RTG SEMINAR ON RELATIVE LANGLANDS PROGRAM

NOTES TAKEN BY GUANJIE HUANG

Contents

1. HOMOGENEOUS SPHERICAL VARIETIES - ROBERT

1.1. **Basic concepts.** Let G be a connected reductive group defined over k, which is algebraically closed characteristic 0. Fix $T \subseteq B$.

Definition. A *spherical variety* is a G-variety X that is normal with an open (dense) orbit X_B.

Proposition 1.1. *If X is affine, X is spherical if and only if $k[X]$ is multiplicity free.*

Example 1.1. The following are spherical

1. G/P with P parabolic.
2. G/H with $H \supset \mathrm{rad}_u(B)$.
3. Toric varieties with $G = T$. $X^*(T)$ contains a subgroup $X^*_X(T)$ consists characters in $k[X]$.
4. Symmetric varieties G/G^θ with $\theta^2 = 1$. Say $G = H \times H$ with θ the "swap".
5. Wonderful varieties.

Definition. A subgroup $H \subseteq G$ is *spherical* if G/H is spherical.

A *spherical embedding* is an embedding $G/H \hookrightarrow X$ into a normal G-variety with $G/H = X_G^\circ$.

1.2. **Classification of homogeneous spherical varieties.** The classification of homogeneous spherical varieties can be decomposed into following steps:

1. If H is spherically closed, there is a unique embedding G/H into wonderful variety. So we need to classify wonderful variety.
2. Classify H with embedding into a fix wonderful variety.

Definition. A G-variety is *wonderful of rank r* if

1. it is proper and smooth;
2. X has an open G-orbit whose complement is r G-divisors $D_1, \cdots, D_r$;
3. the D_i's have non-empty transversal intersections which are the G-orbit closures.

Example 1.2.
1. rank 0: G/P (since they are proper)
2. rank 1: $\mathrm{SL}_2 \curvearrowright \mathbb{P}^1 \times \mathbb{P}^1$; $\mathrm{GL}_2 \times \mathrm{GL}_2 \curvearrowright \mathbb{P}(M_{2\times 2})$.

1.3. **Invariants of spherical varieties.** Let $X_X^* \subseteq X(B)$ be the subgroup of B weights in $k[X]$.

Proposition 1.2. *For $r \in X_X^*$, $\dim\{f \in k(X) \mid b \cdot f = r(b)f\} = 1$.*

Proof. Suffices to take $r = 1$. X has an open B-orbit, so we have

$$1 \to k^\times \to k(X)^{(B)} \to X_X^* \to 1.$$

□

- For $r \in X_X^*$, pick $f_r \in k[X]$ satisfying Proposition 1.2.
- Let Δ_X be the set of B-stable, but not G-stable prime divisors.
- We can define $\rho_X : \Delta_X \to (X_X^*)^*$ by $\langle \rho_X(D), r\rangle = V_D(f_r)$. Let V_X be the set of all G-invariant discrete valuations on $k(X)$. There is an embedding

  $$V_X \hookrightarrow (X_X^*)^* \otimes \mathbb{Q}, \langle v, r\rangle = v(f_r),$$

  that makes V_X a convex cone.
- Denote by P_X the stabilizer of X_B°. In particular $B \subseteq P_X$.

Theorem 1.1 (Losev)**.** *These invariants classify essentially determine spherical H up to isomorphism.*

- Spherical roots: let $\Sigma_X \subseteq X_X^*$ be a set of primitive generators such that $V_X = \{v \in X_X^* \mid \langle v, \sigma\rangle \geq 0, \forall \sigma \in \Sigma_X\}$. Alternatively, take $Y \supseteq X_G^\circ$ spherical embedding, then Σ_X is the set of spherical roots of rank 1 wonderful subvarieties of such Y. For rank 1 wonderful variety Z of rank 1, we have $\{z\} = Z^{B^-}$. T acts on $T_Z(z)/T_Z(G\cdot z)$ which is 1-dimensional by the character called the spherical root of Z. In particular, this shows that Σ_G. the set of all possible spherical roots for any spherical G-variety X, is finite.

Let G_α be the associated parabolic for $\alpha \in S$, the set of simple roots associated to B.

- Let $\Delta_X(\alpha) = \{D \in \Delta_X \mid D \text{ not stable by } G_{\{\alpha\}}\}$
- Let $A_X \subseteq \Delta_X$ be $\{\bigcup \Delta_X(\alpha) \mid \alpha \in S \cap \Sigma_X\}$ (equivalently this means $|\Delta_X(\alpha)| = 2$).
- $S_X^P = \{\alpha \in S \mid \Delta_X(\alpha) = \emptyset\} = \{\alpha \in S \mid G_{\{\alpha\}} \text{ stabilizes all } D \in \Delta_X\}$.

If X is wonderful, $(S_X^P, \Sigma_X, A_X \to (X_X^*)^* = (\mathbb{Z}\Sigma_X)^*)$ is a *spherical G-system*, i.e., a triple $(S^P \subseteq S, \Sigma \subseteq \Sigma_G, A \to (\mathbb{Z}\Sigma)^*)$ with some compatibility condition.

Theorem 1.2 (Luna;Bravi-Pezzini)**.** *Spherical G-systems are in 1-1 correspondence with wonderful varieties.*

Definition. $H \subseteq G$ is *spherically closed* if the kernel of $\mathrm{Aut}_{G/H}$ acting on $\Delta_{G/H}$ (the spherical closure) is equal to H.

If H is spherically closed, G/H has a unique wonderful compactification. So this allow us to classify spherically closed subgroup using Theorem 1.2. To obtain the classification of spherical subgroups, we need an augmentation of spherical G-system called *homogeneous spherical datum*: $(S^P, \Sigma, A, X' \subseteq X^*(T), \rho')$ with $\mathbb{Z}\Sigma = X \subseteq X'$, $\rho' : A \to (X')^*$ such that $A \xrightarrow{\rho'} (X')^* \to (X)^*$ is ρ.

Theorem 1.3 (Luna)**.** *If Theorem 1.2 is true, homogeneous spherical data are in 1-1 correspondence with spherical subgroups given by*

$$H \mapsto (S_{G/H}^P, \Sigma_{G/H}, A_{G/H}, X_{G/H}^*, \rho_{G/H}).$$

2. Luna-Vust Theory: Classification of Spherical Embeddings - Calvin

2.1. **Recall.** Last time we studied when G/H is spherical. We defined

- Ξ_X the set of B-characters in $K(X)$
- Ξ_X^* the dual of Ξ_X
- Δ_X the set of colors, i.e., B-stable but not G-stable irreducible divisors

Today we are going to study how we can embed $G/H \hookrightarrow X$ and classify them.

- $V(G/H)$ be the set of G-invariant valuations on $k(G/H)$.
- $D(X) = \{B\text{-stable irreducible divisors in } X\}$.

2.2. **Techinical tools.**

(1) If X is spherical, there is finitely many G-orbits (B-orbits as well).

(1)' If $Y \subseteq X$ is G-stable, then Y is also a spherical variety (if G is connected).

(2) $f \in k[B_{X_0}]$, where B_{X_0} is the dense Borel orbit, $v_0 \in V(G/H)$, then we can find $f' \in k(G)^{(H)}$??? such that
   - $v_0(f') = v_0(f)$
   - $v(f') \geq v(f)$ for any $v \in V(G/H)$
   - $v_D(f') \geq v_D(f)$ for any $D \in D(X)$.

   (in characteristic p need to replace f by f^n for some n; in characteristic 0 we have $n = 1$).

2.3. **Closed G-orbit in X.** Let Y be a closed G-orbit in X, we can set

$$X_{Y,G} = \{x \in X \mid Y \subseteq \bar{G}_x\}$$

In particular, we see that it is non-empty, G-stable with Y the smallest closed G-orbit. It's open because its complement is a (finite becuase X is spherical) union of closed G-orbits. We can cover X be $X_{Y,G}$ as Y varies.

Definition. A *simple spherical variety* X is a spherical variety with a unique closed G-orbit.

2.4. **Classification of simple embeddings.**

Definition. A *colored cone* is a cone (positive linear combination of finitely many vectors does not contain a linear subspace) $\mathcal{C} \subseteq V$ over $\mathbb{Q}$ with finitely many *colors* $\mathcal{F}$ such that $\rho(\mathcal{F}) \subseteq \mathcal{C}$.

Theorem 2.1. *There is a 1-1 correspondence*

$$\{\textit{simple } G/H \hookrightarrow X\} \leftrightarrow \left\{ \textit{colored cones } (\mathcal{C}, F) \in \Xi^*_{G/H} \;\middle|\; \begin{array}{l} \mathcal{F} \subseteq \rho(\Delta_{G/H}) \\ \mathcal{C} \textit{ finitely generated by } \mathcal{F} \textit{ and elements in } \rho(V(G/H)) \\ \mathcal{C}^\circ \cap \rho(V(G/H)) \neq \emptyset \textit{ unless } \mathcal{C} = \{*\} \end{array} \right\}.$$

Example 2.1. (1) G/P has only trivial embeddings since $\Xi_{G/P} = \{0\}$.

(2) $\mathrm{SL}_2/U \cong \mathbb{A}^2 \backslash \{0\}$. In this case $\Xi_{G/U} \cong \mathbb{Q}$ with $\Delta_{G/U} = \{\frac{1}{2}\}$.
   - If $\frac{1}{2}$ is colored, $X = \mathbb{A}^2$. In this case, X has a unique closed G-orbit $Y = \{0\}$ and we can associate $D_Y(X)$ the set of divisors not containing Y, $\mathcal{F}_Y(X)$ the set of B-stable but not G-stable divisors containing Y, and $B_Y(X)$ the set of G-stable divisors containing Y.
   - If $\frac{1}{2}$ is not colored, $X = \mathrm{Bl}_0(\mathbb{A}^2)$.
   - If we choose $\mathcal{C}$ to be the negative rationals, we have $X = \mathbb{P}^2 \backslash \{0\}$.

Now lets how to go from the right to the left. Given the data, there is an embedding of $\varphi : G/H \hookrightarrow \mathbb{P}(W^\vee)$, where W is the G-linear span of $f_0, \cdots, f_z$ with

- f_0 an H-eigenfunction with $v(f_0) = \bigcup_{D \in D(G/H) \backslash \mathcal{F}} D$;
- $f_i = f_0 \cdot g_i$, where g_i generates the dual cone $\mathcal{C}^\vee$.

φ is quasi finite, so by Zariski main lemma, it factors through an simple spherical embedding X with finite map to $\varphi(\bar{G/H})$.

Let Y be a G-stable divisor, we have $X_{Y,G} \supset X_{Y,B} = \{x \in X \mid \bar{B_X} \supset Y\} = X \backslash \bigcup_{D \in D(X) \backslash D_Y(X) D}$.

Proposition 2.1. *(1) $X_{Y,B}$ is affine open.*
(2) $k[X_{Y,B}] = \{f \in k(X) \mid v(f) > 0, \forall v \in \mathcal{C}\}$.
(3) The B-eigenvalue in $k[X_{Y,B}]^{(B)} = \mathcal{C}^\vee$.
(4) The center of $v = Y$ iff $v \in \mathcal{C}^\circ \cap V(G/H)$

Remark. The center of a valuation v on X is $\{p \in X \mid \mathcal{O}_{X,p} \subseteq \mathcal{O}_v\}$.

2.5. **General spherical embeddings.** Fix G/H.

Definition. A *colored fan* is a set $\mathcal{S}$ of colored cones such that

1. each face of a colored cone in S is in S.
2. any $v \in V$ is in the interior of at most one cone

Theorem 2.2. *There is a 1-1 correspondence*

$$\{G/H \hookrightarrow X\} \leftrightarrow \{\textit{colored fans } (\mathcal{C}_i, \mathcal{F}_i)\}$$

given by the taking the colored cones from the closed G-orbits of X.

Example 2.2. For $G/H = \mathrm{SL}_2/H$, we have 2 unsimple embeddings

1. if $\frac{1}{2}$ is colored, we have $\mathbb{P} \backslash \{0\} \cup \mathbb{A}^2 = \mathbb{P}^2$.
2. If $\frac{1}{2}$ is not colored, we have $\mathbb{P}^2 \backslash \{0\} \cup \mathrm{Bl}_0 \mathbb{A}^2 = \mathrm{Bl}_0(\mathbb{P}^1)$.

2.6. **Morphisms of spherical embeddings.** Let $\varphi : G/H \to G/H'$ be a surjective G-equivariant map, $\varphi^* : k(G/H') \to k(G/H)$ restricts to $k(G/H')^{(B)} \to k(G/H)^{(B)}$, giving rise to $\varphi_* : \Xi^*_{G/H} \to \Xi^*_{G/H'}$ such that $\varphi_*(V(G/H)) \subset V(G/H")$ and $\varphi_*(\Delta_{G/H} \backslash \{\text{colors mapping dominantly}\}) \subseteq \Delta(G/H')$.

Theorem 2.3. *$\varphi : G/H \to G/H'$ extends to $X \to X'$ of embeddings iff φ_* is morphisms of colored fans maps $(\mathcal{C}, \mathcal{F})$ to $(\mathcal{C}', \mathcal{F}')$, meaning $\varphi_*(\mathcal{C}) \subset \mathcal{C}'$ and $\varphi(\mathcal{F} \backslash \{\textit{colors maps dominantly}\}) \subseteq \mathcal{F}'$.*

Proposition 2.2. *φ is proper iff* $\mathrm{Supp}(\mathcal{C}) = \varphi^{-1} \mathrm{Supp}(\mathcal{C}')$.

3. The Dual Groups of Spherical Varieties - Charlotte

Let G be connected reductive group over algebraically closed k of characteristic 0. Fix a choice of $T \subseteq B$, giving rise to base root datum $(\Gamma, S, \Gamma^\vee, S^\vee)$.

3.0.1. *Spherical datum.* Recall that the classification of shperical varieties X (normal G-varieties with an open G-orbit) can be decomposed into 2 steps:

1. Classify wonderful varieties $\leftrightarrow (S^p, \tilde{\Sigma}, A)$ (Theorem 1.2)
2. Classify homogeneous varieties $\leftrightarrow (S^p, \tilde{\Sigma}, A, \Xi', \rho')$ (Theorem 1.3)
3. Classify spherical embeddings $\leftrightarrow$ colored fans (Theorem 2.2)

A spherical datum is a triple (Ξ, Σ, S^p) where S^p is a set of simple roots, Σ is a renormalization of $\tilde{\Sigma}$ called *weak spherical roots* and $\Xi = \Xi' + \mathbb{Z}[\Sigma]$.

In other words, we have a triple (Ξ, Σ, S^p) where:

- Ξ is a subgroup of Γ;
- Σ is a finite subset of elements of Ξ (contained in the root lattice);
- S^p subset of S.

Proposition 3.1. *To any G-variety, one can associate a weak spherical datum.*

Remark. Not any weak spherical datum can be obtained from a G-variety.

3.1. **Task.** Fix $X = (\Xi, \Sigma, S^p)$, we want to associate

- $G_X^\vee$ coming from the weak spherical roots;
- $G_X^\wedge$ coming from the associated roots.

We want to construct

$$G_X^\vee \times \mathrm{SL}_2 \to G^\vee.$$

in serveral steps:

1. Construct $\varphi : G_X^\vee \to G_X^\wedge \subseteq G^\vee$.
2. Study centralizer of $\varphi(G_X^\vee)$, which contains a canonical SL_2.
3. Get $G_X^\vee \times \mathrm{SL}_2 \to G^\vee$.

3.2. The 2 dual groups.

3.2.1. *Weak Spherical roots.*

Proposition 3.2. $(\Xi, \Sigma, \Xi^\vee, \Sigma^\vee)$ *is a based root datum.*

Definition. $G_X^\vee$ is defined to be the connected reductive group over $\mathbb{C}$ with based root datum $(\Xi, \Sigma, \Xi^\vee, \Sigma^\vee)$.

3.2.2. *Associated roots.*

Proposition 3.3. *If* $\sigma \in \Sigma \backslash \Phi$*, then there exists a unique set of* $\{\gamma_1, \gamma_2\} \subseteq \Phi^+$ *such that*

(1) $\sigma = \gamma_1 + \gamma_2$*;*
(2) γ_1, γ_2 *are strongly orthogonal, i.e.,* $\mathbb{Q}\gamma_1 + \mathbb{Q}\gamma_2 \cap \Phi = \{\pm\gamma_1, \pm\gamma_2\}$*;*
(3) $\gamma_1^\vee - \gamma_2^\vee = \sigma_1^\vee - \sigma_2^\vee$ *for some* $\sigma_1, \sigma_2 \in S$*.*

They are called associated roots of σ*.*

Remark. This allows us to define $\Sigma^\vee$ and $\Sigma^\wedge$ more carefully:

- $\Sigma^\vee = \{\sigma^\vee \mid \sigma \in \Sigma\}$ where

$$\sigma^\vee = \begin{cases} \sigma^\vee & \text{if } \sigma \in \Phi \\ \gamma_1^\vee + \gamma_2^\vee & \text{if } \sigma \notin \Phi. \end{cases}$$

- $\Sigma^\wedge = \bigcup_{\sigma\in\Sigma} \sigma^\wedge$ where

$$\sigma^\wedge = \begin{cases} \{\sigma^\vee\} & \text{if } \sigma \in \Phi \\ \{\gamma_1^\vee, \gamma_2^\vee\} & \text{if } \sigma \notin \Phi. \end{cases}$$

Definition. $G_X^\wedge$ is the unique subgroup of $G^\vee$ containing T with $\Sigma^\wedge$ as root system.

Example 3.1. In D_4, $\sigma = 2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 2e_1$ is a spherical root. $\gamma_1 = e_1 - e_4$, $\gamma_2 = e_1 + e_4$.

3.3. **A map $G_X^\vee \to G_X^\wedge$.** Recall that the set $\sigma^\wedge$ for $\sigma \in \Sigma$ form a partition of $\Sigma^\wedge$ into subsets of size at most 2. So there exists a unique $s : \Sigma^\wedge \to \Sigma^\wedge$ involution such that orbits are $\sigma^\wedge$.

Lemma 3.1. *With notation as above:*

(1) *s is a folding, i.e., for all* $\alpha, \beta \in S$*, we have*
  - $\langle \alpha, s(\alpha)^\vee \rangle = 0$ *whenever* $\alpha \neq s(\alpha)$*, and*
  - $\langle \alpha - s(\alpha), \beta^\vee + s(\beta)^\vee \rangle = 0$.

(2) *Every folding is a disjoint union of*
  - *a component of which the folding acts trivially;*
  - *swap 2 isomorphic components;*
  - *one of the four exceptional cases.*

(3) *If s is a graph automorphism, then s induces an automorphism on the adjoint group* $G_{ad}^\wedge$ *associated to* $\Sigma^\wedge$*. Then take* $H_{ad}^\wedge = (G_{ad}^\wedge)^\circ$ *corresponding to folded* $\Sigma^\wedge$ *(which is* Σ*).*

(4) There exists an isogeny

$$G_X^\vee \to (\textit{preim of } H_{ad}^\wedge)^\circ \hookrightarrow G_X^\wedge \twoheadrightarrow (G_X^\wedge)_{ad}$$

Example 3.2. Fold D_4 at α_3, α_4, then $\Sigma = \{\alpha_1, \alpha_2, \alpha_3 + \alpha_4\}$, so we get $\mathrm{Sp}_6 \to \mathrm{O}_8$.
If not, $\Sigma^\wedge \leftrightarrow B_3$, $\Sigma \leftrightarrow G_2$, we have $G_2 \to \mathrm{O}_7$.

3.4. Centralizer of $\varphi(G_X^\vee)$ in $G^\vee$.

Definition. $L_X^\vee \subseteq G^\vee$ be the Levi subgroup determined by S^p.

