otimes\theta_0\big)\cong\tau.$$

It now remains to show that $c=(-1)^{r(\mathbf{S},\phi)}$ and that a character θ_0 as above is in fact necessarily trivial. By applying Proposition 4.11 to $\mathrm{Ind}_{K^d}^{SG_{\mathbf{x},0}}((\rho'_d\otimes\phi_d)\otimes\tilde{\theta}_0)$ (which is the representation "${}^\circ\tau_d$" arising from the twisted pair $(\mathbf{S},\phi\otimes\tilde{\theta}_0)$), for all $\gamma\in S_{0,\mathrm{vreg}}$, we have

$$\begin{aligned}\Theta_{\mathrm{Ind}_{S_0G_{\mathbf{x},0+}\cap K^d}^{G_{\mathbf{x},0}}((\rho'_d\otimes\phi_d)\otimes\theta_0)}(\gamma)&=\Theta_{\mathrm{Ind}_{K^d}^{SG_{\mathbf{x},0}}((\rho'_d\otimes\phi_d)\otimes\tilde{\theta}_0)}(\gamma)\\&=(-1)^{r(\mathbf{S},\phi)}\sum_{w\in W_{G_{\mathbf{x},0}}(\mathbf{S})}\theta^w(\gamma)\theta_0^w(\gamma).\end{aligned}$$

(Note that the character $\varepsilon^{\mathrm{ram}}[\phi\otimes\tilde{\theta}_0]$ associated to the twisted character $\phi\otimes\tilde{\theta}_0$ is the same as $\varepsilon^{\mathrm{ram}}[\phi]$ since $\tilde{\theta}_0$ is of depth zero.) Thus, by the assumption on τ, we get

$$(-1)^{r(\mathbf{S},\phi)}\sum_{w\in W_{G_{\mathbf{x},0}}(\mathbf{S})}\theta^w(\gamma)\theta_0^w(\gamma)=c\sum_{w\in W_{G_{\mathbf{x},0}}(\mathbf{S})}\theta^w(\gamma).$$

Since ϕ is toral by assumption, $\phi|_{S_{0+}}=\theta|_{S_{0+}}$ has trivial $W_{G_{\mathbf{x},0}}(\mathbf{S})$-stabilizer by Lemma 3.8 (recall that $\varepsilon^{\mathrm{ram}}[\phi]|_{S_{0+}}=\mathbb{1}$). We therefore may apply Lemma 5.16 and conclude that $c=(-1)^{r(\mathbf{S},\phi)}$ and $(\theta|_{S_0})\cdot\theta_0=\theta^w|_{S_0}$ for some unique $w\in W_{G_{\mathbf{x},0}}(\mathbf{S})$. By restricting to S_{0+}, we get $\theta|_{S_{0+}}=\theta^w|_{S_{0+}}$. Then the triviality of the stabilizer of $\theta|_{S_{0+}}$ in $W_{G_{\mathbf{x},0}}(\mathbf{S})$ implies that $w=1$. Thus finally we get $(\theta|_{S_0})\cdot\theta_0=\theta|_{S_0}$, which implies that θ_0 is trivial. □

6. Deligne–Lusztig varieties for parahoric subgroups

Up to this point in the paper, our discussion has been centered on algebraically constructed representations of parahoric subgroups whose compact induction to G is irreducible (and hence supercuspidal). Contrary to every other section of this paper, everything in this section holds with no assumptions on the residue characteristic p of F.

In this section, we introduce a class of representations of parahoric subgroups that arise geometrically, via the cohomology of Deligne–Lusztig varieties for parahoric subgroups [CI21b]. These varieties are associated to a maximal torus $\mathbf{S}\hookrightarrow\mathbf{G}$ such that

$$\mathbf{S}\text{ is unramified but not necessarily elliptic.}$$

When $\mathbf{S}$ is elliptic, these representations are closely related to tame supercuspidal representations; Sections 7 and 8 are devoted to understanding this relationship and to discussing the interesting consequences of having such a comparison.

6.1. **The varieties X_r.** We review a generalization of Deligne–Lusztig varieties for $G_{\mathbf{x},0}$ defined in [CI21b]. In the setting that $G_{\mathbf{x},0} = \mathbb{G}(\mathcal{O}_F)$ for a reductive group $\mathbb{G}$ over $\mathbb{F}_q$, these varieties were originally defined and studied by Lusztig in [Lus04] (for F equal characteristic) and later by Stasinski in [Sta09] (for F mixed characteristic). We warn the reader that there is a change of convention between our present work and the papers [Lus04, Sta09, CI21b]: in our normalization, $\mathbb{G}_0$ is a reductive group over a finite field.

Let $\mathbf{S} \hookrightarrow \mathbf{G}$ be an unramified maximal torus (not necessarily elliptic) defined over F and let $\mathbf{x} \in \mathcal{B}(\mathbf{G}, F)$ be a point which belongs to the set $\mathcal{A}(\mathbf{S}, F)$ of Frobenius-fixed points of the apartment $\mathcal{A}(\mathbf{S}_{\gamma,F^{\mathrm{ur}}}, F^{\mathrm{ur}})$. Then for each $r \in \mathbb{R}_{\geq 0}$, there exist $\mathcal{O}_F$-models $\mathcal{G}_{\mathbf{x},r}, \mathcal{G}_{\mathbf{x},r+}$ of $\mathbf{G}$ (à la Bruhat–Tits [BT84] and Yu [Yu15]). Say $\mathbf{S}$ splits over the degree-n unramified extension F_n of F and let $\mathbf{U}$ be the unipotent radical of a F_n-rational Borel subgroup of $\mathbf{G}_{F_n}$ containing $\mathbf{S}_{F_n}$. Following [CI21b, Section 2.4-2.6], for $r \in \mathbb{Z}_{\geq 0}$, there exist group schemes $\mathbb{S}_r \subset \mathbb{G}_r$ defined over $\mathbb{F}_q$ and a group scheme $\mathbb{U}_r \subset \mathbb{G}_{r,\mathbb{F}_{q^n}}$. The group scheme $\mathbb{G}_r$ is constructed by considering the functor of positive loops applied to the $\mathcal{O}_F$-group schemes $\mathcal{G}_{\mathbf{x},r}$. To each smooth closed F_m-subgroup of $\mathbf{G}_{F_m}$, one may construct an associated subgroup scheme of $\mathbb{G}_{r,\mathbb{F}_{q^m}}$ [CI21b, Section 2.6]; the subgroup schemes $\mathbb{S}_r \subset \mathbb{G}_r$ and $\mathbb{U}_r \subset \mathbb{G}_{r,\mathbb{F}_{q^n}}$ are obtained by this construction applied to $\mathbf{S}, \mathbf{U}$. These group schemes have the property that

$$\mathbb{G}_r(\mathbb{F}_q) = \mathcal{G}_{\mathbf{x},0}(\mathcal{O}_F)/\mathcal{G}_{\mathbf{x},r+}(\mathcal{O}_F) = G_{\mathbf{x},0:r+}, \qquad \mathbb{S}_r(\mathbb{F}_q) = S_{0:r+},$$
$$\mathbb{U}_r(\mathbb{F}_{q^n}) = (\mathbf{U}(F_n) \cap \mathcal{G}_{\mathbf{x},0}(\mathcal{O}_{F_n}))/(\mathbf{U}(F_n) \cap \mathcal{G}_{\mathbf{x},r+}(\mathcal{O}_{F_n})).$$

Let $\sigma\colon \mathbb{G}_r \to \mathbb{G}_r$ denote the Frobenius morphism associated to the $\mathbb{F}_q$-rational structure on $\mathbb{G}_r$.

Definition 6.1. For $r \in \mathbb{Z}_{\geq 0}$, we define the following $\mathbb{F}_{q^n}$-subscheme of $\mathbb{G}_r$:

$$X_r := \{x \in \mathbb{G}_r \mid x^{-1}\sigma(x) \in \mathbb{U}_r\}.$$

The scheme X_r is separated, smooth, and of finite type over $\mathbb{F}_{q^n}$ [CI21b, Lemma 3.1]. For $(g,t) \in G_{\mathbf{x},0:r+} \times S_{0:r+}$ and $x \in X_r$, the assignment $(g,t) * x = gxt$ defines an action of $G_{\mathbf{x},0:r+} \times S_{0:r+}$ on X_r which pulls back to an action of $G_{\mathbf{x},0} \times S_0$.

We fix a prime number ℓ which is not equal to p. By functoriality, the cohomology groups $H_c^i(X_r, \overline{\mathbb{Q}}_\ell)$ are representations of $G_{\mathbf{x},0} \times S_0$. If $\theta\colon S \to \overline{\mathbb{Q}}_\ell^\times$ is a $\overline{\mathbb{Q}}_\ell^\times$-valued character trivial on S_{r+}, the subspace $H_c^i(X_r, \overline{\mathbb{Q}}_\ell)[\theta]$ of $H_c^i(X_r, \overline{\mathbb{Q}}_\ell)$ on which S_0 acts by multiplication by $\theta|_{S_0}$ is a representation of $G_{\mathbf{x},0}$. We define a virtual representation $R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta)$ of $G_{\mathbf{x},0}$ with $\overline{\mathbb{Q}}_\ell$-coefficient by

$$R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta) := \sum_{i \geq 0} (-1)^i H_c^i(X_r, \overline{\mathbb{Q}}_\ell)[\theta].$$

In the special case that $r = 0$, the variety X_0 is an affine fibration over a classical Deligne–Lusztig variety and hence their cohomology is the same up to an even degree shift. Hence $R_{\mathbb{S}_0,\mathbb{U}_0}^{\mathbb{G}_0}(\theta)$ is exactly the usual Deligne–Lusztig representation of the finite reductive group $\mathbb{G}_0(\mathbb{F}_q)$ attached to the character $\theta|_{S_0}$ of $\mathbb{S}_0(\mathbb{F}_q)$.

The next two results are $r > 0$ analogues of known $r = 0$ theorems. Proposition 6.2 in the $r = 0$ setting is a special case of the Deligne–Lusztig character formula [DL76, Theorem 4.2] and Theorem 6.3 in the $r = 0$ setting is a special case of the Mackey formula for Deligne–Lusztig varieties [DL76, Theorem 6.8].

Proposition 6.2 ([CI21b, Theorem 1.2])**.** *Let $\theta\colon S \to \overline{\mathbb{Q}}_\ell^\times$ be any character trivial on S_{r+} and let $\gamma \in G_{\mathbf{x},0}$ be an unramified very regular element. If γ is not $G_{\mathbf{x},0}$-conjugate to an element of S, then the character value $\Theta_{R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta)}(\gamma)$ is equal to zero. If γ belongs to S, we have*

$$\Theta_{R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta)}(\gamma) = \sum_{w \in W_{G_{\mathbf{x},0}}(\mathbf{S})} \theta^w(\gamma).$$

Theorem 6.3 ([CI21b, Theorem 1.1])**.** *Let $r > 0$ and let $\theta\colon S \to \overline{\mathbb{Q}}_\ell^\times$ be a character trivial on S_{r+} such that $\theta|_{\mathrm{Nr}_{F_n/F}(\alpha^\vee(E_r^\times))} \not\equiv \mathbb{1}$ for all $\alpha \in R(\mathbf{S},\mathbf{G})$. Let $\mathbf{U}' \subset \mathbf{G}_{F_n}$ be another unipotent radical of a Borel subgroup of $\mathbf{G}_{F_n}$ containing $\mathbf{S}$. Then for any character $\theta'\colon S_{0:r+} \to \overline{\mathbb{Q}}_\ell^\times$,*

$$\big\langle R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta), R_{\mathbb{S}_r,\mathbb{U}'_r}^{\mathbb{G}_r}(\theta')\big\rangle_{G_{\mathbf{x},0:r+}} = \sum_{w \in W_{G_{\mathbf{x},0}}(\mathbf{S})} \langle \theta, \theta'^w \rangle_{S_{0:r+}}. \tag{6}$$

In particular, if θ has trivial $W_{G_{\mathbf{x},0}}(\mathbf{S})$-stabilizer, then $R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta)$ is irreducible (up to a sign) and is independent of the choice of $\mathbf{U}$.