Let W_X be the Weyl group of $G_X^\vee$ and W the Weyl group of $G^\vee$. There is a unique map $W_X \to W$ defined by

$$s_\sigma \mapsto \begin{cases} s_\sigma & \text{if } \sigma \in \Sigma \cap \Phi \\ s_{\gamma_1} s_{\gamma_2} & \text{if } \sigma \notin \Phi. \end{cases}$$

Proposition 3.4. *$(L_X^\vee)^{W_X} \subseteq L_X^\vee$ looks like one of:*

- $1 \subseteq \mathbb{G}_m$
- $H \subseteq H$ *simple*
- $\mathrm{SL}_2 \subseteq \mathrm{SL}_2^n$
- $\cdots$

Theorem 3.1. *There is an adaption (i.e., facts through the map in Lemma 3.1(4))*

$$\varphi : G_X^\vee \to G^\vee$$

such that $\varphi(G_X^\vee)$ and L_X^n (a finite index subgroup of $(L_X)^{W_X}$) commute. In this case, φ is called very adapted

Proposition 3.5. *Let $\psi : \mathrm{SL}_2 \to L_X^\vee$ be the principal SL_2 and $\varphi : G_X^\vee \to G^\vee$ adapted homomorphism, then $\varphi \otimes \psi : G_X^\vee \times \mathrm{SL}_2 \to G^\vee$ is a group homomorphism iff φ is very adapted.*

3.5. **Functoriality.** If we replace $X = (\Xi, \Sigma, S^p)$ by $X_0 = (\Xi, \Sigma_0, S^p)$ with $\Sigma_0 \subseteq \Sigma$, then this is again a weak spherical datum with $G_{X_0}^\vee$ a Levi of $G_X^\vee$. This again corresponds to boundary degeneration of X_{Σ_0}.

4. Local Geometry and Asymptotics - Alex

4.1. **Review of notations.** k a finite extension of $\mathbb{Q}_p$, $\mathbb{G}$ a connected split reductive group over k. Fix a Borel $\mathbb{B} \subseteq \mathbb{G}$ with unipotent radical $\mathbb{U}$ and maximal torus $\mathbb{A} = \mathbb{B}/\mathbb{U}$. Take spherical variety $\mathbb{X} = \mathbb{H}\backslash\mathbb{G}$ that is quasi-affine. Let $\mathbb{X}^\circ$ be the open $\mathbb{B}$-orbit. Let $\mathbb{A}_\mathbb{X}$ be the quotient of $\mathbb{B}$ acting faithfully on $\mathbb{X}/\mathbb{U}$. Let $X(\mathbb{X})$ be the weights of $\mathbb{B}$ occuring in $k(\mathbb{X})$ with dual $X(\mathbb{X})^* = X(\mathbb{A}_X)$. $V \subseteq X(\mathbb{X})^*$ the cone coming from $\mathbb{G}$-stable valuations. $(-V)^\vee = \{x \in X(\mathbb{X}) \otimes \mathbb{R} \mid \langle x, v\rangle \leq 0, \forall v \in V\}$ is a strongly convex cone, with the set of minimal elements Σ_X of $X(\mathbb{X})$ along each generating ray.

There is a bijection

$$\{\text{open } \mathbb{G}\text{-embedding } \mathbb{X} \hookrightarrow \bar{\mathbb{X}}\} \leftrightarrow \{\text{colored fans}\}.$$

Definition. $\mathbb{X} \hookrightarrow \bar{\mathbb{X}}$ is *toroidal* if no $\mathbb{G}$-orbit in $\bar{\mathbb{X}}$ is contained in a color.

In particular, we have

$$\{\text{toroidal } \mathbb{G}\text{-embedding } \mathbb{X} \hookrightarrow \bar{\mathbb{X}}\} \leftrightarrow \{\text{fans}\}.$$

Let $\mathbb{P}$ be the stabilizer of $\mathbb{X}^\circ$ and $\mathbb{U}_\mathbb{P}$ the unipotent radical. Pick $x_0 \in \mathbb{X}^\circ$, defines $\mathbb{A}_\mathbb{X} \hookrightarrow \mathbb{X}^\circ$ by $a \mapsto x_0 \cdot a$ for $\bar{\mathbb{X}}$ toroidal. $\bar{\mathbb{X}}_B = \bar{\mathbb{X}}\backslash\{\text{colors}\}$ and $\mathbb{Y}$ the colsure of $\mathbb{A}_X$ in $\bar{\mathbb{X}}_B$.

Theorem 4.1 (Brion, Luna, Vust)**.** *The map $\mathbb{Y} \times \mathbb{U}_\mathbb{P} \to \bar{\mathbb{X}}_B$, $(y, u) \mapsto yu$ is an isomorphism.*

Remark. There are several consequences of Theorem 4.1.

(1) $\mathbb{X}$ is smooth iff every cone C in the fan F is generated by a subset of a basis for $X(\mathbb{X})^*$.
(2) $\mathbb{X}$ is complete iff $\operatorname{supp}(F) = V$.
(3) $\mathbb{G}$-orbits in $\bar{\mathbb{X}}$ are in bijection wtih cones in F.

4.2. **Boundary degeneration.** Let $\mathbb{Z} \subseteq \bar{\mathbb{X}}$ be a $\mathbb{G}$-orbit in a complete, smooth, toroidal embedding of $\mathbb{X}$, let $\Theta \subseteq \Sigma_X$ be set of elements of Σ_X orthogonal to the convex cone corresponding to $\mathbb{Z}$. Consider $N_{\mathbb{Z}}\bar{\mathbb{X}}$ has a $\mathbb{G}$-action which makes $N_{\mathbb{Z}}\bar{\mathbb{X}}$ has a $\mathbb{G}$-spherical variety. Let $\mathbb{X}_\Theta$ be the open $\mathbb{G}$-orbit.

Proposition 4.1. *With the notation above:*

(1) $\mathbb{A}_{\mathbb{X}} \cong \mathbb{A}_{\mathbb{X}_\Theta}$.
(2) $\mathbb{P}(\mathbb{X}_\Theta) = \mathbb{P}(\mathbb{X})$.
(3) $\Sigma_{\mathbb{X}_\Theta} = \Theta$.
(4) $\mathbb{X}_\Theta^\circ \cong \mathbb{X}^\circ$.
(5) Let $\mathbb{A}_{\mathbb{X},\Theta} = \operatorname{Aut}_{\mathbb{G}}(\mathbb{X}_\Theta)^\circ$*, then* $X_*(\mathbb{A}_{\mathbb{X},\Theta}) = \Theta^\perp \subseteq X(\mathbb{X})^* = X_*(\mathbb{A}_X)$. $\mathbb{A}_{\mathbb{X},\Theta}$ *acts on* $\mathbb{X}_\Theta$.
(6) $\mathbb{X}_\Theta$ *is independent of the choices of* $\mathbb{Z}$ *and* $\bar{\mathbb{X}}$ *(so it only depends on* Θ*).*

Remark. X_Θ are "parabolically induced".

Example 4.1. We have $\mathrm{PGL}(V) \times \mathrm{PGL}(V)$ acting on $\mathrm{PGL}(V)$ by left and right multiplication. Let $n = \dim(V)$.

When $n = 2$, $\mathbb{X} = \mathbb{P}(M_2(k))$ and $\mathbb{B} = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \times \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right)$ with

- $\mathbb{X}^\circ = \{\left(\begin{smallmatrix} * & * \\ * & a \end{smallmatrix}\right) \mid a \neq 0\}$
- $\mathbb{X}^\circ/\mathbb{U} \cong \mathbb{G}_m$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \frac{d^2}{\det}$.

So

- towards ∞, approach $\bar{\mathbb{X}}/\mathbb{X}$;
- towards 0, approach $d = 0$ (color).

For $n > 2$, $\bar{\mathbb{X}}$ is a "variety of complete collections", $\Theta \subseteq \Sigma_X \Leftrightarrow$ flag type, i.e., $0 = d_0 < \cdots < d_{k+1} = n$. $\mathbb{X}_\Theta$ classify triples (K, I, φ) where

- $V = K_0 \supset K_1 \supset \cdots \supseteq K_{k+1} = 0$ with $\operatorname{codim} K_i = d_i$;
- $0 = I_0 \subset \cdots \subseteq I_{k+1} = V$ with $\dim(I_i) = d_i$.
- $\varphi : \mathrm{gr}^K(V) \cong \mathrm{gr}^I(V)$ up to homothety.

$\mathbb{Z}_\Theta$ classify triples up to homothety on each graded piece.

4.3. **Local geometry.** Let $X = \mathbb{X}(k)$. Assume $\mathbb{X}$ admits $\mathbb{G}$-eigenvolume form ω. Consider $L^2(X) = L^2(X, |\omega|)$ as a G-representation.

Proposition 4.2. *For any* $\Theta \subseteq \Sigma_X$*, there exists* $\mathbb{A}_{\mathbb{X}.\Theta} \times \mathbb{G}$*-eigenvolume form on* $\mathbb{X}_\Theta$ *with same* $\mathbb{G}$*-eigencharacter, via identification in Proposition 4.1(4).*

We want to compare X and X_Θ. Let $J \subseteq G$ be a open compact subgroup.

Proposition 4.3. *There is a* J*-stable neighbourhood* U *of* $Z \subseteq \bar{X}$*, and* $U/J \to N_Z\bar{X}/J$ *which is continuous, measure preserving and bijection near* Z*.*

Proposition 4.4. *Let* $\mathbb{Z} \subseteq \bar{\mathbb{X}}$ *be an orbit closure,* U *a neighbourhood of* $Z = N_Z(\bar{X})$*,* $\mathcal{K} =$ *set of locally analytic* $\varphi : U \to \bar{X}$ *inducing isomorphism on normal bundles to* Z*, preserving* G*-orbits.*

(1) $\mathcal{K} \neq \emptyset$.
(2) $\varphi \in \mathcal{K}$ *descends to* $U'/J \to \bar{X}/J$ *for* $U' \subseteq U$.
(3) If $\mathcal{M} \subseteq \mathcal{K}$ *compact (in compact open topology), there exists* U'' *neighbourhood of* Z*,* $U'' \subseteq U$ *such that for any* $\varphi \in \mathcal{M}$ *agree as maps* $U''/J \to \bar{X}/J$

This is called the *exponential maps*, $\exp_{\Theta,J} : N_Z(\bar{X}) \to \bar{X}$.

4.4. **Asymptotics.** Assume $\mathbb{X}$ is *wavefront*, i.e., V is the image of the negative Weyl chamber under the quotient map. Also assume $Z(\mathbb{G})^\circ \twoheadrightarrow Z(\mathbb{X}) = \mathrm{Aut}_{\mathbb{G}}(\mathbb{X})^\circ$.

We want to show that, for each $\Theta \subseteq \Sigma_X$, there is a unique G-equivariant

$$e_\Theta : C_c^\infty(X_\Theta) \to C_c^\infty(X)$$

such that for any open compact $J \subseteq G$, if N_Θ small neighbourhood of Z, $f \in C_c^\infty(N_\Theta)^J$, then $e_\Theta(f) = \exp_{\Theta,J,*}(f)$.

Dually, $e_\Theta^* : C^\infty(X) \to C^\infty(X_\Theta)$ such that $f \in C^\infty(X)^J$ then

$$e_\Theta^*(f)|_{N_\Theta} = \exp_{\Theta,J}^*(f|_{N_\Theta}).$$

As a consequence, let π be a smooth representation of G, we get

$$\mathrm{Hom}(\pi, C^\infty(X)) \to \mathrm{Hom}(\pi, C^\infty(X_\Theta)), M \mapsto M_\Theta$$

called the *asymptotic map.* We then have

$$M(v)|_{N_\Theta} = M_\Theta(v)|_{N_\Theta}, v \in \pi^J, N_\Theta \text{ small enough.}$$

Theorem 4.2. *Let π be a smooth irreducible,* $\dim \mathrm{Hom}(\pi, C^\infty(X)) < \infty$.

Example 4.2. In the example of $\mathbb{X} = \mathrm{PGL}(V)$ and $\mathbb{G} = \mathrm{PGL}(V) \times \mathrm{PGL}(V)$. For $\bar{A} \in X = \mathrm{PGL}(V)(k)$, choose a lift $A \in \mathrm{GL}(V)(k)$. There exists a basis e_i, f_i, $1 \le i \le n$ for $\mathcal{O}^n$ such that $Ae_i = \lambda_i f_i$ and $|\lambda_1| > \cdots > |\lambda_n|$. For Θ corresponding to $0 = d_0 < \cdots < d_{k+1} = n$, $J \subseteq G$ open compact, $T \gg 0$, say A is Θ-*large* if $|\lambda_{d_s}/\lambda_{d_s+1}| \ge T$, $K_i = k\langle e_n, ..., e_{d_i}\rangle$ and $I_i = k\langle f_1, ..., f_{d_i}\rangle$ with φ induced by A. So we have a map

$$\{A, \Theta\text{-large}\} \to \{(K, I, \varphi)\}/J.$$

5. Bernstein Morphisms, Scattering and Plancherel Formula - Guanjie

5.1. **Introduction.**

5.1.1. *Notation and review.*

- k: p-adic field;
- G: connected split reductive group over k
  - $A \subseteq B$: maximal split torus of a Borel
  - $(\Xi(A), \Delta, \Xi(A)^*, \Delta^\vee)$: based root data
  - $G^\vee$: (complex) dual group of G
- $X = H\backslash G$ quasi-affine, wavefront (i.e. the valuation cone $\mathcal{V}$ is the image of the negative Weyl chamber) spherical variety;
  - $\Xi(X)$: the lattice of B-weights in $k(X)$
  - Δ_X: simple spherical roots
  - W_X: little Weyl group
  - $A_X = \mathrm{Hom}(\Xi(X), \mathbb{G}_m) \twoheadleftarrow A$, so $\Xi(X) = \Xi(A_X)$
  - $G_X^\vee$: the (complex) dual group of X
  - $\iota : G_X^\vee \times \mathrm{SL}_2 \to G^\vee$
  - $Z(X) = \mathrm{Aut}_G(X)^\circ = (N(H)/H)^\circ \twoheadleftarrow Z(G)^\circ$.
- X_Θ: boundary degeneration associated to $\Theta \subseteq \Delta_X$;
  - $\Xi(X_\Theta) = \Xi(X) \Rightarrow A_{X_\Theta} = A_X$
  - $\Delta_{X_\Theta} = \Theta$
  - $G_{X,\Theta}^\vee$: the Levi of $G_X^\vee$ with simple roots $\Theta^\vee$.
  - $Z(X_\Theta) = A_{X,\Theta} \subseteq A_X$: the maximal subtorus whose cocharacter group is orthogonal to Θ
  - $A_{X,\Theta}^+ = \{a \in A_{X,\Theta} \mid |\gamma(a)| \geq 1 \text{ for all } \gamma \in \Delta_X \backslash \Theta\}$
  - $A_{X,\Theta}^{+,\circ} = \{a \in A_{X,\Theta} \mid |\gamma(a)| > 1 \text{ for all } \gamma \in \Delta_X \backslash \Theta\}$
  - $e_\Theta : C_c^\infty(X_\Theta) \to C_c^\infty(X)$: G-equivariant "asymptotics" map

5.1.2. *Abstract Plancherel decomposition.* Any separable unitary representation admits an abstract Plancherel decomposition

$$L^2(X) = \int_{\hat{G}} \mathcal{H}_\pi \mu(\pi)$$

where μ is a positive measure on $\hat{G}$, $\mathcal{H}_\pi$ is π-isotypic, together with a family of measurable sections $\eta \mapsto \eta_\pi \in \mathcal{H}_\pi$.

It's proved in [Ber88] that $C_c^\infty(X)$ is a pointwise defined subspace, i.e., there is a family of morphisms

$$L_\pi : C_c^\infty(X) \to \mathcal{H}_\pi$$

for μ-almost every π such that the section $\alpha \mapsto L_\pi(f)$ represents f for every $f \in C_c^\infty(X)$. By pullback, we have a seminorms $\| \bullet \|_\pi$ on $C_c^\infty(X)$; the space $\mathcal{H}_\pi$ are completions of $C_c^\infty(X)$ (and so $C_c^\infty(X)_\pi$).

Therefore, we can describe a Plancherel decomposition of $L^2(X)$ with the following set of data:

- a positive measure μ on $\hat{G}$;
- a measurable set of G-invariant, non-zero seminorms $\|\bullet\|_\pi$ on $C_c^\infty(X)_\pi$, for μ-almost every π, so that for every $f \in C_c^\infty(X)$:

$$\|f\|^2 = \int_{\hat{G}} \|f\|_\pi^2 \mu(\pi).$$

5.1.3. *Outline.* Our goal is to develop the Plancherel decomposition of $L^2(X)$. To be more preicse, we will describe $L^2(X)$ in terms of the discrete series of X and X_Θ's. There will be four parts of the talk:

(1) First we will explain what "discrete" means and an additional condition.
(2) Then we derive the existence of Bernstein morphism $\iota : L^2(X_\Theta) \to L^2(X)$ characterized by its asymptotic properties.
(3) We analyze interactions between Bernstein morphisms and decompose $L^2(X)$ in terms of discrete series of X_Θ's using scattering theory.
(4) Finally we derive explicit formulas for the morphisms in some cases.

5.2. **Discrete series.** Viewing as a representation of $Z(X)$, $L^2(X)$ admits a direct integral decomposition

$$L^2(X) = \int_{\widehat{Z(X)}} L^2(X, \omega) d\omega$$

$$f = \int_{\widehat{Z(X)}} f_\omega d\omega.$$

where $L^2(X, \omega)$ is the completion of $C_c^\infty(X, \omega)$, the space of ω-eigenfunctions compact supported modulo $Z(X)$.

Definition. $L^2(X)_{\text{disc}}$ consists of f for which almost every f_ω belongs to the direct sum of all irreducible subrepresentations of $L^2(X, \omega)$.

A *X-discrete series representation* is a pair (π, M) where π is an irreducible smooth representation of G with unitary central character ω and $M : \pi \to C^\infty(X)$ with image in $L^2(X, \omega)$. The image of all such M, spans the *discrete spectrum* $L^2(X, \omega)_{\text{disc}}$.

Conjecture 5.1 (Discrete Series Conjecture)**.** *There exists a parabolic $P = LU$, a torus D^* of unramified characters of L, a countable collections of families of X-discrete series representations $(I_P^G(\sigma^i \otimes \omega), M_\omega^i)_{\omega \in D^*}$ (with some extra conditions) such that the norm of $L^2(X)_{disc}$ admits a decomposition*

$$\|\Phi\|^2_{disc} = \sum_i \int_{D^*_{unitary}} \|\tilde{M}_\omega^i(\Phi)\|^2 d\omega$$

where $I_P^G(\bullet)$ is the normalized parabolic induction and $\tilde{M}$ is the adjoint of M.

Remark. (1) Discrete Series Conjecture holds for *factorizable* spherical varieties (i.e., $\mathfrak{h} = (\mathfrak{h} \cap Z(\mathfrak{g})) \oplus (\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}])$). In particular, it holds for symmetric case and group case.
(2) This is NOT a direct integral decomposition, as the image of M_ω^i's could be non-orthogonal for different i's and ω's.

5.3. **Bernstein morphisms.** From now on, X and all its boundary degeneration are supposed to satisfy the Discrete Series Conjecture 5.1.

Recall that the smooth asymptotic map

$$e_\Theta : C_c^\infty(X_\Theta) \to C_c^\infty(X)$$

is characterized by its asymptotic behaviour near ∞_Θ, i.e., for J-invariant functions supported on a J-good neighbourhood, coincides with the identification of J-orbits under the exponential map.

Theorem 5.1. *For every $\Theta \subseteq \Delta_X$, there is a G-equivariant morphism $\iota_\Theta : L^2(X_\Theta) \to L^2(X)$ such that for any $a \in A_{X,\Theta}^{+;\circ}$ and $f \in C_c^\infty(X_\Theta)$,*

$$\lim_{n \to \infty} \|\iota_\Theta(a^{-n} \cdot f) - e_\Theta(a^{-n} \cdot f)\| = 0 \tag{5.1}$$

(here $a^{-n} \cdot f$ is the normalized action of a^{-n} by right translation).

Remark. One can think of ∞_Θ as the set of poles of all eigenfunctions f_λ's for $\lambda \in \Delta_X \backslash \Theta$, so the set of strictly antidominant elements $A^{+,\circ}_{X,\Theta}$ pushes a point (and so a^{-1} for a function) towards ∞_Θ.

Sketch of Proof.

- The idea is to ”complete” $e_\Theta : C_c^\infty(X_\Theta) \to C_c^\infty(X)$. However, we need some modification to make e_Θ bounded (recall that completion is the initial object in the category of complete metric spaces with a uniformly continous map to it).
- The modification is to ”project out” the subunitary part of each π component.
- $C_c^\infty(X_\Theta)_\pi$ has a generalized eigenspace decomposition
$$C_c^\infty(\bullet)_\pi = C_c^\infty(\bullet)_\pi^{<1} \oplus C_c^\infty(\bullet)_\pi^{1} \oplus C_c^\infty(\bullet)_\pi^{\nleq 1}$$
with exponents χ, respectively, satisfying $|\chi^{-1}| < 1$, $|\chi^{-1}| = 1$ and $|\chi^{-1}| \nleq 1$ on $A^{+,\circ}_{X,\Theta}$.
- Fix a Plancherel decomposition of $L^2(X)$:
$$\|f\|^2 = \int_{\hat{G}} H_\pi(f)\mu(\pi),$$
with H_π the Hermitian form on $\mathcal{H}_\pi$, the completion of $C_c^\infty(X)_\pi$. There is a corresponding Plancherel decomposition of $L^2(X_\Theta)^J$ for any open compact J:
$$\|f\|^2 = \int_{\hat{G}} (e_\Theta^* H_\pi)^S(f)\mu(\pi),$$
Denote by $\mathcal{H}_\pi^\Theta$ the corresponding completion of $C_c^\infty(X_\Theta)_\pi$. By $A_{X,\Theta}$-invariance, the norm of $\mathcal{H}_\pi^\Theta$ factors through $C_c^\infty(X_\Theta)_\pi^1$.
- Consider
$$\iota_{\Theta,\pi} : C_c^\infty(X_\Theta)_\pi \xrightarrow{\text{proj}} C_c^\infty(X_\Theta)_\pi^1 \xrightarrow{e_{\Theta,\pi}} C_c^\infty(X)_\pi.$$
The square of the norm of $\iota_{\Theta,\pi}$ is bounded by the number of distinct exponents of $A_{X,\Theta}$ on $C_c^\infty(X_\Theta)_\pi^1$. So it extends to a bounded map on Hilbert spaces:
$$\iota_{\Theta,\pi} : \mathcal{H}_\pi^\Theta \to \mathcal{H}_\pi.$$
- Lastly we integrate to obtain
$$\iota_\Theta = \int_{\hat{G}} \iota_{\Theta,\pi}.$$
The relevant issues here is measurability of $\iota_{\Theta,\pi}$ and boundedness of $\int \|\iota_{\Theta,\pi}\|^2$.
- To see the desired property (5.1) of ι_Θ, we look at the generalized eigenspace
$$C_c^\infty(X_\Theta)_\pi = C_c^\infty(\Theta)_\pi^{<1} \oplus C_c^\infty(X_\Theta)_\pi^{1} \oplus C_c^\infty(X_\Theta)_\pi^{\nleq 1}.$$
On ”< 1” component, $a^{-n} \cdot f \to 0$; on ”$= 1$” component, $\iota_{\Theta,\pi} = e_{\Theta,\pi}$; on ”$\nleq 1$” component, $e_\Theta^* H_\pi$ vanishes.

□

Proposition 5.1. *For each $\Omega \subseteq \Theta \subseteq \Delta_X$, let ι_Ω^Θ denote the Bernstein morphism $L^2(X_\Omega) \to L^2(X_\Theta)$. Then the following diagram commutes:*

$$\begin{tikzcd}
L^2(X_\Omega) \arrow[r, "\iota_\Omega^\Theta"] \arrow[dr, "\iota_\Omega"] & L^2(X_\Theta) \arrow[d, "\iota_\Theta"] \\
& L^2(X)
\end{tikzcd}$$

Proof. Follows from similar property of e_Θ. □

Corollary 5.1. *The map*

$$\bigoplus_{\Theta\subseteq\Delta_X} \iota_{\Theta,disc} : \bigoplus_{\Theta\subseteq\Delta_X} L^2(X_\Theta)_{disc} \to L^2(X) \tag{5.2}$$

is surjective.

Proof. Induction on the cardinality of Δ_X. Assume the statement to be true if we replace X by X_Θ, $\Theta \subsetneq \Delta_X$. By Proposition 5.1, the orthogonal complement $\mathcal{H}'$ of $\sum_{\Theta\subsetneq\Delta_X} \iota_\Theta(L^2(X_\Theta)_{\text{disc}})$ is orthogonal to $\iota_\Theta(L^2(X_\Theta))$ for any $\Theta \subsetneq \Delta_X$. By the definition of ι_Θ, this means

$$\mathcal{H}' = \int_{\hat{G}} \mathcal{H}'_\pi \mu(\pi)$$

where the norms for all $\mathcal{H}'_\pi$ are decaying in all directions at infinity (i.e., they have only subunitary, no unitary exponents). By the generalization of Casselman's square integrability criterion, $\mathcal{H}' \subseteq L^2(X)_{\text{disc}}$. ☐

Remark. The map ι is not injective. Moreover, the spaces $\iota_\Theta(L^2(X_\Theta)_{\text{disc}})$'s are not distinct for different Θ's. This is to be expected if one believes in Sakellaridis-Venkatesh conjecture. In terms of parameters, parameters that are not conjugate in $G^\vee_{X,\Theta}$ (or into different $G^\vee_{X,\Theta}$) maybe conjugate in $G^\vee_X$.

5.4. **Scattering theory.** The next question we can ask is: what is the kernel of the surjective map (5.2)? By duality, this is equivalent to finding the image of the injective map

$$\bigoplus_{\Theta\subseteq\Delta_X} \iota^*_{\Theta,\text{disc}} : L^2(X) \to \bigoplus_{\Theta\subseteq\Delta_X} L^2(X_\Theta)_{\text{disc}}. \tag{5.3}$$

5.4.1. *Generic injectivity.* First we need an extra assumption on X called generic injectivity:

Assumption. For every isomorphism $Z(G^\vee_{X,\Theta}) \to Z(G^\vee_{X,\Omega})$ induced by an element in $W_{G^\vee}$, there is an element of W_X that induces the same isomorphism.

Example 5.1. Let $X = \mathrm{Sp}_{2n}\backslash\mathrm{GL}_{2n}$ with dual group $G^\vee_X = \mathrm{GL}_n \hookrightarrow \mathrm{GL}_{2n} = G^\vee$, and a diagonal element $\mathrm{diag}(a_1, \cdots, a_n)$ embedded as $\mathrm{diag}(a_1, a_1, a_2, a_2, \cdots, a_n, a_n)$. For $\Theta, \Omega \subseteq \Delta_X$, $Z(G^\vee_{X,\Theta})$ are identified with $Z(G^\vee_{\tilde\Theta})$, where $G^\vee_{\tilde\Theta}$ is the standard Levi corresponding to $\tilde\Theta = (2\cdot\Theta) \cup \{\text{odd roots}\}$. Any isomorphism between $Z(G^\vee_{\tilde\Theta})$ and $Z(G^\vee_{\tilde\Omega})$ is induced by $w \in W(\tilde\Omega, \tilde\Theta) = W_X(\Omega, \Theta)$. Therefore X satisfies the injectivity assumption.

5.4.2. *Scattering morphisms.* . For $\Theta, \Omega \subseteq \Delta_X$, consider

$$\iota^*_{\Omega,\text{disc}}\iota_{\Theta,\text{disc}} : L^2(X_\Theta)_{\text{disc}} \xrightarrow{\iota_{\Theta,\text{disc}}} L^2(X) \xrightarrow{\iota^*_{\Omega,\text{disc}}} L^2(X_\Omega)_{\text{disc}}.$$

Aside. X_Θ is parabolic induced: let $L_{\tilde\Theta}$ and $P_{\tilde\Theta}$ denote the standard Levi with simple roots $\tilde\Theta = \Delta(X) \cup \mathrm{supp}(\Theta)$, and the corresponding standard parabolic. There exists a spherical variety $X^L_{\tilde\Theta}$ of $L_{\tilde\Theta}$ such that $X_\Theta \cong X^L_{\tilde\Theta} \times^{P^-_{\tilde\Theta}} G$.
The "action on the left" induces a natural map $Z(L_{\tilde\Theta})^\circ \to A_{X,\Theta}$ which is surjective over $\bar{k}$. The image (of k-points) is denoted by $A'_{X,\Theta}$.

Theorem 5.2 (Scattering theorem)**.** *Assume X is a wavefront spherical variety satisfies generic injectivity, and all degenerations (including X) satisfies Discrete Series Conjecture 5.1.*

(1) *The map* $\iota^*_{\Omega,disc}\iota_{\Theta,disc} : L^2(X_\Theta)_{disc} \to L^2(X_\Omega)_{disc}$ *has a unique decomposition*

$$\iota^*_\Omega \iota_\Theta = \sum_{w\in W_X(\Omega,\Theta)} S_w$$

where

$$S_w : L^2(X_\Theta)_{disc} \to L^2(X_\Omega)_{disc}$$

is an $A'_{X,\Theta} \times G$*-equivariant isometry and* $A'_{X,\Theta}$ *acts on* $L^2(X_\Omega)$ *via* $A'_{X,\Theta} \hookrightarrow A_{X,\Theta} \xrightarrow{w} A_{X,\Omega}$.

(2) $\iota_\Omega \circ S_w = \iota_\Theta$, *i.e., the following diagram commutes*

$$\begin{tikzcd}
L^2(X_\Theta)_{disc} \arrow[r, "S_w"] \arrow[dr, "\iota_{\Theta,disc}"] & L^2(X_\Omega)_{disc} \arrow[d, "\iota_{\Omega,disc}"] \\
& L^2(X)
\end{tikzcd}$$

(3) $S_{w'} \circ S_w = S_{w'w}$.

(4) *The map*

$$\bigoplus_{\Theta\subseteq\Delta_X} \frac{\iota^*_{\Theta,disc}}{\sqrt{c(\Theta)}} : L^2(X) \to \bigoplus_{\Theta\subseteq\Delta_X} L^2(X_\Theta)_{disc}$$

(where $c(\Theta) = \sum_\Omega |W(\Omega,\Theta)|$*) is an isometric isomorphism onto the subspace consisting of* $(f_\Theta)_\Theta \in \bigoplus_\Theta L^2(X_\Theta)_{disc}$ *satisfying*

$$S_w f_\Theta = f_\Omega \text{ for every } w \in W_X(\Theta,\Omega).$$

Corollary 5.2. *For* $|\Theta| \neq |\Omega|$, $\iota_\Theta(L^2(X_\Theta)_{disc}) \perp \iota_\Omega(L^2(X)_{disc})$.

Remark. Corollary 5.2 shows that $L^2(X)$ has a direct sum decomposition

$$L^2(X) = \bigoplus_i L^2(X)$$

where $L^2(X)_i$ is the iamge of

$$\bigoplus_{|\Theta|=i} L^2(\Theta)_{\mathrm{disc}}.$$

This can be thought of as decomposition by "degree of continuity".

Proof. It suffices to show that $\iota^*_\Omega\iota_\Theta = 0$, which immediately follows from Theorem 5.2(1). ☐

5.5. **Explicit formula.** We only discussed the existence and propeties of Bernstein morphisms and scattering morphisms. Now, we want to obtain an explicit formula.

5.5.1. *Horocycles.* Recall that the associated parabolic $P(X)$ is $\{g \in G \mid X^\circ \cdot g = X^\circ\}$.

Definition. For $\Theta \subseteq \Delta_X$, the space of Θ-*horocycles* X^h_Θ is the G-variety classifying pairs $(Q,\mathcal{O})$ where Q is a parabolic in the conjugacy class of $P_{\tilde{\Theta}}$ and $\mathcal{O}$ an orbit in U_Q contained in the open Q-orbit on X.

Proposition 5.2. *If* X *is wavefront, there is an natural identification*

$$X^h_\Theta \cong (X_\Theta)^h_\Theta$$

compatible with identification of open Borel orbits.

That being said, even though X and X_Θ are quite different, their spaces of Θ-horocycles are naturally identified. Therefore we may also write X_Θ^h for $(X_\Theta)_\Theta^h$.

An explicit formulas for Bernstein morphisms and scattering morphisms can be obtained from this identification by a suitable transform of functions on X and X_Θ to functions on X_Θ^h, by integrating over the horocycles. Recall that ι_Θ and S_w are determined by the e_Θ, so one just need to describe e_Θ.

5.5.2. *Radon transform.* Defined by integration over generic $U_{\tilde{\Theta}}$-orbits, we have a well-defined "Radon transformatio"

$$R_\Theta : C_c^\infty(X) \xrightarrow{R_\Theta} C^\infty(X_\Theta^h, \delta_\Theta)$$

where $C^\infty(X_\Theta, \delta_\Theta)$ is the smooth sections of a line bundle over X_Θ^h where the stabilizer of each point on X_Θ^h acts on the fiber via the modular character δ_Θ. The map e_Θ^* fits into the diagram

(5.4)

$$\begin{tikzcd}
C_c^\infty(X) \arrow[rr, "e_\Theta^*"] \arrow[dr, "R_\Theta"'] & & C^\infty(X_\Theta) \arrow[dl, "R_\Theta"] \\
& C^\infty(X_\Theta^h, \delta_\Theta) &
\end{tikzcd}$$

5.5.3. *Example: the group case.* $X = H = \mathrm{SL}(V)$ equipped with $G = H \times H$ action (H acting on the right on the 2-dimensional k-vector space V).

- Set $V^* = V - \{0\}$, and $V^h = \{\text{affine lines missing the origin}\}$.
- $X_\emptyset^h$ can be identified $\mathbb{G}_m \backslash V^* \times V^h$, sending $(B \times B', Y)$ to $(v, Y(v))$ where B stabilizes kv and Y is thought of as a class of endomorphism of V (so $Y(v) = U_{B'} \cdot y(v)$ for some $y \in Y$ is an affine lines whose gradient is fixed by B').
- The boundary degeneration $X_\emptyset$ can be identified with the subspace of $\mathrm{End}(V)$ of rank 1 elements. Therefore one has identification $X_\phi = \mathbb{G}_m^{\mathrm{diag}} \backslash (V^h \times V^*)$, mapping a pair (Y, v) to the unique rank 1 endomorphism sending Y to v.
- Fix a symplectic form ω on V, we can identify V^* with V^h by $u \mapsto \{v \in V \mid \omega(v, u) = 1\}$. This allow us to identify
$$X_\phi = \mathbb{G}_m^{\mathrm{diag}} \backslash (V^h \times V^*) = \mathbb{G}_m^{\mathrm{adiag}} \backslash (V^* \times V^*).$$
- Set $\tilde{V} = \{(u, v) \in V^* \times V^* \mid \omega(u, v) = 1\}$. We have two correspondences
$$\tilde{V} \times V^* \xrightarrow[s_1]{t_1} V^* \times V^* \text{ and } V^* \times \tilde{V} \xrightarrow[s_2]{t_2} V^* \times V^*.$$
Take
$$R_i = t_{i,!} s_i^* : \mathcal{S}(V^* \times V^*) \to C^\infty(V^* \times V^*)$$
where s^* is the pullback of functions under s, and $t_!$ is integration over fibers of t (which are $\mathbb{G}_a$-torsors). The map
$$R = R_1 \boxtimes R_2 : \mathcal{S}(V^* \times V^*) \to C^\infty(V^* \times V^*)$$
descends to $\mathbb{G}_m^{\mathrm{adiag}} \backslash V^* \times V^*$, i.e., to a map
$$R : \mathcal{S}(H_\emptyset) \to C^\infty(H_\emptyset).$$
Explicity,
$$Rf(x, y) = \int_{\omega(x, x')=1} \int_{\omega(y', y)=1} f(x', y') dx' dy'.$$
Notice that, in particular, for $A_{X,\emptyset} = A_X = A_H = \mathbb{G}_m$ (acting on $V^* \times V^*$ by multiplication on both factors) eigenfunction f with eigencharacter χ (L^2-normailzed action)
$$R(t \cdot f) = t^{-1} \cdot (Rf).$$
In particular this shows that R_χ (or $R_{1,\chi}$, $R_{2,\chi}$): $C^\infty(H_\emptyset, \chi^{-1}) \to C^\infty(H_\emptyset, \chi)$.

- Now let's describe S_w^* for the nontrivial $w \in W_X = W_H$. It admits an decomposition with respect to the action of $A_H = \mathbb{G}_m$
$$S_w^* = \int_{\widehat{A_H}} S_{w,\chi}^* d\chi$$
where $S_{w,\chi}^* : C^\infty(H_\emptyset, {}^w\chi = \chi^{-1}) \to C^\infty(H_\emptyset, \chi)$. It's shown in [DHS14, Proposition 15.2] that
$$S_{w,\chi} = R_{1,\chi} \boxtimes R_{2,\chi^{-1}}^{-1} = \gamma(\chi^{-1}, 0, \psi)\gamma(\chi, 0, \psi^{-1})R_\chi.$$
where γ is the gamma factor
$$\gamma(\chi, 1-s, \psi) = \frac{\epsilon(\chi, 1-s, \psi)L(\chi^{-1}, s)}{L(\chi, 1-s)},$$
and ψ is a fixed nontrivial unitary additive character so that (k, dx) is self-dual.

5.5.4. *Explict Plancherel formula.* Recall that X_Θ is induced, so
$$L^2(X_\Theta) = I_{\tilde{\Theta}^-}(L^2(X_\Theta^L)).$$
Say if we have a Plancherel decomposition
$$L^2(X_\Theta^L) = \int_{\sigma \in \widehat{L_{\tilde{\Theta}}}} \mathcal{I}_\sigma \mu(\sigma).$$
By induction we then have
$$L^2(X) = \int_{\sigma \in \widehat{L_{\tilde{\Theta}}}} \mathcal{H}_\sigma \mu(\sigma). \tag{5.5}$$
where $\mathcal{H}_\sigma$ is the induction of $\mathcal{I}_\sigma$. If we have a formula
$$\iota_\Theta f(x) = \int_{\sigma \in \widehat{L_{\tilde{\Theta}}}} (\iota_\Theta^\sigma f)(x)\mu(\sigma), f \in L^2(X_\Theta)_{\text{disc}},$$
we get a Plancherel decomposition for $\iota_\Theta(L^2(X_\Theta)_{\text{disc}})$. It turns out that this is known with some assumption.

Theorem 5.3. *Assume $f \in L_2(X_\Theta)_{disc}$ has a pointwise decomposition*
$$f(x) = \int_{\widehat{L_{\tilde{\Theta}}}} f^\sigma(x)\mu_{disc}(\sigma), \quad f^\sigma \in C^\infty(X_\Theta)^\sigma,$$
then
$$\iota_\Theta f(x) = \int_{\widehat{L_{\tilde{\Theta}}}} E_{\Theta,\sigma} f^\sigma(x)\mu(\sigma)$$
where $E_{\Theta,\sigma}$, the Eisenstein integral, is the dual to the composition
$$C_c^\infty(X) \xrightarrow{R} C^\infty(X_\Theta^h, \delta_\Theta) \twoheadrightarrow C^\infty(X_\Theta^h, \delta)_\sigma \xrightarrow{RT_\Theta^{-1}} C_c^\infty(X_\Theta)_\sigma$$
where

- *$C_c^\infty(X_\Theta)_\sigma = C_c^\infty(X_\Theta^L)_\sigma$ is a quotient of the $I_{\Theta^-}(\sigma)$-coinvariant of $C_c^\infty(X_\Theta)$ (again, here we use the fact that X_Θ is parabolically induced).*
- *$C^\infty(X_\Theta)^\sigma$ is the dual of $C_c^\infty(X_\Theta)_\sigma$, viewed as a subspace of $C^\infty(X_\Theta)$.*
- *RT_Θ^{-1} is the inverse of RT_Θ induced by standard intertwining operator*
$$T_\Theta : I_\Theta(\sigma^-) \to \int_{U_\Theta} f(u\bullet)du \in I_\Theta(\sigma)'$$
$$RT_\Theta : C_c^\infty(X_\Theta)_\sigma \twoheadleftarrow C_c^\infty(X_\Theta)_{I_{\Theta^-}(\sigma)} \xrightarrow{R_\Theta T_\Theta} C^\infty(X_\Theta^h, \delta_\Theta)_{I_\Theta(\sigma)'} = C^\infty(C_\Theta^h, \delta_\Theta)_\sigma.$$

Under this assumption, the norm on $\iota_\Theta(L^2(X_\Theta)_{disc})$ *admits a Plancherel decomposition*

$$\|f\|^2_\Theta = \frac{1}{|W_X(\Theta,\Theta)|}\int_{\widehat{L_\Theta}} \|E^*_{\Theta,\sigma}f\|^2\mu_{disc}(\sigma)$$

where the measure and norms here are the discrete part of the Plancherel decomposition of (5.5).