Remark 6.4. Let $\mathbf{S}$ be an unramified maximal torus of $\mathbf{G}$ and θ a character of S which is trivial on S_{r+} for $r \in \mathbb{R}_{>0}$. Then recall that we may associate a sequence of root subsystems $\{R_\bullet\}$ (or reductive subgroups) to θ (see Section 3.3). In terms of this sequence of root subsystems, the condition that

$$\theta|_{\mathrm{Nr}_{E/F}(\alpha^\vee(E_r^\times))} \not\equiv \mathbb{1} \text{ for all } \alpha \in R(\mathbf{S},\mathbf{G})$$

is equivalent to the condition that

$$R_0 = \cdots = R_r = \varnothing.$$

On the other hand, since $\theta|_{S_{r+}}$ is trivial, we have $R_{r+} = R(\mathbf{S},\mathbf{G})$. Thus the assumption of Theorem 6.3 is satisfied if and only if its associated sequence of reductive subgroups is given by $(\mathbf{G}^0(\mathbf{S},\theta), \mathbf{G}^1(\mathbf{S},\theta))$, where $\mathbf{G}^0(\mathbf{S},\theta) = \mathbf{S}$ and $\mathbf{G}^1(\mathbf{S},\theta) = \mathbf{G}$, in other words, θ is 0-toral. Note that, if we furthermore assume the four conditions on p from Section 2.1 (i.e., odd, not bad for $\mathbf{G}$, $p \nmid |\pi_1(\mathbf{G}_{\mathrm{der}})|$, and $p \nmid |\pi_1(\widehat{\mathbf{G}}_{\mathrm{der}})|$), the torality implies that $\theta|_{S_{0+}}$ has trivial $W_G(\mathbf{S})$-stabilizer by Lemma 3.8; in particular, θ has trivial $W_{G_{\mathbf{x},0}}(\mathbf{S})$-stabilizer. In summary, when the four conditions on p from Section 2.1 are satisfied, Theorem 6.3 asserts that the virtual representation $R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta)$ is irreducible (up to sign) for any 0-toral character θ of depth r.

6.2. **The Drinfeld stratification.** In this section, we introduce subschemes of X_r associated to certain twisted Levi subgroups of $\mathbf{G}$, following [CI21c, Definition 3.3.1]. The work of this paper will allow us to formulate a conjecture (Conjecture 6.5) about the Drinfeld stratification which had previously (in *op. cit.*) only been conjectured for GL_n. We will prove part of Conjecture 6.5 in Section 7 (see Theorem 7.6).

Let $\mathbb{G}_r^+$ be the kernel of the natural reduction map $\mathbb{G}_r \to \mathbb{G}_0$. Note that $\mathbb{G}_r^+(\mathbb{F}_q) = G_{\mathbf{x},0+:r+}$. For a twisted Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ containing $\mathbf{S}$, we let $\mathbb{L}_r$ denote the corresponding subgroup scheme of $\mathbb{G}_r$. Attached to the twisted Levi $\mathbf{L}$, one can naturally associate the following subscheme of X_r:

$$X_r^{(\mathbf{L})} := \{x \in \mathbb{G}_r \mid x^{-1}\sigma(x) \in (\mathbb{L}_r \cap \mathbb{U}_r)\mathbb{U}_r^+\}.$$

It is straightforward to show ([CI21c, Lemma 3.3.3]) that $X_r^{(\mathbf{L})}$ is a disjoint union over $G_{\mathbf{x},0}/(L_{\mathbf{x},0}G_{\mathbf{x},0+})$ copies of the scheme $X_r \cap \mathbb{L}_r\mathbb{G}_r^+$. It follows ([CI21c, Lemma 3.3.4]) from this that for any character $\theta\colon \mathbb{S}_r(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$,

$$H_c^i(X_r^{(\mathbf{L})}, \overline{\mathbb{Q}}_\ell)[\theta] \cong \operatorname{Ind}_{\mathbb{L}_r(\mathbb{F}_q)\mathbb{G}_r^+(\mathbb{F}_q)}^{\mathbb{G}_r(\mathbb{F}_q)} \left(H_c^i(X_r \cap \mathbb{L}_r\mathbb{G}_r^+, \overline{\mathbb{Q}}_\ell)[\theta]\right) \qquad \text{for all } i \geq 0.$$

Conjecture 6.5. *Let $\mathbf{L} \subset \mathbf{G}$ be a twisted Levi subgroup of $\mathbf{G}$ containing $\mathbf{S}$. If $R_{0+} \subset R(\mathbf{S}, \mathbf{L})$ (or, equivalently, $\mathbf{G}^0(\mathbf{S}, \theta) \subset \mathbf{L}$), then*

$$H_c^*(X_r, \overline{\mathbb{Q}}_\ell)[\theta] \cong H_c^*(X_r^{(\mathbf{L})}, \overline{\mathbb{Q}}_\ell)[\theta].$$

Remark 6.6. Let $\mathbf{G} = \mathrm{GL}_n$. In the setting $\mathbf{L} = \mathbf{S}$, Conjecture 6.5 was proved in [CI19, Theorem 4.1] using geometric techniques (note that this forces $\mathbf{G}^0(\mathbf{S}, \theta) = \mathbf{S}$). On the other hand, in the GL_n setting, it is expected that Conjecture 6.5 holds without any assumptions on p. By [Kal19, Lemma 3.7.7], the notions of a tame elliptic regular pair and of an admissible pair coincide. Then one can translate between Kaletha's generalized Howe factorization for tame elliptic regular pairs and the classical Howe factorization for admissible characters (see [OT21, Section 5.2]). Then Conjecture 6.5 is the same assertion of [CI21c, Conjecture 7.2.1].

The proof of Theorem 6.3 [CI21b, Theorem 1.1] on the cohomology of X_r can be applied almost identically to obtain an analogous statement for the cohomology of $X_r \cap \mathbb{L}_r\mathbb{G}_r^+$.

Theorem 6.7. *Let $r > 0$ and let $\theta\colon S_0 \to \overline{\mathbb{Q}}_\ell^\times$ be a 0-toral character trivial on S_{r+}. Let $\mathbf{U}' \subset \mathbf{G}_{F_n}$ be another unipotent radical of a Borel subgroup of $\mathbf{G}_{F_n}$ containing $\mathbf{S}$ and let X_r' be the associated variety. Then for any $\theta'\colon S_{0:r+} \to \overline{\mathbb{Q}}_\ell^\times$,*

$$\left\langle H_c^*(X_r \cap \mathbb{L}_r\mathbb{G}_r^+, \overline{\mathbb{Q}}_\ell)[\theta], H_c^*(X_r' \cap \mathbb{L}_r\mathbb{G}_r^+, \overline{\mathbb{Q}}_\ell)[\theta'] \right\rangle_{L_{\mathbf{x},0:r+}G_{\mathbf{x},0+:r+}} = \sum_{w \in W_{L_{\mathbf{x},0}}(\mathbf{S})} \langle \theta, \theta'^w \rangle_{S_{0:r+}}.$$

In particular, if θ has trivial $W_{L_{\mathbf{x},0}}(\mathbf{S})$-stabilizer, then $H_c^*(X_r \cap \mathbb{L}_r\mathbb{G}_r^+, \overline{\mathbb{Q}}_\ell)[\theta]$ is irreducible (up to a sign) and is independent of the choice of $\mathbf{U}$.

Remark 6.8. Let $\mathbf{S}$ be an unramified maximal torus of $\mathbf{G}$ and θ a 0-toral character of S which is trivial on S_{r+} for $r \in \mathbb{R}_{>0}$. Let us consider the cohomology $H_c^*(X_r \cap \mathbb{L}_r\mathbb{G}_r^+, \overline{\mathbb{Q}}_\ell)[\theta]$ for $\mathbf{L} = \mathbf{S}$. Then, since the group $W_{L_{\mathbf{x},0}}(\mathbf{S})$ is trivial, Theorem 6.7 asserts that $H_c^*(X_r \cap \mathbb{S}_r\mathbb{G}_r^+, \overline{\mathbb{Q}}_\ell)[\theta]$ is irreducible for any 0-toral character θ of S of depth r. Note that, in contrast to Remark 6.4, we do not need any assumption on p for this irreducibility result for 0-toral characters.

7. Geometric toral supercuspidal representations

We are now ready to prove one of our main theorems of this paper: an explicit comparison of Yu's algebraic construction of supercuspidal representations (Section 3) and the geometric construction of parahoric representations coming from Deligne–Lusztig-type varieties (Section 6). To do this, we will apply the results of Section 5—that certain representations are characterized by their character values at unramified very regular elements. These comparison methods will result in several theorems as quick corollaries of the work already established in previous sections of this paper.

As such, we inherit the assumptions on p and the assumptions on $(\mathbf{S}, \theta)$ that have been needed for these preceding sections. In this section, we will assume ($\mathfrak{vreg}$) (i.e., $S_{0,\mathrm{vreg}}$ generates S_0 as a group) in addition to our usual assumptions on the residue characteristic of F (i.e., p is odd, p is not bad for $\mathbf{G}$, $p \nmid |\pi_1(\mathbf{G}_{\mathrm{der}})|$ and $p \nmid |\pi_1(\widehat{\mathbf{G}}_{\mathrm{der}})|$). Aside from the GL_n setting discussed in Section 7.3), in this section we consider $(\mathbf{S}, \theta)$ where

- $\mathbf{S}$ is unramified elliptic,
- θ is 0-toral; equivalently, $\mathbf{G}^0(\mathbf{S},\theta)=\mathbf{S}$ and $\mathbf{G}^1(\mathbf{S},\theta)=\mathbf{G}$.

We note that such a pair $(\mathbf{S},\theta)$ is necessarily a tame elliptic regular pair in the sense of Kaletha (Section 3.2) by definition. The first condition is inherited from Section 6, though there we considered unramified tori $\mathbf{S}$ which were not necessarily elliptic. The second condition is inherited from Section 5, though there we considered θ satisfying the weaker condition that θ is toral.

Fix a prime number ℓ which is not equal to p and an isomorphism $\iota\colon \mathbb{C}\cong\overline{\mathbb{Q}}_\ell$. Let $\theta\colon S\to\mathbb{C}^\times$ be any character which is trivial on S_{r+}. The following lemma demonstrates that the choice of ι is innocuous from the perspective of considering the *a priori* ℓ-adic representation $R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta)=H_c^*(X_r,\overline{\mathbb{Q}}_\ell)[\theta]$ as a complex representation.

Lemma 7.1. *Let $\theta_{\ell,\iota}$ denote the $\overline{\mathbb{Q}}_\ell^\times$-valued character obtained by composing θ with the isomorphism $\iota\colon\mathbb{C}^\times\cong\overline{\mathbb{Q}}_\ell^\times$. The complex $SG_{\mathbf{x},0}$-representation $\iota R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta_{\ell,\iota})$ is independent of ℓ and ι.*

Proof. The scheme X_r (Definition 6.1) is separated and of finite type over $\overline{\mathbb{F}}_q$ (see [CI21b, Lemma 3.1]); therefore by [DL76, Proposition 3.3] (thanks to Andy Gordon for reminding the authors about this), we know that for any $(s,g)\in S_0\times G_{\mathbf{x},0}$, the trace $\mathrm{Tr}((s,g)^*,H_c^*(X_r,\overline{\mathbb{Q}}_\ell))$ is an algebraic integer independent of ℓ. By definition,

$$\mathrm{Tr}\big(g^*,R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta_{\ell,\iota})\big)=\frac{1}{|S_{0:r+}|}\sum_{s\in S_{0:r+}}\theta(s)^{-1}\cdot\mathrm{Tr}\big((s,g)^*,H_c^*(X_r,\overline{\mathbb{Q}}_\ell)\big).$$

As a character of the subquotient $S_{0:r+}$, the S-character θ is valued in the algebraic integers; it follows from the above that the character of $R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta_{\ell,\iota})$ considered as a representation of $G_{\mathbf{x},0}$ takes values in $\overline{\mathbb{Q}}$. It follows that $\iota R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta_{\ell,\iota})\cong\iota' R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta_{\ell',\iota'})$ as $G_{\mathbf{x},0}$-representations, where ℓ' is a prime not equal to p and $\iota'\colon\mathbb{C}\cong\overline{\mathbb{Q}}_{\ell'}$ is a(ny) chosen isomorphism. Since the $Z_{\mathbf{G}}$-action on $\iota R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta_{\ell,\iota})$ is given simply by multiplication by $\theta|_{Z_{\mathbf{G}}}$, it now further follows that $\iota R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta_{\ell,\iota})$ is independent of ℓ and ι as an $SG_{\mathbf{x},0}$-representation. □

7.1. **Comparison for X_r.**

Theorem 7.2 is one of the central theorems of this paper. It not only gives an explicit description of the geometrically arising representation $R_{\mathbb{S}_r}^{\mathbb{G}_r}(\theta)$, but puts it in the framework of algebraic constructions of supercuspidal representations, allowing us passage between these drastically different realizations of these representations.

From Theorem 7.2, we get the irreducibility (and hence supercuspidality) of the G-representation c-$\mathrm{Ind}_{SG_{\mathbf{x},0}}^{G}(R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta))$ as a free consequence. We note that this irreducibility was previously only known for inner forms of GL_n and was highly nontrivial to establish: a geometric proof in the 0-toral setting is in [CI21a] (see Theorems 9.1 and 12.5) and a geometric proof in the toral setting is the subject of [CI19], which further relies on [CI21c] (see Section 7.3 for more details).

We focus on the 0-toral case (see Remark 7.4 for comments on relaxing this assumption to toral characters). When $\theta\colon S\to\mathbb{C}^\times$ is 0-toral, then by Theorem 6.3, the virtual representation $R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta)$ does not depend on the choice of $\mathbf{U}$, so we replace this notation by the simpler $R_{\mathbb{S}_r}^{\mathbb{G}_r}(\theta)$.

Theorem 7.2. *Let* $\mathbf{S}$ *be an unramified elliptic maximal torus of* $\mathbf{G}$ *and* $\theta\colon S \to \mathbb{C}^\times$ *a 0-toral character of depth* $r \in \mathbb{R}_{>0}$*. We let* ϕ *be the unique character of* S *satisfying*[4] $\theta = \phi \cdot \varepsilon^{\mathrm{ram}}[\phi]$*. Then*

$$R^{\mathbb{G}_r}_{\mathbb{S}_r,\mathbb{U}_r}(\theta) \cong (-1)^{r(\mathbf{S},\phi)} {}^{\circ}\tau_d, \tag{7}$$

where ${}^{\circ}\tau_d$ *in the right-hand side is Yu's representation associated to the pair* $(\mathbf{S},\phi)$ *(see Section 3). In particular:*

- *(i)* $\text{c-Ind}^{G}_{SG_{\mathbf{x},0}}((-1)^{r(\mathbf{S},\phi)} R^{\mathbb{G}_r}_{\mathbb{S}_r}(\theta)) \cong \pi_{(\mathbf{S},\phi)}$,
- *(ii)* $\text{c-Ind}^{G}_{SG_{\mathbf{x},0}}((-1)^{r(\mathbf{S},\phi)} R^{\mathbb{G}_r}_{\mathbb{S}_r}(\theta))$ *is irreducible and supercuspidal.*

Proof. By the 0-torality of θ and the assumption on p, Theorem 6.3 implies that $R^{\mathbb{G}_r}_{\mathbb{S}_r}(\theta)$ is an irreducible virtual representation of $SG_{\mathbf{x},0}$ (see Remark 6.4). Hence either $R^{\mathbb{G}_r}_{\mathbb{S}_r}(\theta)$ or its negative is a genuine representation; call it $c \cdot R^{\mathbb{G}_r}_{\mathbb{S}_r}(\theta)$. By Proposition 6.2, for $\gamma \in S_{\mathrm{vreg}}$,

$$\Theta_{c\cdot R^{\mathbb{G}_r}_{\mathbb{S}_r}(\theta)}(\gamma) = c \cdot \sum_{w \in W_{G_{\mathbf{x},0}}(\mathbf{S})} \theta^w(\gamma).$$

Therefore Theorem 5.17 is applicable to $\tau := c \cdot R^{\mathbb{G}_r}_{\mathbb{S}_r}(\theta)$ and we get $c = (-1)^{r(\mathbf{S},\phi)}$ and $\tau \cong {}^{\circ}\tau_d$. □

Remark 7.3. Suppose that θ is toral and that the virtual $SG_{\mathbf{x},0}$-representation $R^{\mathbb{G}_r}_{\mathbb{S}_r,\mathbb{U}_r}(\theta) = H^*_c(X_r, \overline{\mathbb{Q}}_\ell)[\theta]$ is irreducible. Suppose that $H^i_c(X_r, \overline{\mathbb{Q}}_\ell)[\theta]$ is in fact concentrated in a single degree $i = r^{\mathrm{geom}}(\theta)$. (This is known to be the case when $\mathbf{G}$ is a division algebra; see [Cha18, Cha20].) Then Theorem 7.2 gives a *cohomological* meaning to the parity of the purely root-theoretically defined integer $r(\mathbf{S},\phi)$ (Proposition 4.9):

$$r(\mathbf{S},\phi) \equiv r^{\mathrm{geom}}(\theta) \pmod 2.$$

(In the division algebra case, one can check this directly—compare the formula for $r(\mathbf{S},\phi)$ in Proposition 4.9 to the formula for the nonvanishing cohomological degree in [Cha20, Theorem 5.1.1].)

Remark 7.4. The 0-torality condition on θ is only needed because we need the irreducibility of $R^{\mathbb{G}_r}_{\mathbb{S}_r}(\theta)$ ([CI21b, Theorem 1.1], presented as Theorem 6.3 in the present paper). On the other hand, it is expected (and known in certain cases) that Theorem 6.3 holds beyond the 0-toral case as long as $\mathbf{S}$ is elliptic. When $\mathbf{G}$ is an inner form of GL_n, Theorem 6.3 is known without any constraints on θ (see Theorem B of [CI19]), and we will discuss this in more detail in Section 7.3. For $\mathbf{G}$ arbitrary and $\mathbf{S}$ Coxeter, establishing Theorem 6.3 is recent work of Dudas–Ivanov (see Theorem 3.2.3 of [DI20]) announced during the preparation of the present paper. With this, when $\mathbf{S}$ is Coxeter, the 0-torality condition on θ in Theorem 7.2 can be relaxed to torality, with identical proof.

7.2. Comparison for $X_r \cap \mathbb{S}_r\mathbb{G}_r^+$.

As in Section 7.1, there is a unique extension of $H^*_c(X_r \cap \mathbb{S}_r\mathbb{G}_r^+, \overline{\mathbb{Q}}_\ell)[\theta]$ to a (virtual) representation of $SG_{\mathbf{x},0+}$ on which $Z_{\mathbf{G}}$ acts via $\theta|_{Z_{\mathbf{G}}}$. We again denote this representation by $H^*_c(X_r \cap \mathbb{S}_r\mathbb{G}_r^+, \overline{\mathbb{Q}}_\ell)[\theta]$.

Theorem 7.5. *Let* $\mathbf{S}$ *be an unramified elliptic maximal torus of* $\mathbf{G}$ *and* $\theta\colon S \to \mathbb{C}^\times$ *a* 0*-toral character of depth* $r \in \mathbb{R}_{>0}$*. We put* $\phi := \theta \cdot \varepsilon^{\mathrm{ram}}[\theta]$*. Then we have*

$$H^*_c(X_r \cap \mathbb{S}_r\mathbb{G}_r^+, \overline{\mathbb{Q}}_\ell)[\theta] \cong (-1)^{r(\mathbf{S},\phi)} \operatorname{Ind}^{SG_{\mathbf{x},0+}}_{K^d}(\rho'_d \otimes \phi_d),$$

[4]Note that we may equivalently set $\phi = \theta \cdot \varepsilon^{\mathrm{ram}}[\theta]$. Indeed, since $\varepsilon^{\mathrm{ram}}[\theta]$ is a tamely ramified quadratic character determined by $\theta|_{S_{0+}}$, we have $\theta|_{S_{0+}} = \phi|_{S_{0+}}$ and $\varepsilon^{\mathrm{ram}}[\theta] = \varepsilon^{\mathrm{ram}}[\phi] = \varepsilon^{\mathrm{ram}}[\phi]^{-1}$.

where the right-hand side is Yu's representation associated to the pair $(\mathbf{S}, \phi)$ (see Section 3).

Proof. By the 0-torality of θ, Theorem 6.7 implies that $H_c^*(X_r \cap \mathbb{S}_r\mathbb{G}_r^+, \overline{\mathbb{Q}}_\ell)[\theta]$ is an irreducible virtual representation of $SG_{\mathbf{x},0+}$ (see Remark 6.8). The proof of Proposition 6.2 in this setting yields the result that if $\gamma \in SG_{\mathbf{x},0+}$ is an unramified very regular element, then

$$\Theta_{H_c^*(X_r \cap \mathbb{S}_r\mathbb{G}_r^+, \overline{\mathbb{Q}}_\ell)[\theta]}(\gamma) = \begin{cases} \theta(g\gamma g^{-1}) & \text{if } \gamma \in g^{-1}Sg \text{ for some } g \in G_{\mathbf{x},0+}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the same proof as in Theorem 7.2 works by using the above formula and Theorem 5.13 in place of Proposition 6.2 and Theorem 5.17, respectively. □

As a straightforward corollary of Theorem 7.2 and 7.5, we resolve Conjecture 6.5 for 0-toral characters:

Theorem 7.6. *Let $\mathbf{S}$ be an unramified elliptic maximal torus of $\mathbf{G}$ and $\theta\colon S \to \mathbb{C}^\times$ a 0-toral character of depth $r \in \mathbb{R}_{>0}$. Then*

$$R_{\mathbb{S}_r}^{\mathbb{G}_r}(\theta) \cong H_c^*(X_r^{(\mathbf{S})}, \overline{\mathbb{Q}}_\ell)[\theta] \cong \operatorname{Ind}_{SG_{\mathbf{x},0+}}^{SG_{\mathbf{x},0}} \big(H_c^*(X_r \cap \mathbb{S}_r\mathbb{G}_r^+, \overline{\mathbb{Q}}_\ell)[\theta]\big).$$

Proof. The second isomorphism in the assertion holds in general (see the paragraph before Conjecture 6.5), so it suffices to check that $R_{\mathbb{S}_r}^{\mathbb{G}_r}(\theta)$ is isomorphic to $\operatorname{Ind}_{SG_{\mathbf{x},0+}}^{SG_{\mathbf{x},0}} \big(H_c^*(X_r \cap \mathbb{S}_r\mathbb{G}_r^+, \overline{\mathbb{Q}}_\ell)[\theta]\big)$. But this isomorphism holds by Theorem 7.2 and 7.5 together with the fact that

$$^\circ\tau_d \cong \operatorname{Ind}_{SG_{\mathbf{x},0+}}^{SG_{\mathbf{x},0}} \operatorname{Ind}_{K^d}^{SG_{\mathbf{x},0+}} (\rho'_d \otimes \phi_d).$$ □