6. The Sakellaridis-Venkatesh Conjecture - Kartik

6.1. **Waldspurger's theorem.** Let f be a modular form of weight 2. We can write f in terms of Mellin transform

$$\int_{-\infty}^{\infty} f(iy)y^{s-1}dy = (2\pi)^{-s}\Gamma(s)L(f,s).$$

By replacing s by k, we have

$$\int_{-\infty}^{\infty} f(iy)y^{k-1}dy = (2\pi)^{-s}\Gamma(s)L(k,s).$$

Equivalently, one can write

$$\int_{\mathbb{A}^\times\backslash T(\mathbb{A})} \varphi_f = +\cdot L(\pi_f,\frac{1}{2}).$$

where $T = E^\times$, E imaginary quadratic.

In the case of real quadratic, Shitani shows that these periods occur in Fourier coefficients of modular forms of half integer weight.

In the case of imaginary quadratic, $\chi : E^\times \to \mathbb{C}^\times$ of infinity type $(l', -l')$ $(l' \geq l)$

$$\int_{\mathbb{A}^\times\backslash E(\mathbb{A})^\times} \varphi_f$$

period is the sum of values of f at CM points.

6.1.1. *Tunnell-Saito & Waldspurger.* Let $G = \mathrm{GL}_2(\mathbb{A})$ and $H = T = \mathbb{A}_E^\times \to G$. A period is an element in $\mathrm{Hom}_H(\pi\otimes\chi,\mathbb{C})$.

- $\dim_{H(k_v)}(\pi_v\otimes\chi_v,\mathbb{C}) \leq 1$.
- If π_v is not a discrete series representation, then dim $= 1$; if π_v is discrete series representation, then dim $= 0$.
- Let B_v be a quaternion algebra ramified at v, π'_v on B_v and $E_v^\times \to B_v^\times$, then
$$\dim \mathrm{Hom}_{H(K_v)}(\pi_v\otimes\chi_v,\mathbb{C}) + \dim \mathrm{Hom}_{H(k_v)}(\pi'_v\otimes\chi_v,\mathbb{C}) = 1$$
and
$$\epsilon(\frac{1}{2},\pi_v,\chi_v) = (-1)^i\chi_v(-1)\eta_v(-1) = \pm 1$$
where $i = 0$ in the first case, and $i = 1$ in the second, η quadratic character of $E_v^\times/\mathbb{Q}_v^\times$.

Choose $\mathcal{L} \in \mathrm{Hom}_H(\pi,\chi) = \otimes_v \mathrm{Hom}_{H(k_v)}(\pi_v\otimes\chi_v,\mathbb{C})$, we can write $\mathcal{L}(x) = \prod_v \mathcal{L}_v(x)$ for $\mathcal{L}_v \in \mathrm{Hom}_{H(k_v)}(\pi_v\otimes\chi_v,\mathbb{C})$. But there are two issues: convergence and choice of $\mathcal{L}_v$.

So instead we can look at

$$\alpha_v \in \mathrm{Hom}_H(\pi\otimes\chi,\mathbb{C})\otimes\mathrm{Hom}_H(\tilde{\pi}\otimes\bar{\chi},\mathbb{C}).$$

The natural pairing

$$\langle -,-\rangle : \pi_v \times \tilde{\pi}_v \to \mathbb{C}$$

gives

$$\alpha_v(f_v,\tilde{f}_v) = \int_{Z(K_v)\backslash H(k_v)} \langle \pi_v(h)f_v,\tilde{f}_v\rangle\chi_v(h)dh, \quad f_v\in\pi_v, \tilde{f}_v\in\tilde{\pi}_v.$$

For $a, b \in H(k_v)$, we have

$$\alpha_v(\pi_v(a)f_v, \pi_v(b)\tilde{f}_v) = \chi^{-1}(a)\chi(b)\alpha_v(f_v, \tilde{f}_v).$$

So we get

$$\alpha_v : (\pi_v \otimes \chi_v) \otimes (\tilde{\pi}_v \otimes \tilde{\chi}_v) \to \mathbb{C}.$$

6.1.2. *Key computation.* When everything in sight is unramified,

$$\frac{\alpha_v(f_v, \tilde{f}_v)}{\langle f_v, \tilde{f}_v \rangle} = \frac{\zeta_{k_v}(2)L(\frac{1}{2}, \pi_{v,E_v}, \chi_v)}{L(1, \pi_v, \mathrm{ad})L(1, \eta_v)}$$

and

$$W(f_v)\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} = \frac{\chi_1(t_1\tilde{w}) - \chi_2(t_2\tilde{w})}{\chi_1(\tilde{w} - \chi_2(\tilde{w}))}\left|\frac{t_1}{t_2}\right|^{\frac{1}{2}}\mathbb{I}_{\mathcal{O}}(\frac{t_1}{t_2}).$$

Theorem 6.1.

$$\frac{(\int f \cdot \chi)(\int \tilde{f} \cdot \bar{\chi})}{\langle f, \tilde{f} \rangle} = \frac{1}{\textit{powers of 2}} \cdot \frac{\zeta_k(2)L(\frac{1}{2}, \pi_f, \chi)}{L(1, \pi, ad)L(1, \eta)} \prod_v \frac{\alpha_v(f_v, \tilde{f}_v)}{\langle f_v, \tilde{f}_v \rangle \mathcal{L}_v(\pi_v, \chi_v)}$$

Remark. Two ways to approach Theorem 6.1:

(1) Theta correspondence;
(2) Relative trace formula.

6.2. **Gan-Gross-Prasad and Ichino-Ikede.** Now $E^\times \hookrightarrow \mathbb{GL}_2$ is replaced by either one of the following

- $\mathrm{SO}(2) \hookrightarrow \mathrm{SO}(3)$
- $U(1) \to U(2)$.

Consider $W \hookrightarrow V$ with $\dim V = \dim W + 1$ so we have

$$H = U(W) \hookrightarrow U(W) \times U(V) = G.$$

Let $X = H\backslash G$, $\pi = \pi_1 \boxtimes \pi_2$ be a tempered representation of G. We can ask whether π is H-distinguished.

We need to look at G and all its pure inner form at once.

- $U(V) \leftrightarrow U(V')$, $\dim V' = \dim V$;
- $\mathrm{SO}(V) \leftrightarrow \mathrm{SO}(V')$. $\dim V' = \dim V$ and $\mathrm{disc}(V') = \mathrm{disc}(V)$.
- V and V' are relevant: $V' = (W') \oplus (W')^\perp$, $V = W \oplus W^\perp$, $W^\perp \cong (W')^\perp$.

Conjecture 6.1 (Gan-Gross-Prasad)**.** *There exists unique (G', π'), G' is a relavant pure inner form of G such that*

$$\mathrm{Hom}_{H'}(\pi', \mathbb{C}) \neq 0 \textit{ (and dim = 1).}$$

There is a recipe to identify (G', π').

Conjecture 6.2 (Ichino-Ikeda)**.**

$$\frac{(\int f \cdot \chi)(\int \tilde{f} \cdot \bar{\chi})}{\langle f, \tilde{f} \rangle} = \frac{1}{\textit{powers of 2}} \cdot \frac{(\textit{prod. of Artin L-functions})L(\frac{1}{2}, \pi_f, \chi)}{L(1, \pi, ad)L(1, \eta)} \prod_v \frac{\alpha_v(f_v, \tilde{f}_v)}{\langle f_v, \tilde{f}_v \rangle \mathcal{L}_v(\pi_v, \chi_v)}$$

6.3. **Sakellaridis-Venkatesh conjecture.** Let $X = H\backslash G$ (not in GGP setting). The problem is that $\int\langle -,-\rangle$ converges badly. The Hermitian forms used here come from Plancherel formula.

Let's first find some substitute for α_v. Last time we have

$$L^2(X) = \int_{\hat{G}} H_\pi \mu(\pi).$$

In the GGP case, this form agrees with α_v. The first question is: what is the support of μ?

Example 6.1. For $\mathrm{SO}(m,2)$, one can associate $\mathrm{SO}(m-1,2)$ and $\mathrm{SO}(m,1)$. Kudla-Millson showed that the distinguished representations are lifts from SL_2 or $\widetilde{\mathrm{SL}}_2$. There is a Arthur packet of size 2 $\{\pi^{\mathrm{ds}}, \pi^{\mathrm{nt}}\}$.

Let $\mathcal{L}_k$ be either the Weil group when k is archimedean, or the Weil-Deligne group when k is non-archimedean.

Definition. We say $\psi : \mathcal{L}_k \times \mathrm{SL}_2 \to G^\vee$ is *X-distinguished* if ψ factors through the canonical map $\iota : G_X^\vee \times \mathrm{SL}_2 \to G^\vee$.

Conjecture 6.3 (Weak form)**.** *Support of Plancherel measure is contained in the set of representations π such that π is contained in an A-packet associated to an X-distinguished parameter ψ.*

Conjecture 6.4.

$$L^2(X) = \int_{\psi:\mathcal{L}_k\times \mathrm{SL}_2 \to \hat{G}/G_X^\vee\text{-}conj} \mathcal{H}_\psi \mu(\psi).$$

Moreover, if we take into consideration all inner forms:

$$\bigoplus_\alpha L^2(X^\alpha) = \int_{[\psi]} \mathcal{H}_\psi \mu(\psi), G = \prod_\beta G^\beta.$$

where

- $\mathcal{H}_\psi \neq 0$ *for almost every all* ψ.
- $\mathcal{H}_\psi$ *is multiplicity free*
- $\mathcal{H}_\psi$ *is the direct sum of some* π*'s.*

7. Unramified Spectrum of Spehrical Varieties - Jialiang

7.1. **Notation and convention.**

- k a p-adic field, $\mathcal{O}_k$ the ring of integers, $\bar{\omega}$ the uniformizer, the residue field $\mathcal{O}_k/\bar{\omega}\mathcal{O}_k \cong F_q, q = p^l$, $\bar{k}$ the algebraic closure of k;
- $\mathbb{G}$ be a split reductive group over k, $\mathbb{B}$ a Borel subgroup of $\mathbb{G}$, $\mathbb{A}$ the maximal split torus of $\mathbb{B}$;
  - $G = \mathbb{G}(k), B = \mathbb{B}(k), A = \mathbb{A}(k), K = \mathbb{G}(\mathcal{O}_k)$, $\mathcal{H}(G,K)$ the spherical Hecke algebra of G
  - $(\Xi(A), \Delta, \Xi(A)^*, \Delta^\vee)$: based root data for G
  - $G^\vee$: (complex) dual group of G, $A^\vee = \Xi(A^\vee)^* \otimes \mathbb{C}$ the split torus of $G^\vee$, an identification of $\Xi(A) \cong \Xi(A^\vee)^*$ and $\Xi(A)^* \cong \Xi(A^\vee)$
  - W the Weyl group of G and $G^\vee$
- $\mathbb{X} = \mathbb{G}/\mathbb{H}$ a homogenious spherical variety, $\mathring{\mathbb{X}}$ the $\mathbb{B}$ open orbit. We assume that $\mathbb{X}$ is quasi-affine.
  - $\bar{k}(\mathbb{X})$: the rational function on $\mathbb{X}$, $\Xi(\mathbb{X})$: the lattice of $\mathbb{B}$-weights in $\bar{k}(\mathbb{X})$, $\Xi(\mathbb{X}) \subset \Xi(A)$,
  - Δ_X: simple spherical roots
  - W_X: little Weyl group
  - $A_X^\vee = \Xi(\mathbb{X}) \otimes \mathbb{C}$ and $A_X^\vee \hookrightarrow A^\vee$
  - $G_X^\vee$: the (complex) dual group of X

- $X = \mathbb{X}(k), \mathring{X} = \mathring{\mathbb{X}}(k)$
- $\mathbb{P} = \{p \in \mathbb{G} | \mathring{\mathbb{X}} \cdot g = \mathring{\mathbb{X}}\}$ and $W_{\mathbb{P}}$ the associated Weyl group.

7.2. **Preliminary: Satake isomorphism.** The Sakake isomorphism

$$\mathcal{S} : \mathcal{H}(G, K) \cong \mathcal{R}(G^\vee)$$

where $\mathcal{R}(\widehat{G})$ is the Grothedick ring of the finite dimensional representation of $G^\vee$. If $\pi \in \mathrm{Irr}(G)_{\mathrm{un}}$, then $\dim(\pi^K)$ is one-dimensional and defines a $\mathcal{H}(G, K)$-character χ_π, which gives rise to a character of $\mathcal{R}(\widehat{G})$ via the Satake isomorphism $\mathcal{S}$. There is a perfect pairing between $A^\vee /\!\!/ W$ and $\mathcal{R}(G^\vee)$ given by the trace function

$$(t, \sigma) \mapsto \mathrm{Tr}_\sigma(t) \quad t \in A^\vee, \sigma \in \mathcal{R}(G^\vee).$$

We get the Satake paramater for π:

$$\mathrm{Irr}(G)_{\mathrm{un}} \cong A^\vee /\!\!/ W$$
$$\pi \mapsto s(\pi)$$

by requiring $\chi_\pi(f) = \mathrm{Tr}_{\mathcal{S}(f)}(s(\pi))$ for $f \in \mathcal{H}(G, K)$.

Next we make this map more explicit. Some facts

- Each unramified principle series $I_B^G(\chi)$ has a unique unramified subquotient
- Each umramified representation is the unique unramified subquotient of an unramified principle series $I_B^G(\chi)$ for an unramified character χ of A;
- $I_B^G(\chi)$ and $I_B^G(\chi')$ has common unramified subquotient if and only of $\chi = \chi'^w$ for some $w \in W$.

The Satake isomorphism for torus: we have an exact sequence

$$1 \longrightarrow \mathbb{A}(\mathcal{O}_k) \longrightarrow A \xrightarrow{r} \Xi(A)^* \longrightarrow 1$$

with $\gamma(t)$ the cocharacter satisfying

$$\langle \gamma(t), \chi \rangle = \mathrm{ord}(\chi(t))$$

for all $\chi \in \Xi(A)$. The splitting of this sequence is given by

$$\Xi(A)^* \cong A/\mathbb{A}(\mathcal{O}_k)$$
$$\lambda^\vee \mapsto \lambda^\vee(\bar{\omega}).$$

Then we have the Satake isomorphism for torus

$$H(A, \mathbb{A}(\mathcal{O}_k)) \cong \mathbb{C}[\Xi(A)^*] \cong \mathbb{C}[\Xi(A^\vee)] \cong \mathcal{R}(A^\vee).$$

We get the Satake parameter for unramified characters of A:

$$\mathrm{Ind}(A)_{\mathrm{un}} \cong A^\vee$$
$$\chi \mapsto s(\chi)$$

by requiring $\chi(\lambda^\vee(\bar{\omega})) = \lambda(s(\chi))$ for $\lambda^\vee \in \Xi(A)^*$. Here $\lambda^\vee \mapsto \lambda$ is our fixed isomorphism beween $\Xi(A)^*$ and $\Xi(A^\vee)$.

Lemma 7.1. *If π is the unique unramified subquotient of an unramified principle series $I_B^G(\chi)$ for an unramified character χ of A, then $s(\pi) = s(\chi)$.*

7.3. **The questions.** The action of G on X induce an action of G on $C^\infty(X)$ and $C_c^\infty(X)$. For $\pi \in \mathrm{Irr}(G)_{\mathrm{un}}$, an unramified representation, we can ask

1. The unramified spectrum of X: When is $\dim \mathrm{Hom}_G(\pi, C^\infty(X)) \neq 0$?
2. The spherical function on X: We know that $\dim(\pi^K) = 1$. Choose $v_\pi \in \Pi^K$. If $\phi \in \mathrm{Hom}_G(\pi, C^\infty(X)) \neq 0$, we want to study the properties of the special function $\phi(v_\pi)$.

7.4. **Unramified spectrum of spherical variety.** If $\mathrm{Hom}_G(\pi, C^\infty(X)) \neq 0$, by duality

$$\mathrm{Hom}_G(C_c^\infty(X), \pi^\vee) \neq 0.$$

We can embed $\pi^\vee$ in $\mathrm{Ind}_B^G(\chi)$ for some unramified character of A, so

$$\mathrm{Hom}_G(C_c^\infty(X), \mathrm{Ind}_B^G(\chi)) \neq 0.$$

We first study when $\mathrm{Hom}_G(C_c^\infty(X), \mathrm{Ind}_B^G(\chi)) \neq 0$. Using Frobenius reciprocity, we have

$$\mathrm{Hom}_G(C_c^\infty(X), \mathrm{Ind}_B^G(\chi)) \cong \mathrm{Hom}_B(C_c^\infty(X), \chi\delta^{-\frac{1}{2}}).$$

The problem transferred to find the $\chi^{-1}\delta^{\frac{1}{2}}$ eigen-distribution of B on $D(X)$ (the distributions on X).

We first state a lemma on describing the rational $\mathbb{B}$-weights in $\bar{k}(\mathbb{X})$.

Lemma 7.2. *The rational $\mathbb{B}$-weights in $\bar{k}(\mathbb{X})$ equals to it's rational $\mathbb{B}$-weights in $\bar{k}(\mathring{\mathbb{X}})$.*

Remark. We will see that this principle also holds for the analytic world: the B eigen-distributions on X is determined by the B eigen-distributions on $\mathring{X}$ up to the Weyl group action.

Theorem 7.1 (Yiannis).

1. *If $\mathrm{Hom}_G(C_c^\infty(X), \mathrm{Ind}_B^G(\chi)) \neq 0$, then $\chi \in {}^w(\delta^{-\frac{1}{2}}A_X^\vee)$ for some $w \in [W : W_P]$, where δ is the modular character of $\hat{G}, \hat{B}, \hat{T}$.*
2. *If $\chi \in \delta^{-\frac{1}{2}}A_X^\vee$, then $\chi\delta^{\frac{1}{2}}$ is a character of P and almost all unramified irreducible π admitting a non-zero morphism from $C_c^\infty(X)$ are isomorphic to $\mathrm{Ind}_P^G(\chi\delta^{\frac{1}{2}})$.*
3. *For almost all such π, we have*

$$\dim \mathrm{Hom}(C_c^\infty(X), \pi) = |N_W(\delta^{-\frac{1}{2}}\hat{A}_X)/W_X| \cdot |H^1(k, \mathbb{A}_X)|,$$

where $\mathbb{A}_X$ is the image in $\mathbb{B}/\mathbb{U}$ of the stabilizer of a generic point on $\mathbb{X}$ and $N_W(\delta^{-\frac{1}{2}}\hat{A}_X)$ consists of the elements in W which stabilizes $\delta^{-\frac{1}{2}}\hat{A}_X$.

Remark.