7.3. **The case of** GL_n**.** Let $\mathbf{G}$ be an inner form of GL_n and let $\mathbf{S}$ be an unramified elliptic maximal torus of $\mathbf{G}$. Then there are no bad primes of $\mathbf{G}$ and $|\pi_1(\mathbf{G}_{\mathrm{der}})| = |\pi_1(\widehat{\mathbf{G}}_{\mathrm{der}})| = 1$, so the only baseline assumption on the residue characteristic of F (Section 2.1) is that $p > 2$. In this setting, $G \cong \mathrm{GL}_{n'}(D_{k_0/n_0})$, where $n = n'n_0$ and D_{k_0/n_0} is the division algebra of dimension n_0^2 with Hasse invariant k_0/n_0, and $S \cong E^\times$, where E is the unramified degree-n extension of F (i.e., it is the smallest extension of F which splits $\mathbf{S}$). This is a specialization of the setting of Section 6, and in this setting Theorems 6.3, 6.7 are known by work of the first author and Ivanov without the 0-torality assumption on θ. We record this result here:

Theorem 7.7 ([CI19, Theorem 3.1], [CI21c, Theorem 5.2.1])**.** *Let $r > 0$ and let $\theta, \theta'\colon S_{0:r+} \to \overline{\mathbb{Q}}_\ell^\times$ be any character. Then*

$$\big\langle R_{\mathbb{S}_r}^{\mathbb{G}_r}(\theta), R_{\mathbb{S}_r}^{\mathbb{G}_r}(\theta')\big\rangle_{G_{\mathbf{x},0:r+}} = \sum_{w \in W_{G_{\mathbf{x},0}}(\mathbf{S})} \langle \theta, \theta'^w \rangle_{S_{0:r+}}.$$

Furthermore, $H_c^*(X_r \cap \mathbb{S}_r\mathbb{G}_r^+, \overline{\mathbb{Q}}_\ell)[\theta]$ is an irreducible representation of $S_0G_{\mathbf{x},0+}$.

This means in particular that we may apply Theorem 5.13 to $H_c^*(X_r \cap \mathbb{S}_r\mathbb{G}_r^+, \overline{\mathbb{Q}}_\ell)[\theta]$ and Theorem 5.17 to $R_{\mathbb{S}_r}^{\mathbb{G}_r}(\theta)$ for any toral character $\theta\colon S \to \overline{\mathbb{Q}}_\ell^\times$ of depth $r \in \mathbb{R}_{>0}$ whenever the following conditions are satisfied:

- $p > 2$ (required for Yu's construction),
- $(\mathbf{Gd}^G)$ is satisfied by $\mathbf{T}_\gamma$ for any unramified very regular element $\gamma \in SG_{\mathbf{x},r}$, and
- $(\mathfrak{vreg})$ holds.

The second condition on $(\mathbf{Gd}^G)$ is always satisfied by Remark 4.4 (2). On the other hand, recall that the inequality $(\star)$ $(|\mathbb{S}(\mathbb{F}_q)|/|\mathbb{S}(\mathbb{F}_q)_{\mathrm{nvreg}}| > 2)$ is sufficient, though not necessary, to guarantee the assumption $(\mathfrak{vreg})$ (see Corollary 5.5). We will show that in fact $(\star)$ is satisfied for all (q,n), including even q, illustrating that in this setting, our constraint on q ($\star$) is weaker than the constraints on p in Yu's construction. In [Hen92, Section 2.7], Henniart proves that $|\mathbb{S}(\mathbb{F}_q)|/|\mathbb{S}(\mathbb{F}_q)_{\mathrm{nvreg}}| > 2n$ holds unless $(q,n) \in \{(2,2),(2,4),(2,6),(3,2)\}$. Of course this means that ($\star$) holds for all (q,n) except possibly the four in this list; by direct computation for these last four, one sees that ($\star$) holds for all (q,n). In summary, in our present setting of $\mathbf{G}$ being an inner form of GL_n, Theorems 5.13 and 5.17 hold for all odd q.

Theorem 7.8. *Let θ be a toral character of S of depth $r > 0$. We put $\phi := \theta \cdot \varepsilon^{\mathrm{ram}}[\theta]$. Then we have*

$$\begin{aligned} R_{\mathbb{S}_r}^{\mathbb{G}_r}(\theta) &\cong (-1)^{r(\mathbf{S},\phi)} \operatorname{Ind}_{K^d}^{SG_{\mathbf{x},0}}(\rho_d' \otimes \phi_d), \\ H_c^*(X_r \cap \mathbb{S}_r\mathbb{G}_r^+, \overline{\mathbb{Q}}_\ell)[\theta] &\cong (-1)^{r(\mathbf{S},\phi)} \operatorname{Ind}_{K^d}^{SG_{\mathbf{x},0+}}(\rho_d' \otimes \phi_d), \\ R_{\mathbb{S}_r}^{\mathbb{G}_r}(\theta) &\cong \operatorname{Ind}_{SG_{\mathbf{x},0+}}^{SG_{\mathbf{x},0}} \big(H_c^*(X_r \cap \mathbb{S}_r\mathbb{G}_r^+, \overline{\mathbb{Q}}_\ell)[\theta]\big). \end{aligned}$$

The twist $\varepsilon^{\mathrm{ram}}[\theta]$ is subtle, and we will discuss it later in this subsection. Before we do that, we note that Theorem 7.8 in particular yields an *algebraic* alternative to the geometric approach of [CI19] to establish the irreducibility of the θ-eigenspace of loop Deligne–Lusztig varieties:

Corollary 7.9. *Let θ be a toral character of S of depth $r > 0$. Then* $\text{c-Ind}_{SG_{\mathbf{x},0}}^{G}(R_{\mathbb{S}_r}^{\mathbb{G}_r}(\theta))$ *is (up to sign) irreducible supercuspidal. In particular, the assumption $p > n$ in* [CI19, Theorem A] *can be relaxed to $p > 2$.*

The $p > 2$ assumption in Corollary 7.9 likely cannot be relaxed due to our approach of comparing with Yu's construction, which uses from the start that p is odd. However, given Henniart's work [Hen92], it seems likely that one can obtain the above result for $p = 2$ by considering Gérardin's construction [Gér79] in place of Yu's in this setting.

Remark 7.10. We remark that the $p > n$ assumption in [CI19, Theorem A] is in place for deep technical reasons, and that this assumption is only needed for θ which are toral but not 0-toral. For 0-toral θ, the first author and Ivanov obtain the irreducibility (and hence supercuspidality) of the compact induction $\text{c-Ind}_{SG_{\mathbf{x},0}}^{G}(R_{\mathbb{S}_r}^{\mathbb{G}_r}(\theta))$ for *all* primes p in [CI21a, Theorem 12.5] by purely geometric techniques. In particular, Corollary 7.9 is strictly weaker than *op. cit.* in the 0-toral setting, exactly because Corollary 7.9 says nothing about $p = 2$.

To finish this subsection, we would like to illustrate the subtle nature of $\varepsilon^{\mathrm{ram}}[\theta]$ and the sign $(-1)^{r(\mathbf{S},\phi)}$ by further specializing the present setting to the case $\mathbf{G} = \mathrm{GL}_n$. Then, in general, any twisted Levi of $\mathbf{G}$ is isomorphic to the product of several Weil restrictions of general linear groups. For any cuspidal $\mathbf{G}$-datum, because of the anisotropicity assumption on $\mathbf{Z}_{\mathbf{G}^0}/\mathbf{Z}_{\mathbf{G}}$, each twisted Levi $\mathbf{G}^i$ is given by $\mathrm{Res}_{E_i/F}\,\mathrm{GL}_{n_i}$, where E_i is the unramified degree-n/n_i extension of F. If $\gamma \in S_{0,\mathrm{vreg}}$ is such that the image of γ in $S_{0:0+} \cong \mathbb{F}_{q^n}^\times$ generates all of $S_{0:0+}$, then $\varepsilon^{\mathrm{ram}}[\theta]$ is the unique tamely ramified character of S such that $\varepsilon^{\mathrm{ram}}[\theta]|_{Z_{\mathbf{G}}} = \mathbb{1}$ and

$$\varepsilon^{\mathrm{ram}}[\theta](\gamma) = \prod_{i=0}^{d-1} (-1)^{(\lfloor n_{i+1}/2 \rfloor - \lfloor n_i/2 \rfloor)(r_i - 1)}.$$

Recalling the definition of $\varepsilon^{\mathrm{ram}}[\theta]$ from Definition 4.8, the above formula comes from the calculation that:

$$|(\Gamma_F \times \{\pm 1\})\backslash(R^{\mathbf{G}}_{\mathbf{x},r/2} \cap R(\mathbf{G},\mathbf{S})^{\mathrm{sym}})| = \begin{cases} \lfloor (n-1)/2 \rfloor & \text{if } r \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

$$|\Gamma_F\backslash(R^{\mathbf{G}}_{\mathbf{x},r/2} \cap R(\mathbf{G},\mathbf{S})_{\mathrm{sym,ur}})| = \begin{cases} 1 & \text{if } r \text{ is even and } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

The calculation of the sign $(-1)^{r(\mathbf{S},\phi)}$ is slightly less delicate. We first note that $r(\mathbf{S},\phi)$ equals $r(\mathbf{S},\theta)$ since $r(\mathbf{S},\phi)$ depends only on the positive-depth part $\phi|_{S_{0+}}$ by definition (Proposition 4.9) and we have $\phi|_{S_{0+}} = \theta|_{S_{0+}}$. We can calculate that

$$|\Gamma_F\backslash R^{\mathbf{G}}_{\mathbf{x},r/2}| = \begin{cases} n-1 & \text{if } r \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

and therefore

$$r(\mathbf{S},\phi) = r(\mathbf{S},\theta) \equiv \sum_{i=0}^{d-1} (n_{i+1} - n_i)(r_i - 1) \pmod 2.$$

It is known ([CI21c, Theorem 6.1.1]) that there is a unique and explicitly described cohomological degree $r^{\mathrm{geom}}(\theta)$ such that $H_c^{r^{\mathrm{geom}}(\theta)}(X_r \cap \mathbb{S}_r \mathbb{G}_r^+, \overline{\mathbb{Q}}_\ell)[\theta] \neq 0$. Moreover, if $\mathbf{G}^0(\mathbf{S},\theta) = \mathbf{S}$, then Theorem 7.8 guarantees that $r^{\mathrm{geom}}(\theta) \equiv r(\mathbf{S},\phi)$ modulo 2. We can see this explicitly as well:

$$\begin{aligned} r(\mathbf{S},\phi) &= \sum_{i=0}^{d-1} (n_{i+1} - n_i)(r_i - 1) \\ &\equiv (r_d + 1)(n_d) - n_0(r_0 + 1) + \sum_{i=1}^{d} n_i(r_{i-1} - r_i) \pmod 2, \end{aligned}$$

and if $n_0 = 1$ (i.e., if $\mathbf{G}^0(\mathbf{S},\theta) = \mathbf{S}$), then this last formula is almost exactly equal to $r^{\mathrm{geom}}(\theta)$ in [CI21c, Corollary 6.1.2], the only difference being summands with even values. We note the peculiarity that the algebraically calculated number $r(\mathbf{S},\phi)$ has summands that consider the difference between n_i's (which captures the sequence of twisted Levi subgroups $\mathbf{G}^i$), while the geometrically calculated number $r^{\mathrm{geom}}(\theta)$ has summands that consider the difference between the depths r_i.

These calculations can be made similarly for $\mathbf{G}$ being any inner form of GL_n, keeping in mind the further calculations that if $\mathbf{G}$ is a nonsplit form of GL_n, the point $\bar{\mathbf{x}} \in \mathcal{B}^{\mathrm{red}}(\mathbf{G}, F)$ corresponding to $\mathbf{S} \hookrightarrow \mathbf{G}$ is not superspecial in the sense of [Kal19, Definition 3.4.8].