1. We give an example to illustrate the modular characters appearing in here. $G = \mathrm{PGL}_2$ and $H = \mathrm{PGL}_2$. So we have $\mathrm{C}_c^\infty(X) = 1$. We know that $1 \hookrightarrow \mathrm{Ind}_B^G(\delta^{-\frac{1}{2}})$, $G_X^\vee$ is trivial. $\mathrm{Hom}_G(C_c^\infty(X), \mathrm{Ind}_B^G(\chi)) \neq 0$ iff $\chi = \delta^{-\frac{1}{2}}$.
2. Here we see that multiplicity may fails for two reasons
    - $N_W(\delta^{-\frac{1}{2}}\hat{A}_X)/W_X \neq 1$. We give an example: $G = \mathrm{Sp}_4$ and $H = \mathbb{G}_m \times \mathrm{SL}_2 \hookrightarrow \mathrm{SL}_2 \times \mathrm{SL}_2 \hookrightarrow \mathrm{Sp}_4$. In this case, $G_X^\vee = G$, $A_X^\vee = A^\vee$, but $W_X \subseteq W$ is of index 2. Therefore $|N_W(\delta^{-\frac{1}{2}}\hat{A}_X)/W_X| = 2$,
    - $|H^1(k, \mathbb{A}_X)| > 1$. The factor $|H^1(k, \mathbb{A}_X)|$ equals to the number of B-orbits on $\mathring{X}$. We give an example: $\mathbb{G} = \mathrm{SL}_2$ and $\mathbb{H} = \mathbb{A}$, a maximal torus. We have $\mathbb{A}_X = \{\pm 1\}$, and

$$H^1(k, \mathbb{A}_X) \cong k^\times/(k^\times)^2.$$

7.4.1. *Idea of the proof.* The theorem is proofed in four steps

- Study the $\chi^{-1}\delta^{\frac{1}{2}}$ eigen-distribution of B on each orbit: Mackey theorem
- Compare these eigen-distribution of B on different orbit: Knop action.
- Study how to extend these eigen-distributions of B on a open orbits to it's closure.
- Study the compose of this eigen-distribution with the intertwining operator.

7.5. **Spherical functions on spherical variety.** Next we study the spherical functions on spherical variety. Instead of given the general theorem, we give a example on Whittaker spherical variety and Casselman-Shalika formula.

7.5.1. *Whittaker spherical variety.* Let $X = (N,\psi)\backslash G$ with $B = AN$, $\psi : N \to \mathbb{C}^\times$ unramified generic. In this case,

$$C^\infty(X) = \mathrm{Ind}_N^G(\psi) = \{f : G \to \mathbb{C} \mid f(ng) = \psi(n)f(g)\}.$$

1. For any $\pi \in \mathrm{Irr}(G)_{\mathrm{un}}$, $\dim \mathrm{Hom}_G(\pi, C^\infty(X)) = 1 = \mathrm{Hom}_N(\pi,\psi)$.
2. For $\pi \in \mathrm{Irr}(G)_{\mathrm{un}}$ we define $W_\pi : G \to \mathbb{C}$ called *Whittaker function* by
$$\begin{cases} W_\pi(ng) = \psi(n)W_\pi(g) \\ W_\pi(gk) = W_\pi(g) \\ (\phi * W_\pi)(g) = \chi_\pi(\phi)W_\pi, \phi \in \mathcal{H}(G,K) \\ W_\pi(1) = 1 \end{cases}$$
where $(\phi * W_\pi)(g) = \int_G \phi(h)W_\pi(gh^{-1})dh$ is the convolution. It turns out that $W_\pi = \phi(v_\pi)$ (up to a choice of v_π).

By Iwasawa decomposition $G = ATK$, we know that W_π is uniquely determined by its value on $A/A(\mathcal{O}_F) \cong \Xi(A)^*$.

If we define $\Xi^*(A)^+ = \{\lambda^\vee \in \Xi^*(A) \mid \langle \lambda^\vee, \alpha\rangle > 0, \forall \alpha \in \Delta^+\}$ to be the set of strictly dominant elements. Then W_π is supported on $\Xi^*(A)^+$ under the isomorphism above.

Example 7.1. In SL_2, we have

$$\begin{pmatrix} \varpi^{\lambda_1} & \\ & \varpi^{\lambda_2} \end{pmatrix} \cdot \begin{pmatrix} 1 & q \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \varpi^{\lambda_1-\lambda_2} q \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \varpi^{\lambda_1} & \\ & \varpi^{\lambda_2} \end{pmatrix}$$

So being strictly dominant simply means that $\lambda_1 > \lambda_2$.

Theorem 7.2 (Shitani-Casselman-Shalika)**.** *We have*

$$W_\pi(\lambda^\vee(\varpi)) = q^{-\langle \rho, \lambda^\vee\rangle} \underbrace{\prod_{\alpha^\vee > 0} \frac{1}{1 - q^{\langle \chi, \alpha^\vee\rangle}} \sum_{w \in W} \left(\prod_{\substack{\alpha^\vee > 0 \\ w\alpha^\vee < 0}} q^{-\langle \chi, \alpha^\vee \rangle} \right) q^{-\langle \chi^w, \lambda^\vee\rangle}}_{(*)}$$

where

- $\lambda^\vee \in \Xi(A)^*$
- $\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$
- χ *is associated to* π *in the sense that* π *occurs in* $\mathrm{Ind}_B^G(\chi)$*. Here we view* $\chi \in \Xi(A) \otimes \mathbb{C}$*.*
- $\chi(\lambda^\vee(\varpi)) = q^{-\langle \chi, \lambda^\vee\rangle}$

Remark. By Weyl character formula for finite dimensional representation of complex Lie groups, one can deduce that $(*)$ is equal to $\mathrm{tr}_{V_\lambda^\vee}(s(\pi))$ where $V_{\lambda^\vee}$ is a finite dimensional representation of $\hat{G}$ with highest weight $\lambda^\vee$, and $s(\pi)$ is the Satake parameter of π, so that $\lambda^\vee(s(\pi)) = q^{-\langle \chi, \lambda^\vee\rangle}$. Therefore, we may rewrite Shitani-Casselman-Shalika as

$$W_\pi(\lambda^\vee(\varpi)) = q^{-\langle \rho, \lambda^\vee\rangle} \mathrm{tr}_{V_\lambda^\vee}(s(\pi)).$$

This relates values of spherical Whittaker function on G to the values of finite dimensional complex representations of it's dual group.

We may also normalize the Spherical Whittaker function in another way so that L-functions will appear: Define

$$\Omega_\chi : I_B^G(\chi) \longrightarrow C^\infty(X)$$
$$f \mapsto \Omega_\chi(f) = \int_N f(nx)\psi^{-1}(n)dn.$$

The integral only converges for χ lies in some areas in $\Xi(A)\otimes\mathbb{C}$ and one can show that this can be extended to the whole $\Xi(A)\otimes\mathbb{C}$ with no-poles. We choose the unramified vector $f_\chi^0\in I_B^G(\chi)$ such that $f(tnk)=\delta^{-\frac{1}{2}}\chi(t)$. Then we know that $\Omega_\chi(f_\chi^0)\in(\mathrm{Ind}_N^G(\psi))^K$ and is proportional to W_π, where π is the unique unramified subquotient of $\mathrm{Ind}_B^G(\chi)$. To compute the proportional constant, we just need to evaluate $\Omega_\chi(f_\chi^0)$ at 1.

Theorem 7.3. *We have*

$$\Omega_\chi(f_\chi^0)(1)=\prod_{\alpha>0}(1-q^{-1}\chi(a_\alpha)).$$

Remark. $\prod_{\alpha>0}(1-q^{-1}\chi(a_\alpha))$ is roughly half of the value of the adjoint L-function at 1.

Assume G is split over local field F with a maximum compact subgroup K. Let $X=H\backslash G$ be a spherical variety. Assume that there is only one $G(F)$-orbit of $X(F)$. For $\pi\in\mathrm{Irr}(G)_{\mathrm{un}}$, an unramified representation, we can ask

1. When is $\dim\mathrm{Hom}_G(\pi,C^\infty(X))\neq 0$?
2. Say $\dim(\pi^K)=1$, $\phi\in\mathrm{Hom}_G(\pi.C^\infty(X))\neq 0$ and $v_\pi\in\Pi^K$, we want to study the properties of $\phi(v_\pi)$

8. Periods and Local Multiplicity of Strongly Tempered Spherical Varieties - Chen Wan

8.1. Notation and assumption.

- F - local field of characteristic 0
- G - connected reductive group over F
- H - spherical subgroup of G

In this talk, we assume that

1. G is quasi-split and unramified and $B(F)\backslash G(F)/H(F)$ has a unique open orbit
2. (G,H) is the Whittaake induction of a reductive spherical pair (G,H_0), i.e., there exists a parabolic $P=MN$, and a generic character $\xi:N(F)\to\mathbb{C}^\times$ such that $G_0=M$ and $H_0=M_\xi$. In this case, $H=H_0\ltimes N$, and we extend ξ to H by setting it to be trivial on the reductive part H_0.

Example 8.1. (1) Whittake model.

(2) Shalika model. $G=\mathrm{GL}_{2n}$ and

$$H=\begin{pmatrix}h&\\&h\end{pmatrix}\begin{pmatrix}I_n&\\&I_n\end{pmatrix},\quad \xi\begin{pmatrix}I_n&x\\&I_n\end{pmatrix}=\psi(\mathrm{Tr}(x)).$$

(3) (G_0,H_0) does not have type N spherical root.

(4) (G_0,H_0) is stongly tempered, i.e., all the tempered matrix coefficients of $G_0(F)$ is integrable on $H_0(F)/Z_{G,H}(F)$ where $Z_{G_0,H}$ is the intersection of H with the center Z_{G_0} of G_0.

Example 8.2. $(\mathrm{SO}_{n+2k+1}\times\mathrm{SO}_n,\mathrm{SO}_n\ltimes N)$, $(\mathrm{GL}_2^3,\mathrm{GL}_2)$, $(U_{n+2k+1}\times U_n,U_n\ltimes N)$.

Aside. Let's briefly recall the type of spherical roots. In PGL_2, they corresponde to different spherical subgroups:

- Type G: $H = \mathrm{PGL}_2$;
- Type T: $H = \mathrm{GL}_1$;
- Type N: $H = \mathrm{O}(2)$;
- Type U: $H = U = \begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}$ with trivial character;
- Type (U, ψ): $H = U$ with character ψ.

In the strongly tempered case, only type T, N and (U, ψ) will show up (the first two are reductive, the third is not).

The L-group of $X = (G, H, \xi)$ is

$$^{L}G = {}^{L}(G/Z_{G,H}).$$

8.2. **Result and motivation.** Let π be a irreducible tempered representation of $G(F)$ with trivial central character on $Z_{G,H}(F)$. ϕ be a matrix coefficient of π, define the *local relative character* to be

$$I_H(\phi) = I_{H,\xi}(\phi) = \int_{H(F)/Z_{G,H}(F)} \phi_\pi(h)\xi(h)^{-1}dh.$$

The goal is to compute $I_H(\phi)$ in unramified case.

Theorem 8.1.

$$I_H(\phi) = \frac{\Delta_G(1)}{\Delta_{H_0/Z_{G,H}}(1)} \cdot \frac{L(\frac{1}{2}, \pi, \rho_X)}{L(1, \pi, \mathrm{Ad})}$$

where

- *Δ_G is the L-function of the dual $M^\vee$ to the motive M associated to G introduced by Gross;*
- *ρ_X is self-dual representation of ${}^{L}G/Z_{G,H} = {}^{L}G_X$ of symplectic type.*

There are two reasons why this is important:

1. we need it to form the global Ichino-Ikeda type conjecture, and
2. the computation is closely related to the L-values and local multiplicity of (G, H, ξ).

More explicitly, let K be a number field with $\mathbb{A} = \mathbb{A}_K$, ϕ a cusp form of $G(\mathbb{A})$. We define the *period integral*

$$\mathcal{P}_H(\phi) = \int_{Z_{G,H}(\mathbb{A})H(k)\backslash H(\mathbb{A})} \phi(h)\xi(h)^{-1}dh.$$

Conjecture 8.1 (Sakellaridis-Venkatesh)**.**

$$|\mathcal{P}_H(\phi)|^2 = \mathit{constant} \cdot \prod_v I_{H_v,\xi_v}(\phi_v).$$

But to make sense this conjecture, we need Theorem 8.1 to show that the infinite product is convergent in the sense of analytic continuation (becuase for all but finitely unramified places, the factor $I_{H_v,\xi_v}(\phi_v)$ is given by partial L-function).

8.3. **Strategy of computation.** Let $B = TN$ and the unique open Borel orbit of $B(F)$ is $B(F)\eta H(F)$. Moreover, $H(F) \cap \eta^{-1}B(F)\eta = 1$, i.e., the stabilizer of the open orbit is in the center.

We want to compute $I(\phi_\theta) = \int_{H(F)} \phi_\theta(h)dh$ where ϕ_θ is the unramified matrix coefficient of $I_B^G(\theta)$ with $\phi_\theta(1) = 1$, where θ is a unitary unramified character of $T(F)$.

8.3.1. *Step 1 - reduction.* Let $f_\theta \in I_B^G(\theta)$ be teh unramified vector with $f_\theta(1) = 1$. Therefore we have

$$\phi_\theta(g) = \int_K f_\theta(kg)dk.$$

Therefore,

$$I(\phi_\theta) = \int_K \int_{H(F)} f_\theta(kh)dhdk,$$

so it follows from Fubini's theorem and absolute convergence, we have

$$\int_{H(F)} f_\theta(gh)dh$$

convergent for $g \in B(F)\eta H(F)$.

Define $\mathcal{Y}_\theta$ to be the function suppported on $B(F)\eta H(F)$ thatt is left $(B(F), \theta^{-1}\delta_B^{-\frac{1}{2}})$-invariant and right $H(F)$-invariant with $\mathcal{Y}_\theta(\eta) = 1$.

For $g \in B(F)\eta H(F)$,

$$\int_{H(F)} f_\theta(gh)dh = \int_{H(F)} f_\theta(\eta h)dhh \cdot \mathcal{Y}_{\theta^{-1}}(g).$$

As a consequence

$$I(\phi_\theta) = \int_K \mathcal{Y}_{\theta^{-1}}(k)dk \cdot \int_{H(F)} f_\theta(\eta h)dh. \tag{8.1}$$

So it suffices to compute the two terms on the right hand side of (8.1).

8.3.2. *Step 2 - reduction to one term.*

Lemma 8.1. *For $f \in C_c^\infty(G(F))$, we have*

$$\int_{G(F)} f(g)dg = \frac{\Delta_G(1)}{\Delta_H(1)}\zeta(1)^{-\operatorname{rk}(G)} \int_{H(F)} \int_{B(F)} f(bhg)dgdh$$

where $\operatorname{rk}(G)$ *is the F-rank of G.*

Using Lemma 8.1, we have

$$\begin{aligned}\int_{G(F)} 1_K(g)\mathcal{Y}_\theta(g)dg &= \frac{\Delta_G(1)}{\Delta_H(1)}\zeta(1)^{-\operatorname{rk}(G)} \int_{H(F)} \underbrace{\int_{B(F)} 1_K(b\eta h)\theta^{-1}\delta^{-\frac{1}{2}}(b)db}_{=f_\theta(\eta h)}\,dh \\ &= \frac{\Delta_G(1)}{\Delta_H(1)}\zeta(1)^{-\operatorname{rk}(G)} \int_{H(F)} f_\theta(gh)dh.\end{aligned}$$

So (8.1) becomes

$$I(\phi_\theta) = \frac{\Delta_G(1)}{\Delta_H(1)}\zeta(1)^{-\operatorname{rk}(G)} \int_K \mathcal{Y}_{\theta^{-1}}(k)dk \cdot \int_K \mathcal{Y}_\theta(k)dk. \tag{8.2}$$

Proposition 8.1. Φ^+ *be the set of positive roots of G. There exists a decomposition of the weights of a representation ρ_X of ${}^LG_X$, denoted by $\Theta = \Theta^+ \cup \Theta^-$, called weighted virtual colors, such that*

$$\int_K \mathcal{Y}_\Theta(k)dk = \frac{\Delta_G(1)}{\Delta_H(1)}\zeta(1)^{-\operatorname{rk}(G)}\beta(\theta)$$

where

$$\beta(\theta) = \frac{\prod_{\alpha\in\Phi^+}(1 - q^{-1}e^{\alpha^\vee})}{\prod_{\gamma\in\Theta^+}(1 - q^{-\frac{1}{2}}e^{\gamma^\vee})}(\theta).$$

Moreover,

$$\prod_{\gamma\in\Theta^+}(1-q^{-\frac{1}{2}}(\theta^{-1}))=\prod_{\gamma\in\Theta^-}(1-q^{-\frac{1}{2}}e^{\gamma^\vee}(\theta)).$$

Remark. Sakellaridis-Wang proved this identity for a W_X-invariant subset of weights of $\hat{G}$. In the strongly tempered case, one can show that Θ is the set of weights of a symplectic representation ρ_X

It follows from Proposition 8.1 that (8.2) becomes

$$\begin{aligned}
I(\phi_\theta) &= \frac{\Delta_G(1)}{\Delta_H(1)}\zeta(1)^{-\operatorname{rk}(G)}\int_K \mathcal{Y}_{\theta^{-1}}(k)dk\cdot\int_K\mathcal{Y}_\theta(k)dk \\
&= \frac{\Delta_H(1)}{\Delta_G(1)}\zeta(1)^{\operatorname{rk}(G)}\cdot(\frac{\Delta_G(1)}{\Delta_H(1)}\cdot\zeta(1)^{-\operatorname{rk}(G)})^2\cdot\frac{\prod_{\alpha\in\Phi^+}(1-q^{-1}e^{\alpha^\vee})}{\prod_{\gamma\in\Theta^+}(1-q^{-\frac{1}{2}}e^{\gamma^\vee})}(\theta) \qquad (8.3)\\
&= \frac{\Delta_G(1)}{\Delta_H(1)}\cdot\frac{L(\frac{1}{2},\pi,\rho_X)}{L(1,\pi,\mathrm{Ad})}.
\end{aligned}$$

This proves Theorem 8.1.

8.3.3. *Computation of* Θ^+. There exists $\beta_\alpha^\vee$ such that $-\beta_\alpha^\vee+\alpha^\vee\in\Theta^+$ and

$$\chi_\theta(x_{-\alpha}(a^{-1})b)=\theta(e^{\beta_\alpha^\vee}(1+a^{-1}))\cdot|1+a^{-1}|^{-\frac{1}{2}}.$$

Here $\chi_\alpha F\to N(F)$, $a\in u_{\alpha(a)}$. So we have

$$\begin{aligned}
I_\alpha(\theta) &:= \mathrm{Vol}(I)^{-1}\int_{G(F)}\mathcal{Y}_\Theta(xy)(\Phi_1(x)+\Phi_{w_\alpha}(x))dx \\
&= \frac{1-q^{-1}e^{\alpha^\vee}(\theta)}{(1-q^{-\frac{1}{2}}e^{\beta_\alpha^\vee}(\theta))(1-q^{-\frac{1}{2}}e^{\alpha^\vee-\beta_\alpha^\vee}(\theta))}
\end{aligned}$$

Θ^+ is the subset of Θ such that $\Theta^+-w_\alpha\Theta^+=\{\beta_\alpha^\vee,\alpha^\vee-\beta_\alpha^\vee\}$ for all simple root α.

8.4. **Local multiplicity.** Let π be an irreducible representation of $G(F)$ whose central character is trivial on $Z_{G,H}(F)$. We want to study the *local multiplicity*

$$m(\pi)=\dim\mathrm{Hom}_{H(F)}(\pi,\xi).$$

For example, in the Gan-Gross-Pr asad case

Theorem 8.2 (Waldspurger, Beuzart-Plessis)**.** *In the Gan-Gross-Prasad case, we have strongly multiplicity one on every tempered Vogan L-packet, i.e., for each tempered local L-packet, there exists a unique element with non-zero multiplicity, and the multiplicity is equal to 1. Moreover, one can detect the distinguished one using* ϵ*-dichotomy.*

Sakellaridis-Venkatesh conjectured that, instead of considering one pair (G,H), one needs to consider all its pure inner forms (G_α,H_α), $\alpha\in H^1(F,H/Z_{G,H})$.

8.4.1. *Result.* The following table include all cases of (G,H,ρ_X) of strongly tempered models we consider.