7.4. **Discussion of small residual characteristic.** We illustrate some of the subtleties that arise for small p by considering a particular example: let F have residual characteristic $p = 2$ and $\mathbf{G} = \mathrm{SL}_2$. In this case, Yu's construction is not available since it requires the oddness of p to utilize the theory of Weil representations for Heisenberg groups over $\mathbb{F}_p$ ([Yu01, Sections 10–]). When $p = 2$, the notion of Yu-datum still makes sense, even if Yu's construction cannot be used to build a supercuspidal from such data. On the other hand, when $p = 2$ and $\mathbf{G} = \mathrm{SL}_2$, there does not even exist a toral Yu-datum! To see this, we let $\mathbf{S}$ be the unramified elliptic maximal torus of $\mathbf{G}$; note that $\mathbf{S}$ splits over the degree-2 unramified extension of F which we denote by E. In this setting, one can compute that

$$S_{r:r+} = \mathrm{Ker}(E_r^\times \xrightarrow{\mathrm{Nr}_{E/F}} F_r^\times)/\,\mathrm{Ker}(E_{r+}^\times \xrightarrow{\mathrm{Nr}_{E/F}} F_{r+}^\times) = \mathrm{Ker}(E_{r:r+}^\times \xrightarrow{\mathrm{Nr}_{E/F}} F_{r:r+}^\times) = F_{r:r+}^\times,$$

where the second equality holds since E is an unramified extension of F, so the norm map on higher unit groups is surjective, and the third equality holds because the residue field has even characteristic. In particular, every character θ of $S_{r:r+}$ is stabilized by all of $W_G(\mathbf{S}) = \mathrm{Gal}(E/F)$. A closely related observation is that $W_G(\mathbf{S})$ acts trivially on the reduction-modulo-2 of the character lattice. This implies that Yu's condition (GE2) (see [Yu01, Sections 8]), which is one of the two conditions for a character θ of S to be $\mathbf{G}$-generic of depth r, is never satisfied.

Now let us move to the geometric side. Let $\theta\colon S \to \overline{\mathbb{Q}}_\ell^\times$ be a character of depth r. In this setting, it is automatic that $\theta|_{S_{r:r+}}$ is nontrivial and $\theta|_{\mathrm{Nr}_{E/F}(\alpha^\vee(E_r^\times))} \neq \mathbb{1}$ for all $\alpha \in R(\mathbf{S}, \mathbf{G})$. By Theorem 6.3,

$$\langle R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta), R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta)\rangle_{\mathrm{SL}_2(\mathcal{O}_F/\mathfrak{p}_F^{r+1})} = 2.$$

It necessarily follows that $R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta)$ consists of two non-isomorphic irreducible representations τ_1, τ_2. It seems possible to guess that $\text{c-Ind}_{\mathrm{SL}_2(\mathcal{O}_F)}^{\mathrm{SL}_2(F)}(\tau_1)$ and $\text{c-Ind}_{\mathrm{SL}_2(\mathcal{O}_F)}^{\mathrm{SL}_2(F)}(\tau_2)$ are non-isomorphic irreducible (and therefore supercuspidal) representations of $\mathrm{SL}_2(F)$.

Our expectation is that in general the geometric representations $R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta)$ realize representations which *do not* appear in Yu's construction. In the above setting, we expect that $R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta)$ should be a representation constructed by Gérardin [Gér75a] [Gér75b]. In some cases, this is a theorem of Chen–Stasinski [CS17]; their technique works for all p, and specializing their result to the setting of SL_2 and $p = 2$, Chen and Stasinski prove that if r is odd, then $R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta)$ coincides with the representation corresponding to the character $\theta|_{S_0}$ in Gérardin's construction. It seems reasonable to us that for general depths, the representation $R_{\mathbb{S}_r,\mathbb{U}_r}^{\mathbb{G}_r}(\theta)$ should also correspond to one of Gérardin's representations $(\theta\cdot\varepsilon)|_{S_0}$, where ε is a twisting character which should be independent of θ and p. (For large p, Theorem 5.17 can be applied to compare Gérardin's construction with Yu's construction.)

8. Geometric L-packets of toral supercuspidal representations

In this section, we examine the implications of our results within the context of the local Langlands correspondence. We write W_F for the Weil group, I_F the inertia group, and P_F the wild inertia group. We let I_F^r denote the r-th upper ramification filtration of I_F for $r \in \mathbb{R}_{\geq 0}$. Let $\widehat{\mathbf{G}}$ be the Langlands dual group for $\mathbf{G}$ and write ${}^LG = \widehat{\mathbf{G}} \rtimes W_F$. Following Kaletha [Kal19], we assume F has characteristic 0 in this section.

We first review Kaletha's construction of 0-toral supercuspidal L-packets. As in [Kal19, Definition 6.1.1], define:

Definition 8.1. A *0-toral supercuspidal parameter of generic depth* $r > 0$ is a discrete L-parameter $\varphi\colon W_F \to {}^LG$ satisfying the following conditions:

(1) The centralizer of $\varphi(I_F^r)$ in $\widehat{\mathbf{G}}$ is a maximal torus and contains (the projection from ${}^LG$ to $\widehat{\mathbf{G}}$ of) $\varphi(P_F)$.
(2) $\varphi(I_F^{r+})$ is trivial, i.e., $\varphi(\sigma) = 1 \rtimes \sigma$ for any $\sigma \in I_F^{r+}$.

In the present paper, our focus is on Howe-unramified supercuspidal representations. If a 0-toral supercuspidal parameter φ of generic depth $r > 0$ is such that the centralizer of $\varphi(I_F^r)$ in $\widehat{\mathbf{G}}$ corresponds to an unramified torus in $\mathbf{G}$ [Kal19, Section 5.1], then we additionally say that φ is *Howe-unramified*.

Following [Kal19, Definition 5.2.4, Section 6.1], we define a *Howe-unramified 0-toral supercuspidal L-packet datum of depth* r to be a tuple $(\mathbf{S}, \widehat{\jmath}, \chi, \theta)$ consisting of

- $\mathbf{S}$ an unramified torus of dimension equal to the absolute rank of $\mathbf{G}$, defined over F with anisotropic quotient $\mathbf{S}/\mathbf{Z}_{\mathbf{G}}$
- $\widehat{\jmath}\colon \widehat{\mathbf{S}} \to \widehat{\mathbf{G}}$ is an embedding of complex reductive groups whose $\widehat{\mathbf{G}}$-conjugacy class is Γ_F-stable
- χ is a minimally ramified χ-data for $R(\mathbf{S}, \mathbf{G})$
- $\theta\colon S \to \mathbb{C}^\times$ is a 0-toral character of depth r.

Here, although a priori $\mathbf{Z}_{\mathbf{G}}$ is not a subgroup of $\mathbf{S}$, we may regard $\mathbf{Z}_{\mathbf{G}}$ as a subgroup of $\mathbf{S}$ by choosing an embedding of $\mathbf{S}$ into $\mathbf{G}$ which is *admissible* for $\widehat{\jmath}$. Moreover, the resulting subgroup structure does not depend on the choice of such an embedding. We refer the readers to [Kal19, Section 5.1] for more details including the definition of the admissibility. We note that there is a unique choice of χ because $\mathbf{S}$ is unramified [Kal19, Definition 4.6.1], so we will omit it from the tuple. By [Kal19, Proposition 6.1.2], there is a bijection

$$(\mathbf{S}, \widehat{\jmath}, \theta) \mapsto \varphi_{(\mathbf{S}, \widehat{\jmath}, \theta)}$$

between isomorphism classes of Howe-unramified 0-toral supercuspidal L-packet data of depth r and $\widehat{\mathbf{G}}$-conjugacy classes of Howe-unramified 0-toral supercuspidal parameters of generic depth r. Explicitly, the L-parameter is the composition

$$\varphi_{(\mathbf{S}, \widehat{\jmath}, \theta)}\colon W_F \xrightarrow{\varphi_\theta} {}^L S \xrightarrow{{}^L j} {}^L G$$

where $\varphi_\theta\colon W_F \to {}^L S$ is the L-parameter of θ and ${}^L j\colon {}^L S \to {}^L G$ is the Langlands–Shelstad extension of $\widehat{\jmath}$ determined by the canonical χ-data. A *Howe-unramified* 0*-toral supercuspidal datum of depth* r [Kal19, Definition 5.3.2] is a tuple $(\mathbf{S}, \widehat{\jmath}, \theta, j)$ where $(\mathbf{S}, \widehat{\jmath}, \theta)$ is a Howe-unramified 0-toral supercuspidal L-packet datum of depth r and $j\colon \mathbf{S} \hookrightarrow \mathbf{G}$ is an F-rational embedding admissible for $\widehat{\jmath}$. Here we refer the readers to [Kal19, Section 5.1] for the definition of the admissibility. We crucially observe that the set of all Howe-unramified 0-toral supercuspidal datum of depth r corresponding to a fixed $(\mathbf{S}, \widehat{\jmath}, \theta)$ is indexed by G-conjugacy classes within the stable conjugacy class of a(ny) F-rational embedding $j\colon \mathbf{S} \hookrightarrow \mathbf{G}$ admissible for $\widehat{\jmath}$.

For such an embedding $j\colon \mathbf{S} \hookrightarrow \mathbf{G}$, let $\mathbf{x}_j \in \mathcal{B}(\mathbf{G}, F)$ be a point whose image in $\mathcal{B}^{\mathrm{red}}(\mathbf{G}, F)$ is associated with $j\mathbf{S} := j(\mathbf{S})$. We let $\mathbb{S}_{j,r} \subset \mathbb{G}_{j,r}$ be the group schemes defined over $\mathbb{F}_q$ associated to $j\mathbf{S}$ and $\mathbf{x}_j$ as in Section 6.1 and we assume $(\mathfrak{vreg})$ in Section 5.1. By combining our geometric comparison theorem (Theorem 7.2) with Kaletha's construction of toral supercuspidal L-packets, we obtain the following result:

Theorem 8.2. *Let $(\mathbf{S}, \widehat{\jmath}, \theta)$ be a Howe-unramified* 0*-toral supercuspidal L-packet datum of depth r. Then the corresponding L-parameter $\varphi_{(\mathbf{S}, \widehat{\jmath}, \theta)}$ has L-packet*

$$\{\text{c-Ind}_{jS \cdot G_{\mathbf{x}_j,0}}^{G}(|R_{\mathbb{S}_{j,r}}^{\mathbb{G}_{j,r}}(j\theta)|)\}_{j \in \mathcal{J}_G^{\mathbf{G}}},$$

where

- *$j\theta := \theta \circ j^{-1}$,*
- *$|R_{\mathbb{S}_{j,r}}^{\mathbb{G}_{j,r}}(j\theta)|$ denotes $(-1)^{r(j\mathbf{S}, j\theta)} R_{\mathbb{S}_{j,r}}^{\mathbb{G}_{j,r}}(j\theta)$, and*
- *$\mathcal{J}_G^{\mathbf{G}}$ is a set of representatives of the G-conjugacy classes of the stable conjugacy class within F-rational embeddings $\mathbf{S} \hookrightarrow \mathbf{G}$ admissible for $\widehat{\jmath}$.*

Proof. Kaletha's construction of regular supercuspidal L-packets assigns to $\varphi_{(\mathbf{S}, \widehat{\jmath}, \theta)}$ the finite set of supercuspidal representations of G given by $\{\pi_{(j\mathbf{S}, j\theta')}\}_{j \in \mathcal{J}_G^{\mathbf{G}}}$ where $j\theta'$ is a particular twist of the character $j\theta$ of $jS := j\mathbf{S}(F)$. In [Kal19, Step 3 in Section 5.3], this twist is written as $j\theta' = (\theta \cdot \zeta_S^{-1}) \circ j^{-1} \cdot \epsilon_{f,\mathrm{ram}} \cdot \epsilon^{\mathrm{ram}}$; Kaletha's ϵ^{ram} is the character $\varepsilon^{\mathrm{ram}}[j\theta]$ with respect to $(j\mathbf{S} \subset \mathbf{G}, j\theta)$ in Section 4, and because we assume $\mathbf{S}$ is unramified, the other twists ζ_S and $\epsilon_{f,\mathrm{ram}}$ are trivial. Therefore by Theorem 7.2, $\pi_{(j\mathbf{S},j\theta')}$ is isomorphic to the geometrically arising supercuspidal representation $\text{c-Ind}_{jS\cdot G_{\mathbf{x}_j,0}}^{G}(|R_{\mathbb{S}_{j,r}}^{\mathbb{G}_{j,r}}(j\theta)|)$. □

The fact that the geometric description does not need to be separately twisted by $\varepsilon^{\mathrm{ram}}[j\theta]$ has nontrivial significance. Kaletha's L-packets $\{\pi_{(j\mathbf{S},j\theta')}\}_{j\in\mathcal{J}_G^{\mathbf{G}}}$ are *not* the same as $\{\pi_{(j\mathbf{S},j\theta)}\}_{j\in\mathcal{J}_G^{\mathbf{G}}}$; these packets can be genuinely different (see [DS18, Example 5.5] for an explicit example). In fact, it is known that although $\{\pi_{(j\mathbf{S},j\theta)}\}_{j\in\mathcal{J}_G^{\mathbf{G}}}$ appears more canonical from the perspective of Yu's construction and Kaletha's generalization of the Howe factorization, these sets of supercuspidals do *not* (!) satisfy stability and therefore cannot be L-packets.