G	H	ρ_X
$\mathrm{GL}_4 \times \mathrm{GL}_2$	$\mathrm{GL}_2 \times \mathrm{GL}_2$	$(\wedge^2 \otimes \mathrm{Std}_2) \oplus \mathrm{Std}_4 \oplus \mathrm{Std}_4^\vee$
$\mathrm{GU}_4 \times \mathrm{GU}_2$	$\mathrm{G}(U_2 \times U_2)$	$(\wedge^2 \otimes \mathrm{Std}_2) \oplus \mathrm{Std}_4 \oplus \mathrm{Std}_4^\vee$
$\mathrm{GSp}_6 \times \mathrm{GSp}_4$	$\mathrm{G}(\mathrm{Sp}_4 \times \mathrm{Sp}_2)$	$\mathrm{Spin}_7 \otimes \mathrm{Spin}_5$
GL_6	$\mathrm{GL}_2 \ltimes N$	$\wedge^3$
GU_6	$\mathrm{GU}_2 \ltimes N$	$\wedge^3$
GSp_{10}	$\mathrm{GL}_2 \ltimes N$	Spin_{11}
$\mathrm{GSp}_6 \times \mathrm{GL}_2$	$\mathrm{GL}_2 \ltimes N$	$\mathrm{Spin}_7 \otimes \mathrm{Std}_2$
$\mathrm{GSO}_8 \times \mathrm{GL}_2$	$\mathrm{GL}_2 \ltimes N$	$\mathrm{HSpin}_8 \otimes \mathrm{Std}_2$
$\mathrm{GSO}_{12} \times \mathrm{GL}_2$	$\mathrm{GL}_2 \ltimes N$	HSpin_{12}
E_7	$\mathrm{PGL}_2 \ltimes N$	ω_7

Remark. For all the models except the second one, the pure inner form are in 1-1 correspondence with quaternion algebra

Theorem 8.3. *For the cases shown above,*

1. *assume the local Langlands conjecture and the multiplicity formula holds for (G, H), we have strong multiplicity one on every tempered L-packet.*
2. *If F is p-adic or $\mathbb{C}$, the multiplicity formula holds for all the models except $(E_7, \mathrm{PGL}_2 \ltimes N)$. If $F = \mathbb{R}$, multiplicity formula holds for model 1 to 4.*

8.4.2. *Multiplicity formula.* $m(\pi) = m_{\mathrm{geom}}(\pi)$ is defined via the Harish-Chandra character Θ_π of π.

Example 8.3. In model 4-10,

$$m_{\mathrm{geom}}(\pi) = \iota_{\Theta_\pi}(1) + \int_{\Gamma_{\text{ell. reg.}}(H_0)} \iota_{\Theta_\pi}(t)dt$$

where ι_{Θ_π} is the regular term of Θ_π.

8.4.3. *ϵ-dichotomy.* Let $\phi : W_F \to {}^L G$ be a tempered Langlands parameter, Z_ϕ the centralizer of $\mathrm{Im}(\phi)$ in $\hat{G}$, $S_\phi = Z_\phi / Z_\phi^\circ$. It follows from LLC that there should be a bijection between $\Pi_\phi = \bigcup_{\alpha \in H^1(F,G)} \Pi_\phi(G_\alpha)$ and $\mathrm{Irr}(S_\phi)$.

We now want to define a quadratic character of S_ϕ.

For $s \in S_\phi$, we showed that there exists $s' \in sZ_\phi^\circ$ such that s' belongs to an elliptic extended endoscopic datum $(G', s', {}^L\eta)$ and ϕ factors through ${}^L G'$.

Remark. s' is not unique.

Let $V_{s',-}$ be the -1-eigenspace of $\rho_X(s')$, and so $\mathrm{Im}(\phi)$ stabilizes $V_{s',-}$. So we have

$$\phi_{s',\rho_X} : W_F' \to \mathrm{GL}(V_{s',-}).$$

Definition. $\omega_H(s) = \epsilon(\frac{1}{2}, \phi_{s',\rho_X}) \in \{\pm 1\}$.

Conjecture 8.2 (ϵ-dichotomy)**.** *ω_H is well defined (independent of the choice of s') and it's a character of S_ϕ. ω_H corresponds to the unique distinguished element in the packet.*

Conjecture 8.3 (Weaker ϵ-dichotomy)**.** *For all the models except 2, the unique distinguished element belongs to $\Pi_\phi(G)$ iff $\epsilon(\frac{1}{2}, \Pi_\phi, \rho_X) = 1$.*

Theorem 8.4. *Assume LLC and multiplicity formula.*

1. *Assume Conjecture 8.3 holds for all models smaller than (G, H), then Conjecture 8.2 holds for all tempered packet of (G, H) except when Π_ϕ is discrete and $|\Pi_\phi(G)| = 1$.*
2. *Assume Conjecture 8.3 holds for all models smaller or equal to (G, H), Conjecture 8.2 holds for (G, H).*

Here the order is given by

$$\begin{tikzcd}
(\mathrm{GU}_6, \mathrm{GU}_2 \ltimes N) \arrow[r] & (\mathrm{GU}_4 \times \mathrm{GU}_2, \mathrm{G}(U_2 \times U_2))
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathrm{GSp}_{10}, \mathrm{GL}_2 \ltimes N) \arrow[r] \arrow[d] & (\mathrm{GSO}_8 \times \mathrm{GL}_2, \mathrm{GL}_2 \ltimes N) \arrow[r] & (\mathrm{GL}_4 \times \mathrm{GL}_2, \mathrm{GL}_2 \times \mathrm{GL}_2) \\
(\mathrm{GSP}_6 \times \mathrm{GL}_2, \mathrm{GL}_2 \ltimes N) & (\mathrm{GSO}_{12}, \mathrm{GL}_2 \ltimes \mathrm{N}) \arrow[u] \arrow[r] & (\mathrm{GL}_6, \mathrm{GL}_2 \ltimes \mathrm{N}) \arrow[u] \\
(\mathrm{GSp}_6 \times \mathrm{GSp}_4, \mathrm{G}(\mathrm{Sp}_4 \times \mathrm{Sp}_2)) \arrow[u] \arrow[r] & (\mathrm{GSp}_4 \times \mathrm{GL}_2 \times \mathrm{GL}_2, (\mathrm{GL}_2 \times \mathrm{GL}_2)^{\Delta}) &
\end{tikzcd}$$

where → means "greater than".

Remark. All of the smaller ones are given by Levi subgroup of endoscopic subgroup. The smallest cases are all known (GGP).

9. An Overview of Local Theta Correspondence - Lukas

9.1. **Motivation.** Let $X = H\backslash G$ be a spherical variety of G, with dual group $G_X^\vee$ together with a distinguished morphism

$$\iota_X : G_X^\vee \times \mathrm{SL}_2 \to G^\vee.$$

If one is really optimistic, this gives rise to a map

$$\iota_X^* : \hat{G}_X \to \hat{G}.$$

It's part of Sakellaridis-Venkatesh conjecture that we have the following Plancherel decomposition

$$L^2(H\backslash G) = \int_{\hat{G}_X} W(\pi) \otimes \iota_X^*(\pi) d\mu(\pi)$$

where

- $W(\pi)$ is some multiplicity space, and
- $d\mu$ is the Plancherel decomposition on $\hat{G}_X$.

In some case, (G, G_X) happens to be a reductive dual pair, and the conjectures above can be verified using the machinary of theta correspondence.

9.2. **Setup.** Let k be a local field with odd residual characteristic and $(W, \langle -, - \rangle)$ a symplectic vector space over k.

Definition. The *Heisenberg group* $H(W)$ is defined by

$$H(W)(R) = \langle (w, t) \mid w \in W \otimes_k R, t \in R \rangle$$

with multiplication given by

$$(w_1, t_1) \cdot (w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \frac{1}{2}\langle w_1, w_2 \rangle).$$

One can think of $H(W)(R)$ as the central extension

$$0 \to R \to H(W)(R) \to W \otimes_k R \to 0.$$

Theorem 9.1 (Stone-von Neumann). *For each nontrivial central character $\psi : R \to \mathbb{C}^\times$, there is a unique irreducible representation ρ_ψ of $H(W)(R)$.*

Proof. Choose some polarization $W = X \oplus Y$. Take the Schwartz space $\mathcal{S}(X(R))$ and endow it with the action of $H(W)(R)$ by

- $\rho_\psi(x) f(w) = f(w + x)$, $x \in X(R)$

- $\rho_\psi(y)f(w) = \psi(\langle y, w\rangle)f(w)$, $y \in Y(R)$
- $\rho_\psi(t)f(w) = \psi(t)f(w)$, $t \in R$

□

Let's consider the map

$$\mathrm{Sp}(W)(R) \times H(W)(R) \to H(W)(R), (g, (w, t)) \mapsto (gw, t).$$

Notice that the representation $\rho_\psi(g\bullet)$ also has central cahracter ψ, so it follows from uniqueness of Theorem 9.1 and Schur's Lemma that for each $g \in \mathrm{Sp}(W)(R)$, there exists $\omega_\psi(g)$ in $\mathrm{PGL}(\rho_\psi)$.

Definition. We define the *metaplectic group* to be the fibre product

$$\begin{tikzcd}
\mathrm{Mp}(W)(R) \arrow[r, "\omega"] \arrow[d] & \mathrm{GL}(\rho_\psi) \arrow[d] \\
\mathrm{Sp}(W)(R) \arrow[r, "\omega_\psi"] & \mathrm{PGL}(\rho_\psi)
\end{tikzcd}$$

Remark. $\mathrm{Mp}(W)$ is not an algebraic group, but it is the extension

$$1 \to \mathbb{C}^\times \to \mathrm{Mp}(W) \to \mathrm{Sp}(W) \to 1.$$

Proposition 9.1. *The above short exact sequence doesn't split, and* $\mathrm{Mp}(W)$ *is independent of* ψ.

Proposition 9.2. *Let k be a nonarchimedean local fiel, and Y maximal isotropic subspace of W. Let P_Y be the stabilizer of Y, then*

$$\mathrm{Mp}(W) \times P_Y \to P_Y$$

splits (but not uniquely).

Proposition 9.3. *Let F be a global field, then the embedding* $\mathrm{Sp}(W)(F) \to \mathrm{Sp}(W)(\mathbb{A}_F)$ *lifts to*

$$\begin{tikzcd}
 & \mathrm{Mp}(W)(\mathbb{A}_F) \arrow[d] \\
\mathrm{Sp}(W)(F) \arrow[r] \arrow[ur, dashed] & \mathrm{Sp}(W)(\mathbb{A}_F).
\end{tikzcd}$$

Definition. Define

$$\begin{aligned}\Theta(\mathcal{S}(X(\mathbb{A}_F))) &\to \mathbb{C} \\ f &\mapsto \sum_{x \in X(F)} f(x)\end{aligned}$$

and

$$\begin{aligned}\Theta_f : \mathrm{Mp}(W)(\mathbb{A}_F) &\to \mathbb{C} \\ g &\mapsto \Theta(gf).\end{aligned}$$

Remark. Θ_f is invariant modulo $g \in \mathrm{Sp}(W)(F)$, so we have

$$\Theta_f : \mathrm{Sp}(W)(F)\backslash \mathrm{Mp}(W)(\mathbb{A}_F) \to \mathbb{C}.$$

9.3. Dual reductive pair.

Definition. $G_1, G_2 \subseteq \mathrm{Sp}(W)$ are dual reductive pair it they are mutual centralizer.

Example 9.1. V orthogonal vector space and W symplectic vector space, then we have reductive dual pair:

$$\mathcal{O}(V) \times \mathrm{Sp}(W) \to \mathrm{Sp}(V \otimes W).$$

Let k be a local field and $D = k$, a quadratic extension or quaternion algebra (so equipped with an involution $x \mapsto \bar{x}$).

Definition. V be a right D-module and V' the D-linear functional on V, so we have

$$\mathrm{Hom}_D(W, V) = V \otimes_D W'.$$

Definition. For $\epsilon = \pm 1$, (V, B) is a right ϵ-Hermitian D-module if

- V is a right D-module
- $B : V \times V \to D$ is a sesquilinear, ϵ-Hermitian and nondegenerate form.

Definition. For (V, B) a right ϵ-Hermitian D-module, we can define

$$G(V, B) = \{g \in \mathrm{GL}(V) \mid B(gv, gw) = B(v, w), \forall v, w \in V\}.$$

We define (V^*, B^*) to be the left ϵ-Hermitian form with $V^* = V$ with D-structure given by $av^* = (v\bar{a})^*$ and $B^*(V, W) = \overline{B(W, V)}$.

If (V, B_V) is a right ϵ_V-Hermitian module, (W, B_W) a right ϵ_W-Hermitian module, then $V \otimes_D W^*$ is a symplectic k-vector space with symplectic form

$$B(v_1 \otimes w_1^*, v_2 \otimes w_2^*) = \mathrm{Tr}(B_V(v_1, v_2) B_W(w_1, w_2))$$

if $\epsilon_V \epsilon_W = -1$. Therefore, we have a reductive dual pair

$$G(V) \times G(W) \to \mathrm{Sp}(V \otimes_D W^*).$$

We define $\hat{G}_i$ to be the cover given by

$$\begin{tikzcd} \tilde{G}_i \arrow[r] \arrow[d] & \mathrm{Mp}(W) \arrow[d] \\ G_i \arrow[r] & \mathrm{Sp}(W). \end{tikzcd}$$

Let (π, V_π) be an irreducible admissible representation of $\tilde{G}_1$ with central character ψ, define

$$\mathcal{N}(\pi) = \bigcap_{\lambda \in \mathrm{Hom}_{\tilde{G}_1}(\mathcal{S}, \pi)} \ker(\lambda) \tag{9.1}$$

$$\mathcal{S}(\pi) = \mathcal{S}/\mathcal{N}(\pi). \tag{9.2}$$

Then $\mathcal{S}(\pi)$ is a $\tilde{G}_2$-representation.

Proposition 9.4 (Howe). *There is a smooth representation $\Theta_\psi(\pi)$ of $\tilde{G}_2$ unique up to isomorphism such that*

$$\mathcal{S}(\pi) \cong \pi \otimes \Theta_\psi(\pi)$$

Theorem 9.2 (Howe Duality). *Let π be a smooth irreducible admissible representation of $\tilde{G}_1$.*

(1) $\Theta_\psi(\pi) = 0$ or an admissible representation of $\tilde{G}_2$.
(2) There exists a unique irreducible quotient $\theta_\psi(\pi)$.
(3) If $\theta_\psi(\pi_1) = \theta_\psi(\pi_2) \neq 0$, then $\pi_1 \cong \pi_2$.

Definition. $\mathrm{Howe}_\psi(\tilde{G}_1, \tilde{G}_2) = \{\pi \in \mathrm{Irr}(\tilde{G}_1) \mid \theta_\psi(\pi) \neq 0\}$.

It turns out that there is a bijection

$$\mathrm{Howe}_\psi(\tilde{G}_1, \tilde{G}_2) \leftrightarrow \mathrm{Howe}_\psi(\tilde{G}_2, \tilde{G}_1)$$

9.4. **Global picture.** For $\varphi \in \mathrm{Cusp}(G_1(F)\backslash G_1(\mathbb{A}_F))$, we can define an automorphic form on $G_2(F)\backslash \tilde{G}_2(\mathbb{A}_F)$ by

$$g_2 \mapsto \int_{G_1(F)\backslash G_1(\mathbb{A}_F)} \Theta_f(g_1, g_2)\varphi(g_1) dg_1) \in \mathcal{A}$$

10. Gan-Gomez's Approach towards Sakellaridis-Venkatesh Conjecture - Guanjie

10.1. **Introduction.**

10.1.1. *Goal.* Let k be a local field. Our goal is to show the following conjecture of Sakellaridis-Venkatesh on the local spectrum of spherical varieties $X = H\backslash G$ over k of low ranks:

$$L^2(H\backslash G) \cong \int_{\hat{G}_X} W(\pi) \otimes \iota_*(\pi) d\mu(\pi) \tag{10.1}$$

where

- $\iota_* : \hat{G}_X \to \hat{G}$ is induced from $\iota : G_X^\vee \times \mathrm{SL}_2 \to G^\vee$;
- $W(\pi)$ is some finite-dimensional multiplicity space;
- $d\mu$ is the Plancherel measure on $\hat{G}_X$.

Remark. The spectral measure of $L^2(H\backslash G)$ is contained in the set of X-distinguished Arthur parameters, and the multiplicity space should be related to the number of inequivalent ways an Arthur parameter valued in $G^\vee$ can be lifted to $G_X^\vee$.

10.1.2. *Outline.*

- We will find suitable division algebra D and right D-Hermitian spaces (V, B_V), (W, B_W) so that $G = G(V)$ and $G_X = G(W)$, and so we will have reductive dual pair
  $$G(V) \times G(W) \hookrightarrow \mathrm{Sp}(V \otimes W^*).$$
- By restricting the Weil representation Π of $\mathrm{Mp}(V \otimes W^*)$ (which splits over $G(V)$ and $G(W)$ unless V is odd-dimensional quadratic space, in which case we would redefine $G(W)$ to be the induced double cover), we have L^2-theta correspondence
  $$\Pi|_{G(W)\times G(V)} = \int_{\hat{G}(W)} \pi \otimes \Theta(\pi) d\mu_\theta(\pi) \tag{10.2}$$
  where μ_θ is some measure on $\hat{G}(W)$, and $\Theta(\pi)$ is a (possibly zero, reducible) unitary representation of $G(V)$ called the *L^2-theta lift* of π.

> **Aside**. Θ is different from the *big smooth theta lift* $\Theta^\infty(\pi^\infty)$ Lukas introduced last time (which is the maximal π^∞-isotypic quotient of Π^∞). However, it can be shown that
> $$\Theta(\pi)^\infty \subseteq \theta^\infty(\pi^\infty).$$
> Since $\theta^\infty(\pi^\infty)$ (the *small theta lift*) is the maximal semisimple quotient of $\Theta^\infty(\pi^\infty)$ of finite length, $\Theta(\pi)$ is a direct sum of finitely many irreducible unitary representation of μ_θ-almost all π. In particular, if k is not 2-adic, $\Theta(\pi)$ is irreducible with
> $$\Theta(\pi)^\infty = \theta^\infty(\pi^\infty)$$
> for almost μ_θ-all π.

- We will compute the N-spectrum of Π in two ways:
  - a formal decomposition, which is related to $\Theta(\pi)$ and multiplicity space $W_\chi(\pi)$, and
  - using an explicit model, which is related to $L^2(H\backslash G)$.

  Compare the two decompositions, we will get the desired spectral decomposition modulo the descrption of the measure and the multiplicity space.
- Using mixed model, we will show that the measure is just Plancherel measure as desired.
- Using Bessel-Plancherel Theorem, we will give a description of the multiplicity space in terms of the Plancherel decomposition of the Whittaker variety $(N, \chi\backslash G_X)$.

10.2. **Decomposition of N-spectrum.**

10.2.1. *Siegel parabolic.* Assume $W = E \oplus F$ is a complete polarization, where E, F are complementary totally isotropic subspaces of W. In this way, one can identify the dual of E with F^* through B_W.