Let us elaborate on stability here. In general, it is expected that the local Langlands correspondence not only associates an L-packet $\Pi_\varphi^{\mathbf{G}}$ to each L-parameter φ, but also gives a parametrization of the members of $\Pi_\varphi^{\mathbf{G}}$ in terms of a certain finite group $\mathcal{S}_\varphi$ determined by φ; members of $\Pi_\varphi^{\mathbf{G}}$ are expected to be labelled by irreducible characters of $\mathcal{S}_\varphi$. If we let $\langle\pi, -\rangle$ denote the irreducible character of $\mathcal{S}_\varphi$ corresponding to $\pi \in \Pi_\varphi^{\mathbf{G}}$, then the stability of the L-packet $\Pi_\varphi^{\mathbf{G}}$ asserts that the linear combination of Harish-Chandra characters

$$\sum_{\pi\in\Pi_\varphi^{\mathbf{G}}} \langle\pi, 1\rangle \cdot \Theta_\pi$$

is stable, i.e., constant on every stable conjugacy class of strongly regular semisimple elements (see [Art89, Section 6] for a general formulation of stability). When φ is a 0-toral supercuspidal parameter (or, more generally, regular supercuspidal parameter), the associated group $\mathcal{S}_\varphi$ is abelian (see [Kal19, Section 5.3]). Hence, for such an L-parameter, stability simply asserts that the sum $\sum_{\pi\in\Pi_\varphi^{\mathbf{G}}} \Theta_\pi$ should be stable. Although stability alone cannot characterize the local Langlands correspondence, it has an important role as a touchstone in verifying the validity of the construction of the L-packets.

Given this, is it naturally pressing to ask:

> Does the correspondence $(j\colon \mathbf{S} \hookrightarrow \mathbf{G}, \theta) \mapsto \pi_{(j\mathbf{S},j\theta)}$ map stable conjugacy classes to sets of supercuspidals with stable character sums?

In the setting of Howe-unramified 0-toral supercuspidal representations, this question was posed by Reeder [Ree08], who proved that this correspondence maps stable conjugacy classes to sets of supercuspidals with constant formal degree. DeBacker–Spice [DS18] proved that the answer is in fact no and defined a twisting character $\varepsilon^{\mathrm{ram}}$. Under the additional assumption that F has characteristic zero with sufficiently large residual characteristic, DeBacker–Spice proved that the twisted correspondence $(j\colon \mathbf{S} \hookrightarrow \mathbf{G}, \theta) \mapsto \pi_{(j\mathbf{S},j\theta\cdot\varepsilon^{\mathrm{ram}}[j\theta])}$ does in fact preserve stability ([DS18, Theorem 5.10]). Kaletha [Kal19] defined twisting characters in the more general setting of regular supercuspidal representations and proved the associated stability preservation assertion for 0-toral supercuspidal representations under the same assumptions on F as in DeBacker–Spice ([Kal19, Theorem 6.3.2]). This fact is strong evidence for the validity of Kaletha's construction of the local Langlands correspondence.

The content of the next theorem is that if we replace the correspondence $(j\colon \mathbf{S} \hookrightarrow \mathbf{G}, \theta) \mapsto \pi_{(j\mathbf{S},j\theta)}$ with the geometric construction $(j\colon \mathbf{S} \hookrightarrow \mathbf{G}, \theta) \mapsto \text{c-Ind}_{jS\cdot G_{\mathbf{x}_j,0}}^{G}(|R_{\mathbb{S}_{j,r}}^{\mathbb{G}_{j,r}}(j\theta)|)$, then we do not need to separately define a twisting character. Theorem 8.3 is a corollary of results we have already established in this paper (and [DS18, Theorem 5.10] or [Kal19, Theorem 6.3.2]), but we would like to emphasize and repeat the following point mentioned in the introduction: the geometry seems to innately know about automorphic side of the local Langlands correspondence.

Theorem 8.3. *Assume additionally*[5] *that F has characteristic zero with residual characteristic $p \geq (2+e)n$ where e is the ramification degree of F over $\mathbb{Q}_p$ and n is the dimension of the smallest faithful rational representation of $\mathbf{G}$. The correspondence*

$$(j\colon \mathbf{S} \hookrightarrow \mathbf{G}, \theta) \mapsto \text{c-Ind}_{jS \cdot G_{\mathbf{x}_j,0}}^{G}(|R_{\mathbb{S}_{j,r}}^{\mathbb{G}_{j,r}}(j\theta)|)$$

preserves stability for 0-toral characters θ.

9. Regular supercuspidal representations characterized by S_{vreg}

In this section, we prove that certain regular supercuspidal representations are determined by their Harish-Chandra characters on unramified very regular elements. These results can be viewed as versions of the comparison results of Section 7 in the setting that the group $SG_{\mathbf{x},0}$ is replaced by the p-adic group G; on the other hand, neither result implies the other logically (see Remark 9.3). The advantage to obtaining a characterization result at the level of G is that it allows one to characterize members of certain L-packets by their Harish-Chandra characters on a very small collection of elements of G, as one does for real groups [Lan89], resolving a hole mentioned by Kaletha around [Kal19, (5.3.3)]. Such a characterization problem was also mentioned several years earlier by Adler–Spice [AS09, Section 0.3], motivated by Henniart's characterization of certain supercuspidal representations of GL_n [Hen92, Hen93].

The class of regular supercuspidals for which we establish this characterization are those which correspond to tame elliptic regular pairs $(\mathbf{S}, \phi)$ where $\mathbf{S}$ is unramified and $\mathbf{G}^0(\mathbf{S}, \phi) = \mathbf{S}$—that is, precisely the class of Howe-unramified toral supercuspidal representations. We will additionally assume in this section that $\mathbf{S}$ is such that the assumption ($\mathfrak{vreg}$) introduced in Section 5.2 is satisfied.

We prove the following theorem in Section 9.2.

Theorem 9.1. *Let $\mathbf{S}$ be an elliptic unramified maximal torus of $\mathbf{G}$ and let $\theta\colon S \to \mathbb{C}^\times$ be a toral character. Then there is a unique regular supercuspidal representation π of G such that for every $\gamma \in S_{\mathrm{vreg}}$,*

$$\Theta_\pi(\gamma) = c \cdot \sum_{w \in W_G(\mathbf{S})} \theta({}^w\gamma) \tag{8}$$

for a nonzero constant $c \in \mathbb{C}$. Furthermore, $\pi \cong \pi_{(\mathbf{S},\phi)}$ with $\phi := \theta \cdot \varepsilon^{\mathrm{ram}}[\theta]$ and we must have $c = (-1)^{r(\mathbf{S},\phi)}$, where $\mathbf{x} \in \mathcal{B}(\mathbf{G}, F)$ is a point associated to $\mathbf{S}$ and $r(\mathbf{S},\phi)$ is as in Proposition 4.9.

Theorem 9.1 allows us to formulate the construction of Kaletha's L-packets in the following way. Let $j\colon \mathbf{S} \hookrightarrow \mathbf{G}$ be an unramified elliptic maximal torus defined over F and let $\theta\colon S \to \mathbb{C}^\times$ be a toral character. For any F-rational embedding $j'\colon \mathbf{S} \hookrightarrow \mathbf{G}$ stably conjugate to j, let $\pi_{j'}$ be *the* regular supercuspidal representation of G with Harish-Chandra character

$$\Theta_{\pi_{j'}}(\gamma) = c \cdot \sum_{w \in W_G(j'\mathbf{S})} j'\theta({}^w\gamma), \qquad \text{for } \gamma \in j'S_{\mathrm{vreg}},$$

[5]These assumptions are needed in [DS18] as their proof relies on the logarithm map on $G_{\mathbf{x},0+}$ (exists when $p \geq (2+e)n$ by [DR09, Lemma B.0.3]) and also Waldspurger's results [Wal97] on the fundamental lemma and the transfer conjecture. These are the same assumptions as in [Kal19, Section 6.3].

where $j'\theta := \theta \circ j'^{-1}$ and c is some nonzero constant. Then the L-packet Kaletha constructs [Kal19, Section 5.3] is precisely the collection of all $\pi_{j'}$; moreover, the L-parameter corresponding to this L-packet is the homomorphism $\varphi_{(\mathbf{S},\widehat{j},\theta)} \colon W_F \to {}^L G$ recalled in Section 8. The contribution of Theorem 9.1 in this context is exactly the italicized *the* above; that is, that members of certain L-packets are characterized by their Harish-Chandra characters on unramified very regular elements.

Remark 9.2. Kaletha actually asks for something slightly different—that regular supercuspidal representations are characterized by their characters on *shallow* elements (see around [Kal19, (5.3.3)]). Not all shallow elements are unramified very regular (because shallow elements can have connected centralizer being equal to a torus which is not unramified), and not all unramified very regular elements are shallow (because shallow elements necessarily have order coprime to p). A key point here is that if γ is unramified very regular, then any element of $\gamma \cdot T_{\gamma,0+}$ is also unramified very regular. But this is not the case for shallow elements! Pushing further in this direction, it is in fact possible to find two non-isomorphic regular supercuspidal representations with identical character values on shallow elements. Indeed, if we take two non-G-conjugate tame elliptic regular pairs $(\mathbf{S}, \theta)$ and $(\mathbf{S}, \theta')$ such that the regular generic reduced cuspidal $\mathbf{G}$-data associated to them have the same depth zero part (but have different positive depth part), then we cannot distinguish the regular supercuspidal representations $\pi_{(\mathbf{S},\theta)}$ and $\pi_{(\mathbf{S},\theta')}$ by their characters at shallow elements. So, in order for Kaletha's desired characterization to hold in general (i.e., outside the Howe-unramified setting), one must replace "shallow" by a generalized notion of very regularity.

Remark 9.3. We note that Theorem 9.1 is not strong enough to obtain Theorem 7.2 without the work of Sections 5, 7. The point here is without Sections 5,7, we would not know the irreducibility nor the supercuspidality of the induced representation $\text{c-Ind}_{S \cdot G_{\mathbf{x},0}}^{G}(|R_{\mathbb{S}_r}^{\mathbb{G}_r}(\theta)|)$.

On the flip side, the results of Section 5 are not enough to obtain Theorem 9.1 because the results of Section 5 characterize representations at the level of parahoric subgroups.