Definition. (1) For $A \in \mathrm{End}_D(E)$, define $A^* \in \mathrm{End}_D(F)$ by

$$B_W(e, A^*f) = B_W(Ae, f), \quad \forall e \in E, f \in F.$$

(2) For $T \in \mathrm{Hom}_D(F, E)$, define $T^* \in \mathrm{Hom}_D(F, E)$ by

$$B_W(f_1, T^*f_2) = \epsilon_W B_W(Tf_1, f_2), \quad \forall f_1, f_2 \in F.$$

(3) We define the *Siegal parabolic subgroup* of $G(W)$ to be

$$P = \{g \in G(W) \mid gE = E\}.$$

One can verify immediately that P has a Levi decomposition $P = MN$ where

$$M = \left\{ \begin{pmatrix} A & \\ & (A^*)^{-1} \end{pmatrix} \middle| A \in \mathrm{GL}(E) \right\} \cong \mathrm{GL}(E)$$
$$N = \left\{ \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix} \middle| X \in \mathrm{Hom}_D(F, E), X^* = -\epsilon_W X \right\} = \mathrm{Hom}_D(F, E)_{-\epsilon_W}.$$

10.2.2. *Character of N.* Fix an nontrivial character $\psi : k \to \mathbb{C}^\times$, we have an group isomorphism

$$\mathrm{Hom}_D(E, F)_{-\epsilon_W} \to \hat{N}$$
$$Y \mapsto \left(\chi_Y : \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix} \mapsto \chi(\mathrm{Tr}_{k,F}(YX)) \right).$$

The adjoint action of $M \cong \mathrm{GL}(E)$ on $\hat{N} = \mathrm{Hom}_D(E, F)_{-\epsilon_W}$ can be described by

$$A \cdot Y = (A^*)^{-1} Y A^{-1}, \quad \forall A \in \mathrm{GL}(E), Y \in \mathrm{Hom}_D(E, F)_{-\epsilon_W}.$$

Let $\mathcal{O}_Y$ be the orbit of Y under this action. The stabilizer $M_{\chi_Y} = \mathrm{Stab}_M(\chi_Y)$ can be identified with

$$M_{\chi_Y} = \{g \in \mathrm{GL}(E) \mid Y = A^* Y A\}.$$

10.2.3. *Restriction of* Π *to* $P \times G(V)$. By Mackey theory [Ser77, Proposition 25], for a unitary representation of $G(W)$, we have

$$\pi|_P = \bigoplus_{\mathcal{O}_Y \in \Omega} \mathrm{Ind}^P_{M_{\chi_Y} N} W_{\chi_Y}(\pi) \tag{10.3}$$

where $\Omega = \{\mathcal{O}_Y \mid Y \in \mathrm{Hom}_D(E, F)_{-\epsilon_W}\}$ is the set of all M-orbits of N-characters, and $W_{\chi_Y}(\pi)$ is an $M_{\chi_Y N}$-module on which N acts by $\chi_Y(n)$. Combine with (10.2), we get

$$\Pi = \bigoplus_{\mathcal{O}_Y \in \Omega} \int_{\hat{G}(W)} \mathrm{Ind}_{M_{\chi_Y} N} W_{\chi_Y}(\pi) \otimes \Theta(\pi) d\mu_\theta(\pi). \tag{10.4}$$

10.2.4. *Schrödinger model.* The complete polarization $W = E \oplus F$ induces a complete polarization

$$V \otimes_D W^* = (V \otimes_D E^*) \oplus (V \otimes_D F^*).$$

Then the Weil representation can be realized on the space $L^2(V \otimes_D F^*) = L^2(\mathrm{Hom}_D(E, V))$. This is realization of Π is called the *Schrödinger model.* We describe the action of $P \times G(V)$ as follows.

Let $B_V^\flat : V \to (V^*)'$ be given by

$$(w^*)(B_V^\flat v) = B_V(w, v).$$

Then the action of $P \times G(V)$ on $L^2(\mathrm{Hom}_D(E,V))$ is given by

$$\begin{aligned}
\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix} \cdot \phi(T) &= \psi(\mathrm{Tr}_k(XT^* B_V^\flat T))\phi(T), && \forall X \in \mathrm{Hom}_D(F,E)_{-\epsilon_W}; \\
\begin{pmatrix} A & \\ & (A^*)^{-1} \end{pmatrix} \cdot \phi(T) &= |\det_k(A)|^{-\frac{\dim_D(V)}{2}} \phi(TA), && \forall A \in \mathrm{GL}(E); \\
g \cdot \phi(T) &= \phi(g^{-1}T), && \forall g \in G(V).
\end{aligned} \tag{10.5}$$

Let

$$\begin{aligned}
\Omega_V &= \left\{ \mathcal{O}_Y \subseteq \Omega_V \,\middle|\, \begin{array}{l} \mathcal{O}_Y \text{ is open in } \mathrm{Hom}_D(E,F)_{-\epsilon_W} \\ Y = T^* B_V^\flat T \text{ for some } T \in \mathrm{Hom}_D(E,V) \end{array} \right\} \\
\mathcal{Y}_Y &= \{ T \in \mathrm{Hom}_D(E,V) \mid T^* B_V^\flat T \in \mathcal{O}_Y \}
\end{aligned}$$

In particular we see that

$$\bigcup_{\mathcal{O}_Y \in \Omega_V} \mathcal{Y}_Y \subseteq \mathrm{Hom}_D(E,V)$$

is a dense open subset whose complement in $\mathrm{Hom}_D(E,V)$ has measure 0. Therefore,

$$L^2(\mathrm{Hom}_D(E,V)) \cong \bigoplus_{\mathcal{O}_Y \subseteq \Omega_V} L^2(\mathcal{Y}_Y), \tag{10.6}$$

and each of the direct summand is clearly $P \times G(V)$-invariant from the formulas above.

10.2.5. *$L^2(\mathcal{Y}_Y)$ as induced representation.* We will show that $L^2(\mathcal{Y}_Y)$ are equivalent to some induced representation of $P \times G(V)$.

First notice that, the "geometric" action of $P \times G(V)$ on $L^2(\mathcal{Y}_Y)$

$$\left(\begin{pmatrix} A & X \\ & (A^*)^{-1} \end{pmatrix}, g \right) \cdot T = gTA^{-1}$$

is transitive on $\mathcal{Y}_Y$. Choose $T_Y \in \mathcal{Y}_Y$ such that $T_Y^* B_V^\flat T_Y = Y$, then the stabilizer of T_Y in $P \times G(V)$ is

$$(P \times G(V))_{T_Y} = \left\{ \begin{pmatrix} A & X \\ & (A^*)^{-1} \end{pmatrix} \in P \times G(V) \,\middle|\, gT_Y = T_Y A \right\}.$$

Fix a choice of $g \in G(V)$ such that $gT_Y = T_Y A$ for some $A \in \mathrm{GL}(E)$, then

$$Y = T_Y^* B_V^\flat T_Y = T_Y^* g^* B_V^\flat g T_Y = A^* T_Y^* B_V^\flat T_Y A = A^* Y A,$$

i.e., $A \in M_{\chi_Y}$. That being said, if we define $[T] = TM_{\chi_Y} \subseteq \mathrm{Hom}_D(E,V)$ for $T \in \mathrm{Hom}_D(E,V)$, then

$$(P \times G(V))_{T_Y} \subseteq M_{\chi_y} N \times G(V)_{[T_Y]}.$$

It follows from the formulas (10.5) that

$$\begin{aligned}
L^2(\mathcal{Y}_Y) &\cong \mathrm{Ind}_{(P \times GG(V))_{T_Y}}^{P \times G(V)} \chi_Y \\
&\cong \mathrm{Ind}_{M_{\chi_Y} N \times G(V)_{[T_Y]}}^{P \times G(V)} \mathrm{Ind}_{(P \times G(V))_{T(Y)}}^{M_{\chi_Y} N \times G(V)_{[T_Y]}} \chi_Y.
\end{aligned} \tag{10.7}$$

The short exact sequence

$$1 \to 1 \times G(V)_{T_Y} \to (P \times G(V))_{T_Y} \xrightarrow{q} M_{\chi_Y} N \to 1$$

induces $G(V)_{T_Y}\backslash G(V)_{[T_Y]} \cong M_{\chi_Y}$ in the following way. For each $A \in M_{\chi_Y}$, there is a unique $g \in G(V)_{[T_Y]}/G(V)_{T_Y}$ such that $gT_Y = T_Y A$. Uniqueness is obvious. For existence, notice that the condition $T_Y^* B_V^\flat T_Y = Y$ is equivalent to the following diagram being commutative

$$\begin{tikzcd}
V \times V \arrow[r, "B_V(-,-)"] & D \\
E \times E \arrow[u, "T_Y \times T_Y"] \arrow[ur, "B_W(-,Y(-))"'] &
\end{tikzcd},$$

while the condition $A \in M_{\chi_Y} \Leftrightarrow Y = A^* Y A \Leftrightarrow A$ is an isometry on E equipped with the Hermitian form $B_W(-, Y(-))$. Therefore, one can find g such that $gT_Y = T_Y A$ by extending $A : E \to E$ (E thought of as a subspace of V) to V.

It follows from the short exact sequence above and (10.7), that

$$\begin{aligned} L^2(\mathcal{Y}_Y) &\cong \operatorname{Ind}_{M_{\chi_Y} N \times G(V)_{[T_Y]}}^{P \times G(V)} L^2(G(V)_{T_Y}\backslash G(V)[T_Y]) \\ &\cong \operatorname{Ind}_{M_{\chi_Y} N}^{P} L^2(G(V)_{T_Y}\backslash G(V)) \end{aligned} \tag{10.8}$$

where

- N acts by the character χ_Y;
- $M_{\chi_Y} \cong G(V)_{[T_Y]}/G(V)_{T_Y}$ acts by left translation;
- $G(V)$ acts by right translation.

Combine (10.6) and (10.8), we get

$$\Pi \cong \bigoplus_{\mathcal{O}_Y \in \Omega_V} \operatorname{Ind}_{M_{\chi_Y} N}^{P} L^2(G(V)_{T_Y}\backslash G(V)). \tag{10.9}$$

So by comparing the N-spectrum of (10.4) and (10.9), we get

Proposition 10.1. *As an $M_\chi N \times G(V)$-module,*

$$L^2(G(V)_{T_Y}\backslash G(V)) \cong \int_{\hat{G}(W)} W_{\chi_Y}(\pi) \otimes \Theta(\pi) d\mu_\theta(\pi). \tag{10.10}$$

10.3. Description of μ_θ.

Now we will determine μ_θ. This is purely a matter of theta correspondence. However, in practice, our $G(W)$ has low rank. Therefore, we make the following assumption.

10.3.1. *Stable range.* In this subsection, the dual pair $(G(V), G(W))$ will be assumed to be in the *stable range*, i.e., there is a totally isotropic D-submodule $X \subseteq V$ such that $\dim_D(X) = \dim_D(W)$.

Remark. In this case, the small smooth theta lift $\theta^\infty(\pi) \neq 0$ for any irreducible smooth representation π of $G(W)$.

10.3.2. *Mixed model.* Let X, Y be totally isotropic, complementary subspaces of V such that $\dim_D(X) = \dim_D(W)$, and $U = (X \oplus Y)^\perp$. We will use B_V to identify Y with $(X^*)'$ by

$$(x^*)(y) = B_V(x, y), \quad \forall x \in X, y \in Y.$$

Consider the polarization

$$(V \otimes_D W^*) = (X \otimes W^* \oplus U \otimes F^*) \bigoplus (Y \otimes W^* \oplus U \otimes E^*).$$

As a vector space,

$$L^2(X \otimes W^* \oplus U \otimes F^*) \cong L^2(\operatorname{Hom}_D(W, X)) \otimes L^2(\operatorname{Hom}_D(E, U)).$$

Let $(\omega_U, L^2(\operatorname{Hom}_D(E < U)))$ be the Schrödinger model of the Weil representation associated to $\operatorname{Mp}(U \otimes W^*)$. We will identify the space $L^2(\operatorname{Hom}_D(W, X)) \otimes L^2(\operatorname{Hom}_D(E, U))$ on the right hand side with the space of L^2 functions from $\operatorname{Hom}_D(W, X)$ to $L^2(\operatorname{Hom}_D(E, U))$. This is the so-called *mixed model* of the oscillator representation.

Let us identify $\mathrm{GL}(X) \times G(U)$ with the subgroup of $G(V)$ perserving the direct sum $V = X \oplus Y \oplus U$ by

$$\begin{aligned} \mathrm{GL}(X) \times G(U) &\to \mathrm{GL}(X) \times \mathrm{GL}(Y) \times \mathrm{GL}(U) \\ (A, g) &\mapsto (A, (A^*)^{-1}, g). \end{aligned}$$

Under this idenitification, the action of $G(W) \times \mathrm{GL}(X) \times G(U)$ on this model can be described by: for $T \in \mathrm{Hom}_D(W, X)$ and $S \in \mathrm{Hom}_D(E, U)$, then

$$\begin{aligned} g \cdot \phi(T)(S) &= [\omega_U(g)\phi(Tg)](S), && \forall g \in G(W); \\ h \cdot \phi(T)(S) &= \phi(T)(h^{-1}S), && \forall h \in G(U); \\ A \cdot \phi(T)(S) &= |\det(A)|^{\frac{\dim W}{2}} \phi(A^{-1}T)(S), && \forall A \in \mathrm{GL}(X). \end{aligned} \tag{10.11}$$

Again, we want to write this space as an induced representation. First observe that $G(W) \times \mathrm{GL}(X)$ acts transitively on the invertible elements in $\mathrm{Hom}_D(W, X)$, which form a single open orbit whose complement has measure 0. Fix $T_0 \in \mathrm{Hom}_D(W, X)$ invertible, define ϵ_W-Hermitian form B_{T_0} on X by

$$B_{T_0}(x_1, x_2) = B_W(T_0^{-1}x_1, T_0^{-1}x_2).$$

The group preserves this form is

$$G(X, B_{T_0}) = T_0 G(W) T_0^{-1} \subseteq \mathrm{GL}(X).$$

The stabilizer of T_0 in $G(W) \times \mathrm{GL}(X)$ is

$$(G(W) \times \mathrm{GL}_{T_0})_{T_0} = \{(g, T_0 g T_0^{-1}) \mid g \in G(W)\} \cong G(W).$$

It follows from (10.11) that

$$\begin{aligned} L^2(W \otimes X) \otimes L^2(\mathrm{Hom}_D(E, U)) &\cong \mathrm{Ind}_{(G(W)\times\mathrm{GL}(X))_{T_0}}^{G(W)\times G(X,B_{T_0})} L^2(hom_D(E, U)) \\ &= \mathrm{Ind}_{G(W)\times G(X,B_{T_0})}^{G(W)\times\mathrm{GL}(X)} \mathrm{Ind}_{(G(W)\times\mathrm{GL}(X))_{T_0}}^{G(W)\times G(X,B_{T_0})} L^2(\mathrm{Hom}_D(E, U)). \end{aligned}$$

Here $(G(W) \times \mathrm{GL}(X))_{T_0}$ acts by first projecting into the first factor then acting as the (Schrö) Weil representation of $\mathrm{Mp}(U \otimes W^*)$. But this representation is the same as first projecting into the second component and using Schródinger model of the Weil representation of $\mathrm{Mp}(U \otimes X^*)$ (X equipped with the form B_{T_0}) to define an action of $G(X, B_{T_0})$ on $L^2(\mathrm{Hom}_D(T_0(E), U))$. Therefore, we have

$$\begin{aligned} \Pi &\cong L^2(W^* \otimes X) \otimes L^2(\mathrm{Hom}_D(E, U)) \\ &\cong \mathrm{Ind}_{G(W)\times G(X,B_{T_0})}^{G(W)\times\mathrm{GL}(X)} \mathrm{Ind}_{(G(W)\times\mathrm{GL}(X))_{T_0}}^{G(W)\times G(X,B_{T_0})} L^2(\mathrm{Hom}_D(T_0(E), U)) \\ &\cong \mathrm{Ind}_{G(W)\times G(X,B_{T_0})}^{G(W)\times\mathrm{GL}(X)} \underbrace{(\mathrm{Ind}_{(G(W)\times\mathrm{GL}(X))_{T_0}}^{G(W)\times G(X,B_{T_0})} 1)}_{\cong L^2(G(W)) \curvearrowleft G(W)\times T_0 G(W) T_0^{-1}} \otimes L^2(\mathrm{Hom}_D(T_0(E), U)) \\ &\cong \mathrm{Ind}_{G(W)\times G(X,B_{T_0})}^{G(W)\times\mathrm{GL}(X)} \int_{\hat{G}(W)} \pi^* \otimes (\pi^{T_0} \otimes L^2(\mathrm{Hom}_D(T_0(E), U))) d\mu_{G(W)}(\pi) \\ &\cong \int_{\hat{G}(W)} \pi^* \otimes (\mathrm{Ind}_{G(X,B_{T_0})}^{\mathrm{GL}(X)} \pi^{T_0} \otimes L^2(\mathrm{Hom}_D(T_0(E), U))) d\mu_{G(W)}(\pi) \end{aligned} \tag{10.12}$$

where π^* is the contragradient representation of π, π^{T_0} is the representation of $G(X, B_{T_0})$ given by $\pi^{T_0}(g) = \pi(T_0^{-1} g T_0)$, for $g \in G(X, B_{T_0})$. So (10.12) is just a T_0-twisted version of Harish-Chandra Plancherel decomposition of $L^2(G(W))$ equipped with $G(W) \times G(W)$-action by left and right translation. Notice that the multiplicity of π^* in (10.12) is nonzero for each π in the support of $\mu_G(W)$. Therefore, comparing (10.2) and (10.12), we have

Proposition 10.2. *If $(G(W), G(V))$ is in the stable range, then $\mu_\theta = \mu_{G(W)}$.*

10.4. **Description of multiplicity spaces.** Now we start to describe the multiplicity space $W_{\chi_Y}(\pi)$ in (10.4). Notice that it has nothing to do with theta correspondence, and it's purely a matter about representations of $G(W)$.

10.4.1. *The Bessel-Plancherel theorem.* The main result is the following.

Theorem 10.1 (Bessel-Plancherel Theorem). *Let (W, B_W) be an ϵ_W-Hermitian D-module, and $P = MN$ the Siegel parabolic subgroup associated to a complete polarization $W = E \oplus F$.*

(1) If $\mathcal{O}_\chi$ is open in $\hat{N}$, there is an isomorphism of $M_\chi \times G(W)$-modules

$$L^2(N\backslash G(W);\chi) \cong \int_{\hat{G}(W)} W_\chi(\pi) \otimes \pi d\mu_{G(W)}(\pi).$$

(2) If $\dim_D(W) = 2$, then for open $\mathcal{O}_\chi$, $\dim W_\chi(\pi) < \infty$, and

$$W_\chi(\pi) = \mathrm{Hom}_N(\pi^\infty, \chi),$$

the space of continuous χ-Whittaker functionals on π^∞.

So we get a description of spectral decomposition of $L^2(H\backslash G)$ in terms of the Plancherel decomposition of the Whittaker variety $L^2(N, \chi\backslash G_X)$.

10.4.2. *Local H-period.* By the smooth analog of the computation with the Schrödinger model, one has

Lemma 10.1. *For any irreducible smooth representation σ^∞ of $G(V)$, let $\Theta^\infty(\sigma^\infty)$ denote the big smooth theta lift of σ^∞ to $G(W)$. Then for open $\mathcal{O}_\chi$,*

$$\mathrm{Hom}_N(\Theta^\infty_{W,V}(\sigma^\infty), \chi) \cong \mathrm{Hom}_{G(V)_{T_Y}}(\sigma^\infty, \mathbb{C}).$$

In the cases we are considering ($\dim_D W = 2$, $\epsilon_W = -1$), one can show that $\sigma^\infty = \Theta(\pi)^\infty = \theta^\infty(\pi^\infty)$ is irreducible (even if k is 2-adic), and $\Theta^\infty_{W,V}(\sigma^\infty) = \pi$, for any irreducible tempered representation π of $G(W)$. Therefore, by Lemma 10.1, we have

$$W_\chi(\pi) = \mathrm{Hom}_N(\pi^\infty, \chi) = \mathrm{Hom}_N(\Theta^\infty_{W,V}(\sigma^\infty), \chi) = \mathrm{Hom}_{G(V)_{T_Y}}(\Theta(\pi)^\infty, \mathbb{C}). \tag{10.13}$$

That being said, the multiplicity space can be identified with the space of H-periods on $\Theta(\pi)^\infty$.

10.5. **Examples.** Using previous results, we can obtain certain examples of the Sakellaridis-Venkatesh conjecture.