9.1. **Character formula on unramified very regular elements.** In order to prove Theorem 9.1 we will first need a character formula for regular supercuspidal representations on unramified very regular elements. Such a formula is not contained in the work of Adler–Spice [AS09, Theorem 6.4] because their formula requires the compactness assumption that $\mathbf{G}^{d-1}/\mathbf{Z}_{\mathbf{G}}$ is F-anisotropic, and this hypothesis is not satisfied by every regular supercuspidal representation. In [Kal19, Section 4.4], Kaletha establishes a character formula for shallow elements without this compactness condition. Given the comments in Remark 9.2, neither Kaletha's nor Adler–Spice's formulas suffice for us.

In this section, we prove:

Proposition 9.4. *Let $(\mathbf{S}', \phi)$ be a tame elliptic regular pair of $\mathbf{G}$ and let $\mathbf{x}' \in \mathcal{B}(\mathbf{G}, F)$ be a point associated to $\mathbf{S}' \hookrightarrow \mathbf{G}$. Let $\mathbf{S}$ be an unramified elliptic maximal torus of $\mathbf{G}$. When $\mathbf{S}'$ is not G-conjugate to $\mathbf{S}$, we have $\Theta_{\pi_{(\mathbf{S}',\phi)}}(\gamma) = 0$ for any unramified very regular element $\gamma \in S_{\mathrm{vreg}}$. When $\mathbf{S}'$ equals $\mathbf{S}$, for any unramified very regular element $\gamma \in S_{\mathrm{vreg}}$, we have*

$$\Theta_{\pi_{(\mathbf{S}',\phi)}}(\gamma) = (-1)^{r(\mathbf{G}^0) - r(\mathbf{S}) + r(\mathbf{S},\phi)} \sum_{w \in W_G(\mathbf{S})} \varepsilon^{\mathrm{ram}}[\phi]({}^w\gamma) \cdot \phi({}^w\gamma),$$

where the exponent of (-1) is as in Proposition 4.9.

Before we prove the main result of this subsection (Proposition 9.4) which we will use to prove Theorem 9.1 in the next section, let us fix some notation. From now until the end of the paper, we will use the following notation. Let $(\mathbf{S}', \phi)$ be a tame elliptic regular pair with an associated point $\mathbf{x}' \in \mathcal{B}(\mathbf{G}, F)$ and let $(\vec{\mathbf{G}}, \pi_{-1}, \vec{\phi})$ be a corresponding regular generic reduced cuspidal $\mathbf{G}$-datum (Section 3.2). We caution that at this point it could happen that either $\mathbf{S}'$ is not unramified or ϕ is not toral. We let ${}^{\circ}\rho_i'$, $\tilde{\phi}_i$ denote the "intermediate" representations arising in Yu's construction of the supercuspidal representation associated to $(\vec{\mathbf{G}}, \pi_{-1}, \vec{\phi})$ (see Sections 3.1, 3.4 for recollections). We remind the reader that ${}^{\circ}\rho_i'$ is a representation of ${}^{\circ}K^i = S' G^0_{\mathbf{x}',0}(G^0, \ldots, G^i)_{\mathbf{x}',(s_0,\ldots,s_{i-1})}$ and $\tilde{\phi}_i$ is a representation of $G^i_{\bar{\mathbf{x}}'} \ltimes J^{i+1}$ where $J^{i+1} = (G^i, G^{i+1})_{\mathbf{x}',(r_i,s_i)}$. Furthermore, we fix an unramified elliptic maximal torus $\mathbf{S}$ of $\mathbf{G}$.

Proof of Proposition 9.4. We first note that the central character of $\pi_{(\mathbf{S}',\phi)}$ is given by $\phi|_{Z_{\mathbf{G}}}$ ([Kal19, Fact 3.7.11]) and $\varepsilon^{\mathrm{ram}}[\phi]|_{Z_{\mathbf{G}}} = \mathbb{1}$ by definition (Definition 4.8). Moreover, since $\mathbf{S}$ is unramified, we have $S = S_0 Z_{\mathbf{G}}$. Thus it is enough to treat the case where γ belongs to $S_{0,\mathrm{vreg}}$.

Recall (Sections 3.1, 3.4) that we have

$$\pi_{(\mathbf{S}',\phi)} = \text{c-Ind}_{K_\sigma}^G (\sigma \otimes \phi_d) = \text{c-Ind}_{K_\sigma}^G (\sigma) \otimes \phi_d$$

where $K_\sigma = K_{\sigma_d} = G^{d-1}_{\bar{\mathbf{x}}'} G_{\mathbf{x}',0+}$ and $\sigma = \sigma_d = \mathrm{Ind}_{K^d}^{K_{\sigma_d}} (\rho_d')$. Let Θ_σ denote the character of σ and let $\dot{\Theta}_\sigma$ denote the extension by zero of Θ_σ to a function of G.

Claim 1. For any unramified very regular element $\gamma \in S_{\mathrm{vreg}}$,

$$\Theta_{\pi_{(\mathbf{S}',\phi)}}(\gamma) = \phi_d(\gamma) \sum_{g \in {}^{\circ}K^d \backslash G_{\bar{\mathbf{x}}'}} \dot{\Theta}_{{}^{\circ}\rho_d'}({}^g\gamma).$$

We first argue that the function $g \mapsto \dot{\Theta}_\sigma({}^g\gamma)$ on $G/Z_{\mathbf{G}}$ is supported on a single right coset $(G_{\bar{\mathbf{x}}'}/Z_{\mathbf{G}}) \cdot g_\sigma$ in $G/Z_{\mathbf{G}}$ (and is in particular compactly supported). For this we follow an argument of Kaletha [Kal19, §4.4] (who did it under the assumption that γ is shallow) adapted to our situation (γ is an unramified very regular element of S). By definition, we have that for any $\alpha \in R(\mathbf{S}, \mathbf{G})$, we have $\alpha(\gamma) \not\equiv 1 \pmod{\mathfrak{p}_F}$. Moreover, γ is bounded modulo $Z_{\mathbf{G}}$ since $S = S_0 Z_{\mathbf{G}}$. Hence, by [Tit79, Section 3.6.1], the set of fixed points of γ in $\mathcal{B}^{\mathrm{red}}(\mathbf{G}, F^{\mathrm{ur}})$ is $\mathcal{A}^{\mathrm{red}}(\mathbf{T}_\gamma, F^{\mathrm{ur}}) = \mathcal{A}^{\mathrm{red}}(\mathbf{S}, F^{\mathrm{ur}})$. Hence the fixed points of γ in the rational building $\mathcal{B}^{\mathrm{red}}(\mathbf{G}, F)$ consists of a single point $\bar{\mathbf{x}}$. The same holds for ${}^g\gamma$ in place of γ, with fixed point $g\bar{\mathbf{x}}$. Obviously ${}^g\gamma$ is an element of the stabilizer subgroup $G_{\bar{\mathbf{x}}'}$ of $\bar{\mathbf{x}}'$ if and only if $\bar{\mathbf{x}}' = g\bar{\mathbf{x}}$. In particular, unless $\bar{\mathbf{x}}' = g\bar{\mathbf{x}}$, we have ${}^g\gamma \notin K_\sigma$ and $\dot{\Theta}_\sigma({}^g\gamma) = 0$. Hence $g \mapsto \dot{\Theta}_\sigma({}^g\gamma)$ is supported on $\{g \in G \mid g\bar{\mathbf{x}} = \bar{\mathbf{x}}'\}$, which is either empty (in which we take g_σ arbitrarily) or forms a single right coset of $G_{\bar{\mathbf{x}}'}/Z_{\mathbf{G}}$ in $G/Z_{\mathbf{G}}$ (in which case the coset of g_σ is uniquely determined).

In the following, by replacing $(\mathbf{S}, \gamma)$ with $({}^{g_\sigma}\mathbf{S}, {}^{g_\sigma}\gamma)$, we assume that $g_\sigma = 1$. Note that then the point $\bar{\mathbf{x}}$ associated to $\mathbf{S}$ is necessarily equal to $\bar{\mathbf{x}}'$.

By noting that γ is elliptic, the Harish-Chandra integral character formula (see [HC70, 94 page]) implies that

$$\text{(HC)} \qquad \Theta_{\pi_{(\mathbf{S}',\phi)}}(\gamma) = \frac{\deg \pi_{(\mathbf{S}',\phi)}}{\dim \sigma} \phi_d(\gamma) \int_{G/Z_{\mathbf{G}}} \dot{\Theta}_\sigma({}^g\gamma)\, dg,$$

where $\deg \pi_{(\mathbf{S}',\phi)}$ is the formal degree of the supercuspidal representation $\pi_{(\mathbf{S}',\phi)}$ with respect to a fixed Haar measure of $G/Z_{\mathbf{G}}$. (In [HC70], the characteristic of F is assumed to be zero. See [AS09, Proof of Theorem 6.4] for an expository on the validity of the integral formula in the positive characteristic case.) Since the support of the function $g \mapsto \dot{\Theta}({}^{g}\gamma)$ is contained in $G_{\bar{\mathbf{x}}'}/Z_{\mathbf{G}}$ (note that now $g_\sigma = 1$), we can compute this integral as follows:

$$\begin{aligned}\int_{G/Z_{\mathbf{G}}} \dot{\Theta}_\sigma({}^{g}\gamma)\,dg &= \sum_{g' \in K_\sigma \backslash G_{\bar{\mathbf{x}}'}} \int_{K_\sigma g'/Z_{\mathbf{G}}} \dot{\Theta}_\sigma({}^{g}\gamma)\,dg \\ &= \sum_{g' \in K_\sigma \backslash G_{\bar{\mathbf{x}}'}} \mathrm{meas}(K_\sigma g'/Z_{\mathbf{G}}) \cdot \dot{\Theta}_\sigma({}^{g'}\gamma) \\ &= \mathrm{meas}(K_\sigma/Z_{\mathbf{G}}) \sum_{g \in K_\sigma \backslash G_{\bar{\mathbf{x}}'}} \dot{\Theta}_\sigma({}^{g}\gamma).\end{aligned}$$

As the irreducible supercuspidal representation $\pi_{(\mathbf{S}',\phi)}$ is obtained by the compact induction of σ from K_σ to G, we have

$$\deg \pi_{(\mathbf{S}',\phi)} = \dim \sigma \cdot \mathrm{meas}(K_\sigma/Z_{\mathbf{G}})^{-1}$$

(see, e.g., [BH96, Theorem A.14]). Thus the formula (HC) is simplified to

$$\Theta_{\pi_{(\mathbf{S}',\phi)}}(\gamma) = \sum_{g \in K_\sigma \backslash G_{\bar{\mathbf{x}}'}} \dot{\Theta}_\sigma({}^{g}\gamma).$$

Since we have $\sigma = \mathrm{Ind}_{{}^{\circ}K^{d}}^{K_\sigma} {}^{\circ}\rho'_d \otimes \phi_d$, the Frobenius formula implies Claim 1.

Claim 2. If there is an unramified very regular element $\gamma \in S_{\mathrm{vreg}}$ such that $\Theta_{\pi_{(\mathbf{S}',\phi)}}(\gamma) \neq 0$, then $\mathbf{S}'$ is necessarily G-conjugate to the unramified elliptic maximal torus $\mathbf{S}$.