Taking $D = k, k \times k, M_2(k)$, and W to be skew-Hermitian with $\dim_D(W)$, we have

D	$G = G(V)$	$H = G(V)_{T_Y}$	$G_X = G(W)$
k	SO_n	SO_{n-1}	SL_2 or $\widetilde{\mathrm{SL}}_2$
$k \times k$	GL_n	GL_{n-1}	GL_2
$M_2(k)$	Sp_{2n}	Sp_{2n-2}	SO_4

Taking D to be non-split version of $k \times k$ or $M_2(k)$, i.e., the quadratic extension E/k or quaternion $\mathbb{H}$, we have

D	$G = G(V)$	$H = G(V)_{T_Y}$	$G_X = G(W)$
E	U_n	U_2	U_{n-1}
$\mathbb{H}$	$\mathrm{Sp}_n(\mathbb{H})$	$\mathrm{Sp}_{n-1}(\mathbb{H})$	$\mathcal{O}_2(\mathbb{H})$

We also have some exceptional cases that can be proved using exceptional theta correspondence in execptional groups.

$X = G/H$	G_X	approach
$\mathrm{SL}_3 \backslash G_2$	$\widetilde{\mathrm{SL}}_2$	relate to $\mathrm{SO}_6 \backslash \mathrm{SO}_7$
$G_2\backslash \mathrm{Spin}_7$	SL_2	relate to $\mathrm{SO}_7 \backslash \mathrm{SO}_8$
$\mathrm{SO}_3 \backslash \mathrm{SL}_3$	$\widetilde{\mathrm{SL}}_3$	theta in G_2
$\mathrm{Sp}_6 \backslash \mathrm{SL}_6$	SL_3	theta in E_7
$(J,\psi)\backslash G_2$	PGL_3	theta in E_6
$\mathrm{SU}_3 \backslash \mathrm{Spin}_7$	$(\mathrm{Spin}_3 \times \mathrm{Spin}_5)/\Delta_{\mu_2}$	theta in E_7
$G_2\backslash \mathrm{Spin}_8$	$\mathrm{SL}_2^3 \backslash \Delta\mu_2$	theta in E_7
$\mathrm{Spin}_9 \backslash F_4$	PGL_2	theta in E_7
$F_4\backslash E_6$	SL_3	theta in E_8

10.6. **Smooth story.** Now we focus on the case $X = \mathrm{SO}_{n-1} \backslash \mathrm{SO}_n$ with $G_X = \mathrm{SL}_2$ or $\widetilde{SL}_2$. In stead of the L^2-theory, we now work in the smooth settings (all representations are smooth from now on). In this case, the map $\iota : G_X^\vee \times \mathrm{SL}_2 \to G^\vee$ is given by

- when n is even,

$$\iota : \mathrm{PGL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\mathrm{Sym}^2 \times \mathrm{Sym}^{n-4}} \mathrm{SO}_3(\mathbb{C}) \times \mathrm{SO}_{n-3}(\mathbb{C}) \to \mathrm{SO}_n(\mathbb{C})$$

- when n is odd,

$$\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\mathrm{Sym}^1 \times \mathrm{Sym}^{n-4}} \mathrm{Sp}_2(\mathbb{C}) \times \mathrm{Sp}_{n-3}(\mathbb{C}) \to \mathrm{Sp}_{n-1}(\mathbb{C}).$$

10.6.1. *Relative characters.* Both Theorem 10.1(1) and (10.13) indicate that there should be some functoriality going on between $H\backslash G$ and $(N,\chi)\backslash G_X$. Therefore, one should expect to have certain relative character identities.

We have a map

$$\theta = \iota_* : \mathrm{Irr}(G_X) \to \mathrm{Irr}(G)$$

and there is an expected isomorphism

$$\bigoplus_{\iota_*(\sigma)=\pi} \mathrm{Hom}_N(\sigma,\chi) = \mathrm{Hom}_H(\pi,\mathbb{C}).$$

In our case, $\iota_* = \theta$ is injective (by Theorem 9.2), and the left hand side is 1-dimensional (by uniqueness of Whittaker models). In this case, the spectral decomposition of $L^2(H\backslash G)$ and $L^2(N,\chi\backslash G_X)$ gives a canonical $l_\sigma \in \mathrm{Hom}_N(\sigma,\chi)$, $l_\pi \in \mathrm{Hom}_H(\pi,\mathbb{C})$.

$$\begin{aligned} \mathcal{B}_{\sigma,l_\sigma} : C_c^\infty(G_X) &\to \mathbb{C} \\ f &\mapsto \sum_{v\in \mathrm{ONB}(\sigma)} \overline{l_\sigma(\pi(\bar{f})(v))} \cdot l_\sigma(v) \end{aligned}$$

which factors as

$$\begin{aligned} C_c^\infty(G_X) &\twoheadrightarrow C_c^\infty(N,\chi\backslash G_X) \to \mathbb{C}, \\ f &\mapsto (g \mapsto \int_N f(ng)\bar{\chi}(n)dn), \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}_{\pi,i(l)} : C_c^\infty(G) &\to \mathbb{C} \\ f &\mapsto \sum_{v\in \mathrm{ONB}(\pi)} \overline{l_\pi(\pi(\bar{f})(v))} \cdot l_\pi(v) \end{aligned}$$

which factors as

$$C_c^\infty(G) \twoheadrightarrow C_c^\infty(H\backslash G_X) \to \mathbb{C},$$
$$f \mapsto (g \mapsto \int_H f(hg)dh),$$

Moreover, these extend to the spaces of Schwartz functions.

10.6.2. *Transfer of test functions.* Since $\dim_k W = 2$, we can identify $\Pi^\infty \cong \mathcal{S}(V)$, the space of Schwartz functions on V. Then we have

$$\begin{aligned} p : \mathcal{S}(V) &\to \mathcal{S}(N,\chi\backslash G_X) \\ p(\Phi)(g) &= (g\cdot\Phi)(v_1) \\ q : \mathcal{S}(V) &\to \mathcal{S}(H\backslash G) \\ q(\Phi)(g) &= \Phi(g^{-1}\cdot v_1). \end{aligned}$$

Theorem 10.2 (Relative character identity)**.** *The following diagram commutes:*

$$\begin{tikzcd} \mathcal{S}(V) \arrow[r, "q"] \arrow[d, "p"] & \mathcal{S}(H\backslash G) \arrow[d, "\mathcal{B}_{\iota_*(\sigma),l_{\iota_*(\sigma)}}"] \\ \mathcal{S}(N,\chi\backslash G_X) \arrow[r, "\mathcal{B}_{\sigma,l}"] & \mathbb{C}. \end{tikzcd}$$

10.7. **Global story.** Globally, when k is a number field with ring of adeles $\mathbb{A}$, one can consider the global period

$$\mathcal{P}_H : \mathcal{A}_{\mathrm{cusp}}(G) \to \mathbb{C}$$
$$\phi \mapsto \int_{H(k)\backslash H(\mathbb{A})} \phi(h)dh.$$

The restriction on a cuspidal representation $M = \otimes M_v$ defines $\mathcal{P}_{H,M} \in \mathrm{Hom}_{H(\mathbb{A})}(M,\mathbb{C})$.

10.7.1. *Functorial lift.* The first question we can ask is: when is M the functorial lift from G_X via ι. Gan-Wan proved that: if $\mathcal{P}_{H,M} \neq 0$, then there exists a cuspidal representation Σ of G_X such that $M_v = \iota_*(\Sigma_v)$.

10.7.2. *Decomposition of $\mathcal{P}_{H,M}$.* For each v, the spectral decomposition gives rise to $l_{M_v} \in \mathrm{Hom}_{H(k_v)}(M_v,\mathbb{C})$. So it's natural to compare $\mathcal{P}_{H,M}$ and $\prod_v l_{M_v}$.

More precisely, we need to normalzie l_{M_v} so that the Euler product $\prod_v l_{M_v}(\phi_v)$ converges. For this purpose, we need to evaluate $l_{M_v}(\phi_v^0)$ with ϕ_v^0 the spherical unit vector in M_v. It turns out that if $M_v = \iota_*(\Sigma_v)$ for tempered $\Sigma_v \in \hat{G}_{X,v}$, one has

$$|l_{M_v}(\phi_v^0)|^2 = L^\flat_{X,v}(\Sigma_v) = \Delta_v(0)\cdot\frac{L_{X,v}(0,\Sigma_v)}{L(1,\Sigma_v,\mathrm{Ad})}$$

where

- $\Delta_v(s)$ is a product a local L-factors which only depends on X and not on Σ_v;
- $L_{X,v}(s,\Sigma_v) = \prod_d L(s+d,\Sigma_v,V_X^d)$ associated to the $\frac{1}{2}\mathbb{Z}$-graded finite dimensional algebraic representation $V_X = \oplus_d V_X^d$ of $G_X^\vee$ introduced in [SVV17].

They also determine the constant $c(M)$ such that

$$\mathcal{P}_H = c(M)\cdot L_X(s,\Sigma)\cdot\prod_v |l^\flat_{M_v}(\phi_v)|^2.$$

Remark. One should compare this with Theorem 8.1.

11. An Introduction to the Relative Trace formula - Elad

The goal is to explain Wei Zhang's proof of the following theorem.,

Theorem 11.1. *If* $\varphi_n = \otimes_v \varphi_{n,v}$ *(resp.* $\varphi_{n+1} = \otimes_v \varphi_{n+1,v}$*) be cuspidal automorphic form of* $U(V_n)$ *(resp.* $U(V_{n+1})$*), then we have*

$$\int_{U(V_n)(F)\backslash U(V_n)(\mathbb{A}_F)} \varphi_n(g_n)dg_n = 2^{-B_\beta} L(\frac{1}{2}, \pi_{n+1}, \pi_n) \prod_v \alpha_v^{\#}(\varphi_{n,v}, \varphi_{n+1,v}) \tag{11.1}$$

where

$$\alpha_v^{\#}(f_n, f_{n+1}) = \frac{1}{L(\frac{1}{2}, \pi_{n,v}, \pi_{n+1,v})} \int_{U(V_n)(F_v)} \langle \pi_{n,v}(g_n)f_{n,v}, f_{n,v}\rangle \langle \pi_{n+1,v}(g_n)f_{n+1,v}, f_{n+1,v}\rangle dg_n.$$

11.1. **Periods, L-functions and functoriality.** Let F be a number field. Conjecturally there exists a group L_F called *the global Langlands group* such that the irreducible n-dimensional representations of L_F are in bijection with irreducible cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$.

Let E/F be a quadratic extension of F itself, and $(V, \langle -, -\rangle)$ finite dimensional non-degenerate symmetric or Hermitian space over E, and $G(V)$ the isometry group of V. Irreducible cuspidal automorphic representation π of $G(V)(\mathbb{A}_F)$ has a global Langlands parameter

$$\varphi_\Pi : L_F \xrightarrow{\varphi_\pi} \hat{G}(V)(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C}).$$

Therefore, φ_π corresponds to an irreducible automorphic representation of $\mathrm{GL}_N(\mathbb{A}_F)$ denoted by Π. Assume that Π is cuspidal.

V	$\dim V$	L-functions
symmetric	$2n+1$	$L(s, \Pi, \wedge^2)$
symmetric	$2n$	$L(s, \Pi, \mathrm{Sym}^2)$
Hermitian	$2n+1$	$L(s, \Pi, \mathrm{AS}^+)$
Hermitian	$2n$	$L(s, \Pi, \mathrm{AS}^-)$

If the L-function has a pole at $s = 1$, then the representation coming from the desired group $G(V)(\mathbb{A}_F)$. These L-function conditions can be encoded using periods.

- $L(s, \Pi, \wedge^2)$ has a pole at $s = 1$ iff the Shalika period

  $$\int_{\mathbb{A}_F^\times \backslash \mathrm{GL}_N(\mathbb{A}_F)} \int_{\mathrm{Mat}_{n\times n}(F)\backslash \mathrm{Mat}_{n\times n}(\mathbb{A}_F)} \varphi((\left(\begin{smallmatrix} I_n & x \\ & I_n \end{smallmatrix}\right))(\left(\begin{smallmatrix} g & \\ & g \end{smallmatrix}\right)))\Psi(\mathrm{tr}(x))dxdg \neq 0.$$

  where $\Psi : \mathbb{A}_F \to \mathbb{C}^\times$ is an additive character.

11.2. **Relative trace formula.**

11.2.1. *Motivation.* Let G be a fintie group. The space of functions $G \to \mathbb{C}$ that are invariant under G-conjugation has an orthonormal basis. These are the characteristic functions of the conjugacy classes. There is another natural basis $\{\mathrm{tr}(\pi(g)) \mid \pi \in \mathrm{Irr}(G)\}$.

11.2.2. *Relative representation theory.* Given $H_1, H_2 \leq G$ and characters $\chi_i : H_i \to \mathbb{C}^\times$, study functions satisfying

$$f(h_1 g h_2) = \chi(h_1)\chi(h_2)f(g).$$

Let's add the following assumption: for every $\pi \in \mathrm{Irr}(G)$,

$$\dim \mathrm{Hom}_{H_i}(\pi|_{H_i}, \chi_i) \leq 1, i = 1, 2.$$

So we also have

$$\dim \mathrm{Hom}_{H_i}(\chi_i, \pi|_{H_i}) \leq 1.$$

It follows from Frobenius reciprocity that

$$\dim_G \operatorname{Hom}_G(\pi, \operatorname{Ind}_{H_1}^G \chi_1) \leq 1.$$

Notice that the space of functions satisfying the above condition can be identified with

$$(\operatorname{Ind}_{H_1}^G \chi_1)^{H_2,\chi_2} = \oplus_\pi W(\pi, H_1, \chi_1)^{H_2,\chi_2}$$

where $W(\pi, H_1, \chi_1)$ is the unique subspace of $\operatorname{Ind}_{H_1}^G \chi_1$ isomorphic to π. But the right hand side consists of basis of the form

$$\frac{1}{|H_1|}\frac{1}{|H_2|}\sum_{h_1\in H_1}\sum_{h_2\in H_2} \chi_1^{-1}(h_1)\chi_2^{-1}(h_2) \operatorname{tr}\pi(h_1 g h_2).$$

11.2.3. *General picture.* Let G be a reductive group with H_1, H_2 reductive subgroups. $\chi_i : H_i \to \mathbb{C}^\times$. For $f \in C_c^\infty(G(\mathbb{A}))$, it acts by right convolution

$$R(f)(\varphi)(x) = \int_{G(\mathbb{A})} f(g)\varphi(xy)dy = \int_{G(\mathbb{A})} f(x^{-1}y)\varphi(y)dy.$$

However, notice that

$$\begin{aligned}\int_{G(\mathbb{A})} f(x^{-1}y)\varphi(y)dy &= \int_{G(F)\backslash G(\mathbb{A})}\sum_{\gamma\in G(F)} f(x^{-1}\gamma y)\varphi(\gamma y)dy \\ (\varphi\in L^2([G])) &= \int_{G(F)\backslash G(\mathbb{A})}\sum_{\gamma\in G(F)} f(x^{-1}\gamma y)\varphi(y)dy.\end{aligned}$$

So $R(f)$ is an integral operator with kernel $K_f(x,y) = \sum_{\gamma\in G(F)} f(x^{-1}\gamma y)$. Notice that we have

$$\begin{aligned}\int_{[H_1]}\int_{[H_2]} K_f(h_1,h_2)\chi_1(h_1)\chi_2(h_2)dh_1dh_2 &= \int_{[H_1]}\int_{[H_2]}\sum_\gamma f(h_1^{-1}\gamma h_2)\chi_1(h_1)\chi_2(h_2)dh_1dh_2 \\ &= \gamma \in \Gamma \operatorname{vol}([(H_1\times H_2)_\gamma)] \cdot \operatorname{orb}(f,\gamma)\end{aligned}$$

where for a double coset $H_1\gamma H_2$, the orbital integral is defined to be

$$\operatorname{orb}(f,\gamma) = \int_{(H_1\times H_2)_\gamma(\mathbb{A})\backslash(H_1\times H_2)(\mathbb{A})} f(h_1^{-1}\gamma h_2)\chi_1(h_1)\chi_2(h_2)dh_1dh_2.$$

Let π be an irreducible cuspidal representation of $G(\mathbb{A})$, the projection of $R(f)$ to π is given by

$$R(f)\varphi = \sum_{\psi\in\mathrm{ONB}(\pi)} (R(f)\psi)(x)\langle\varphi,\psi\rangle = \int_{[G]}\sum_{\psi\in\mathrm{ONB}(\pi)} R(f)\psi(x)\bar\psi(y)\varphi(y)dy.$$

So we have a kernel

$$K_{\pi,f}(x,y) = \sum_{\psi\in\mathrm{ONB}(\pi)} R(f)\psi(x)\bar\psi(y).$$

Similarly, we have

$$\begin{aligned}I_\pi(f) &= \int_{[H_1]}\int_{[H_2]} K_{\pi,f}(h_1,h_2)\chi_1(h_1)\chi_2(h_2)dh_1dh_2 \\ &= \sum_{\psi\in\mathrm{ONB}(\pi)}\int_{[H_1]}(R(f)\phi)(h_1)\chi_1(h_1)dh_1\int_{[H_2]}\bar\psi(h_2)\chi_2(h_2)dh_2 \\ &= \sum_{\psi\in\mathrm{ONB}(\pi)} \mathcal{P}_{H_1,\chi_1}(R(f)\psi)\overline{\mathcal{P}_{H_2,\chi_2^{-1}}(\psi)}.\end{aligned}$$

Theorem 11.2 (Relative trace formula)**.**

$$\sum_{\gamma\in\Gamma}\operatorname{vol}([(H_1\times H_2)_\gamma])\cdot\operatorname{orb}(f,\gamma) + \cdots = \sum_{\pi\ cuspidal} I_\pi(f) + \cdots$$

In general, the $\cdots$ terms (which come from the continuous spectrum and the residue spectrum) are very difficult to determine. To avoid them, we can either assume G is anisotropic or f is cuspidal, i.e., $f = \otimes_v f_v$ and there exists v such that for any x, y and unipotent N,

$$\int_{N(F_v)} f_v(xny)dn = 0.$$

11.3. **Ichino-Ikeda.** Let $G = U(V_n) \times U(V_{n+1})$ with $H_1 = H_2 = U(V_n)^{\Delta}$, and $\chi_1 = \chi_2 = 1$.

11.3.1. *Reformulation.* Now we can reformula Theorem 11.1. For $f = \otimes f_v$,

$$I_\pi(f) = 2^{-B_\beta} L(\frac{1}{2}, \pi) \cdot \prod_v I_{\pi_v}(f_v)$$

where

$$I_{\pi_v}(f_v) = \sum_{\psi_v \in \mathrm{ONB}(\pi_v)} \alpha_v^{\#}(\pi_v(f_v)\psi_v, \psi_v).$$

11.3.2. *Filcker-Rallis.* Let $G' = \mathrm{Res}_{E/F} \mathrm{GL}_{n+1} \times \mathrm{Res}_{E/F} \mathrm{GL}_n$, $H_1' = \mathrm{Res}_{E/F} \mathrm{GL}_n^{\Delta}$, $H_2' = \mathrm{GL}_{n+1} \times \mathrm{GL}_n$, where $\chi_1' = |\det|^n$, $\chi_2'(h_{n+1}, h_n) = \chi_{E/F}(\det h_{n+1})^n \chi_{E/F}(\det h_n)^{n-1}$.

11.3.3. *Smooth transfer.* The idea is to compare RTFs for G and G', and use the fact that Ichino-Ikeda is known for GL_n.

First observe that there is a bijection between two sets of "regular semisimple orbits". An element $\gamma \in G(F)$ is called *regular semisimple* if its $H_1 \times H_2$-orbit is Zariski closed and its stabilizer is of minimal dimension. In our case, this means the stabilizer is trivial. If (W_{n+1}, W_n) and (W'_{n+1}, W'_n) are Hermitian, write $(W_{n+1}, W_n) \sim (W'_{n+1}, W'_n)$ if there eixsts λ such that $W_{n+1}^{\lambda} \sim W'_{n+1}$, $W_n^{\lambda} \sim W'_n$, where $\bullet^{\lambda}$ means to multiply the Hermitian form by λ.

There is a natural bijection

$$\bigcup_{W/\sim} \langle G(W)(F) \rangle_{\mathrm{rs}} \cong G'(F)_{\mathrm{rs}}.$$

Under this bijection we can define the transfer of functions from the geometric side of RTFs (the orbital integrals match). Wei Zhang shows that if two functions are transfer of each other, then their spectral sides of RTFs also match.

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