If $\Theta_{\pi_{(\mathbf{S}',\phi)}}(\gamma)$ is not zero, then $\Theta_{\pi_{(\mathbf{S}',\phi)}}(z\gamma) \neq 0$ is also not zero for any $z \in Z_{\mathbf{G}}$. Thus, by $Z_{\mathbf{G}}$-translation, we may suppose that γ belongs to S_0; in particular, γ is bounded. Moreover, if $\Theta_{\pi_{(\mathbf{S}',\phi)}}(\gamma)$ is not zero, there exists $g \in G_{\bar{\mathbf{x}}'}$ such that $\dot{\Theta}_{{}^{\circ}\rho'_d}({}^{g}\gamma)$ is not zero by Claim 1. By replacing γ with ${}^{g}\gamma$, we may assume that $\Theta_{{}^{\circ}\rho'_d}(\gamma) \neq 0$ (in particular, γ belongs to ${}^{\circ}K^{d} \subset S'G_{\mathbf{x}',0}$). As discussed in the proof of Claim 1, now we have $\bar{\mathbf{x}} = \bar{\mathbf{x}}'$. This implies that the unramified very regular element γ of $S \subset SG_{\mathbf{x},0}$ is also unramified very regular as an element of $S'G_{\mathbf{x}',0}$. Hence, by Lemma 4.7, we may furthermore assume that γ belongs to $S'G^{0}_{\mathbf{x}',0} (= {}^{\circ}K^{0})$ by replacing γ with its ${}^{\circ}K^{d}$-conjugate appropriately. By definition, ${}^{\circ}\rho'_d$ is the representation of ${}^{\circ}K^{d} = {}^{\circ}K^{d-1}J^{d}$ descended from the ${}^{\circ}K^{d-1} \ltimes J^{d}$-representation $(\tilde{\phi}_{d-1}|_{{}^{\circ}K^{d-1} \ltimes J^{d}}) \otimes (({}^{\circ}\rho'_{d-1} \otimes \phi_{d-1}|_{{}^{\circ}K^{d-1}}) \ltimes \mathbb{1})$. Thus we have

$$\Theta_{{}^{\circ}\rho'_d}(\gamma) = \Theta_{\tilde{\phi}_{d-1}}(\gamma \ltimes 1) \cdot \Theta_{{}^{\circ}\rho'_{d-1}}(\gamma) \cdot \phi_{d-1}(\gamma).$$

Then the second factor on the right-hand side can be computed in the same way:

$$\Theta_{{}^{\circ}\rho'_{d-1}}(\gamma) = \Theta_{\tilde{\phi}_{d-2}}(\gamma \ltimes 1) \cdot \Theta_{{}^{\circ}\rho'_{d-2}}(\gamma) \cdot \phi_{d-2}(\gamma).$$

Continuing inductively, we see that we must have $\Theta_{{}^{\circ}\rho'_0}(\gamma) \neq 0$. By the boundedness of γ, we can take a topological Jordan decomposition (i.e., a normal (0+)-approximation) $\gamma = \gamma_0\gamma_+$ with a topologically semisimple element $\gamma_0 \in S'G^{0}_{\mathbf{x}',0}$ and a topologically unipotent element $\gamma_+ \in S'G^{0}_{\mathbf{x}',0}$ (see [Spi08, Proposition 1.8]). Then we have

$$\Theta_{{}^{\circ}\rho'_0}(\gamma) = \Theta_{\kappa_{(\mathbf{S}',\phi_{-1})}}(\gamma) = \Theta_{\kappa_{(\mathbf{S}',\phi_{-1})}}(\gamma_0).$$

By Lemma 4.5, this vanishes unless γ_0 is $S'G^{0}_{\mathbf{x}',0}$-conjugate to an element of S'. We take $g \in S'G^{0}_{\mathbf{x}',0}$ such that ${}^{g}\gamma_0 \in S'$. Since γ is unramified very regular, the topologically semisimple part γ_0 is regular semisimple in $\mathbf{G}$. Thus the connected centralizer $\mathbf{T}_{{}^{g}\gamma_0} = {}^{g}\mathbf{T}_{\gamma_0}$ of ${}^{g}\gamma_0 \in S'$ is equal to $\mathbf{S}'$. On the other hand, we note that γ_0 is contained in the closure $\overline{\langle\gamma\rangle}$ of the cyclic group group $\langle\gamma\rangle$ in G (see [Spi08, Proposition 1.7 (2)]). Since $\overline{\langle\gamma\rangle}$ is contained in $T_\gamma = \mathbf{T}_\gamma(F)$, where $\mathbf{T}_\gamma$ is the connected centralizer of γ in $\mathbf{G}$, we get $\gamma_0 \in \mathbf{T}_\gamma$. This implies that we have $\mathbf{T}_{\gamma_0} = \mathbf{T}_\gamma$ by the regularity of γ_0. As $\mathbf{T}_\gamma$ is nothing but $\mathbf{S}$, we get $\mathbf{S} = \mathbf{T}_\gamma = \mathbf{T}_{\gamma_0} = {}^{g^{-1}}\mathbf{S}'$, which establishes Claim 2.

We are now ready to finish the proof of Proposition 9.4. From now on, we assume that $\mathbf{S}'$ is G-conjugate to $\mathbf{S}$. Let us compute the character

$$\Theta_{\pi_{(\mathbf{S}',\phi)}}(\gamma) = \sum_{g \in {}^\circ K^d \backslash G_{\bar{\mathbf{x}}'}} \dot{\Theta}_{{}^\circ\rho'_d}({}^g\gamma).$$

This can be done in a similar way to the proof of Proposition 4.11. By G-conjugation, we may assume that $\mathbf{S}' = \mathbf{S}$. In the following, we omit $'$ from the notation; we simply write $\mathbf{S}$ and $\mathbf{x}$. By the argument in the previous paragraph, we see that

$$\sum_{g \in {}^\circ K^d \backslash G_{\bar{\mathbf{x}}}} \dot{\Theta}_{{}^\circ\rho'_d}({}^g\gamma) = \sum_{\substack{g \in {}^\circ K^d \backslash G_{\bar{\mathbf{x}}} \\ {}^g\gamma \in S}} \Theta_{{}^\circ\rho'_d}({}^g\gamma).$$

The index set of the sum on the right-hand side can be furthermore rewritten as $({}^\circ K^d \cap N_{G_{\bar{\mathbf{x}}}}(\mathbf{S}))\backslash N_{G_{\bar{\mathbf{x}}}}(\mathbf{S})$, which equals $N_{SG^0_{\mathbf{x},0}}(\mathbf{S})\backslash N_{G_{\bar{\mathbf{x}}}}(\mathbf{S})$ by Lemma 4.10 (2). Here note that $N_{SG^0_{\mathbf{x},0}}(\mathbf{S}) = SN_{G^0_{\mathbf{x},0}}(\mathbf{S})$ and $N_{G_{\bar{\mathbf{x}}}}(\mathbf{S}) = N_G(\mathbf{S})$ (the latter equality holds since if an element of G normalizes $\mathbf{S}$, then g stabilizes the point $\bar{\mathbf{x}}$ associated to $\mathbf{S}$). Hence, by Proposition 4.9, we have

$$\begin{aligned}\Theta_{\pi_{(\mathbf{S},\phi)}}(\gamma) &= \sum_{g \in SN_{G^0_{\mathbf{x},0}}(\mathbf{S})\backslash N_G(\mathbf{S})} (-1)^{r(\mathbf{G}^0)-r(\mathbf{S})+r(\mathbf{S},\phi)} \sum_{w \in W_{G^0_{\mathbf{x},0}}(\mathbf{S})} \varepsilon^{\mathrm{ram}}[\phi]({}^{wg}\gamma) \cdot \phi({}^{wg}\gamma) \\ &= (-1)^{r(\mathbf{G}^0)-r(\mathbf{S})+r(\mathbf{S},\phi)} \sum_{w \in W_G(\mathbf{S})} \varepsilon^{\mathrm{ram}}[\phi]({}^{w}\gamma) \cdot \phi({}^{w}\gamma). \qquad \square\end{aligned}$$

Remark 9.5. After we released the first version of this paper, it was announced by Fintzen–Kaletha–Spice that they established, under the additional assumptions that F has characteristic zero and $p > n(2+e)$ (the same assumptions as in Theorem 8.3), a general formula for the characters of regular supercuspidal representations [FKS21, Theorem 4.3.5] based on the work of Spice [Spi18, Spi21].

9.2. **Proof of Theorem 9.1.** We use the notation fixed in the previous subsection. To prove Theorem 9.1 we will use Proposition 9.4 together with the following lemma:

Lemma 9.6. *Let $\theta\colon S \to \mathbb{C}^\times$ be a smooth character such that $\theta|_{S_{0+}}$ has trivial $W_G(\mathbf{S})$-stabilizer. Then for any $\gamma \in S_{\mathrm{vreg}}$, there exists an element $x \in S_{0+}$ such that*

$$\sum_{w \in W_G(\mathbf{S})} \theta({}^w(\gamma x)) \neq 0.$$

In particular, the function $\sum_{w \in W_G(\mathbf{S})} \theta^w$ is not identically zero on S_{vreg}.

Proof. If $\sum_{w \in W_G(\mathbf{S})} \theta({}^w(\gamma x)) = 0$ for all $x \in S_{0+}$, then this implies that

$$\sum_{w \in W_G(\mathbf{S})} \theta({}^w\gamma) \cdot \theta^w|_{S_{0+}} = 0.$$

This implies that the $\theta^w|_{S_{0+}}$ are linearly dependent, which is impossible since they are assumed to be distinct. $\square$

Proof of Theorem 9.1. First, the regular supercuspidal representation associated to the pair $(\mathbf{S}, \phi)$ indeed satisfies (8) by Proposition 9.4. Thus our task is to show uniqueness.

Let π be a regular supercuspidal representation of G. By Proposition 3.3, there is a corresponding tame elliptic regular pair $(\mathbf{S}', \phi')$, which is unique up to G-conjugacy. We assume that the representation π satisfies the equality (8) for a toral character θ of an unramified elliptic maximal torus $\mathbf{S}$ of $\mathbf{G}$. Because $\theta|_{S_{0+}}$ has trivial $W_G(\mathbf{S})$-stabilizer by Lemma 3.8, Lemma 9.6 implies that there is an element $\gamma \in S_{\mathrm{vreg}}$ such that $\Theta_\pi(\gamma) \neq 0$. By Proposition 9.4, this implies that $\mathbf{S}$ and $\mathbf{S}'$ must be G-conjugate. In particular, this means we may assume $\mathbf{S}' = \mathbf{S}$, so that we now have $\pi \cong \pi_{(\mathbf{S},\phi')}$ for some character ϕ' of S.

By Proposition 9.4, for all $\gamma \in S_{\mathrm{vreg}}$, we have

$$\Theta_{\pi_{(\mathbf{S},\phi')}}(\gamma) = (-1)^{r(\mathbf{G}^0)-r(\mathbf{S})+r(\mathbf{S},\phi')} \sum_{w \in W_G(\mathbf{S})} \varepsilon^{\mathrm{ram}}[\phi']({}^w\gamma) \cdot \phi'({}^w\gamma). \tag{9}$$

It is now in relating ϕ' and θ that we will invoke the remaining assumption ($\mathfrak{vreg}$). With this assumption, Lemma 5.16 holds. Therefore we must have that $\varepsilon^{\mathrm{ram}}[\phi'] \cdot \phi' = \theta^w$ for some $w \in W_G(\mathbf{S})$ and $c = (-1)^{r(\mathbf{G}^0)-r(\mathbf{S})+r(\mathbf{S},\phi')}$. As $\varepsilon^{\mathrm{ram}}[\phi']$ is tamely ramified, we get $\phi'|_{S_{0+}} = \theta^w|_{S_{0+}}$. Since $\varepsilon^{\mathrm{ram}}[\phi']$ (resp. $\varepsilon^{\mathrm{ram}}[\theta^w]$) is determined by $\phi'|_{S_{0+}}$ (resp. $\theta^w|_{S_{0+}}$), we have $\varepsilon^{\mathrm{ram}}[\phi'] = \varepsilon^{\mathrm{ram}}[\theta^w]$. By noting that $\varepsilon^{\mathrm{ram}}$ is a sign character, we finally conclude that $\phi' = \theta^w \cdot \varepsilon^{\mathrm{ram}}[\theta^w]\,(= (\theta \cdot \varepsilon^{\mathrm{ram}}[\theta])^w)$. This implies that $\pi_{(\mathbf{S},\phi')} \cong \pi_{(\mathbf{S},\theta\cdot\varepsilon^{\mathrm{ram}}[\theta])}$. □

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