RTG SEMINAR ON SHIMURA VARIETIES

NOTES TAKEN BY GUANJIE HUANG

Abstract. These are the notes from RTG number theory seminar on Shimura varieties at University of Michigan in 2023 Winter. The seminar is organized by Tasho Kaletha, Kartik Prasanna, and Charlotte Chan.

Contents

1. Introduction - Tasho Kaletha

The Langlands program studys the relation between three categories

(1) automorphic representations
(2) Galois representations
(3) motives.

Shimura varieties shed light on all three areas.

1.1. Automorphic representations.

Let F be a global field, i.e., finite extension of $\mathbb{Q}$ or $\mathbb{F}_p(t)$. Let G be a connected reductive F-group. An automorphic representation sits in $L^2(G(F)\backslash G(\mathbb{A}))$. It turns out that $\pi = \otimes_v \pi_v$, where π_v is an irreducible representation of $G(F_v)$. For almost all v, π_v is unramified, i.e., $\pi^{G(\mathcal{O}_{F_v})} \neq 0$ (this may not be true for all placees, but we can find a finite set S of places reductive group scheme $\mathcal{G}$ over $\mathcal{O}_{F,S}$ such that $\mathcal{G} \times F = G$). Under Langlands correspondence, unramified π_v corresponds to semisimple conjugacy class $c(\pi_v)$ in $\hat{G} \rtimes \mathrm{Frob}_v$ called Satake parameter. Given $\rho : \hat{G} \rtimes \Gamma \to \mathrm{GL}(V)$, we can define the local L-factor

$$L(s, \pi_v, \rho) = \det(1 - \rho(c(\pi_v))q_v^{-s})^{-1}$$

and partial L-function

$$L^S(s, \pi, \rho) = \prod_{v \notin S} L(s, \pi_v, \rho).$$

1.2. Galois representation.

For an Artin representation $\rho : \Gamma \to \mathrm{GL}(V)$, where V is a finite dimensional $\mathbb{C}$-vector spaces, one has the Artin L-function

$$L^S(s, \rho) = \prod_{v \notin S} \det(1 - \rho(\mathrm{Frob}_v) \cdot q_v^{-s})^{-1}$$

where S is a finite set of places which ramify the quotient of the representation. In fact, the full L-function is also defined:

$$L(s,\rho)=\prod_v \det(1-\rho(\mathrm{Frob}_v)\mid_{V^{\rho(I_v)}}\cdot q_V^{-s})^{-1}.$$

Moreover, for l-adic representation $\rho:\Gamma\to\mathrm{GL}(V)$, where V is a finite dimensional $\bar{\mathbb{Q}}_l$-vector space, one can also define the notion of a L-function.

1.3. **Motives.** On smooth projective varieties over $\mathbb{Q}$ we have

- Betti cohomology
- de Rham cohomology
- l-adic cohomology

The idea is that, there is going to be a category called motives with realization functors into different cohomology theories. The motives can be made from the category of smooth projective varieties. First we make more morphisms by thinking of a morphism $X\to Y$ as a graph in $X\times Y$, and replace graph by what is called a cycle, and define the equivalence relation among cycles.

1.4. **Shimura varieties.** Given two groups $G\times H$ acting on V, then V has the decomposition

$$V=\bigoplus \pi\boxtimes\rho, \pi\in\mathrm{Irr}(G),\rho\in\mathrm{Irr}(H).$$

We hope $\pi\mapsto\rho$ is a map.

Example 1.1. $G\times H$ acts on a space X, $V=H^*(X)$.

Example 1.2. $G\times H$ acts on a system of spaces X_i, and $V=\varinjlim H^*(X_i)$.

Example 1.3. G acts on X over $\mathbb{Q}$, $G\times\Gamma$ acts on $V=H^i_{\mathrm{et}}(X\times\bar{\mathbb{Q}},\mathbb{Q}_l)$.

1.4.1. *Global Shimura varieties.* Let G be a reductive group, X a $G(\mathbb{R})$-conjugacy class of homomorphisms $\mathbb{C}^\times_{\mathbb{R}}\to\mathbb{G}_{\mathbb{R}}$ subject to axioms. For a compact open subgroup $K\subseteq G(\mathbb{A}_f)$, we can define

$$\mathrm{Sh}_K=G(\mathbb{Q})\backslash X\times G(\mathbb{A}_f)/K=G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty\cdot K.$$

For different K, this is going to be a quasi-projective smooth $\mathbb{C}$-variety. Moreover, Sh_K has a model over $E/\mathbb{Q}$ (reflex field). Then $G(\mathbb{A}_f)$ acts on $\{\mathrm{Sh}_K\}$ and the $\varinjlim_K H^*(\mathrm{Sh}_K\times_E\bar{\mathbb{Q}},\bar{\mathbb{Q}}_l)$ has a action of $G(\mathbb{A}_f)\times\Gamma_E$. To get the action of the infinite place, we can consider

$$V'=\varinjlim_K H^*(\mathrm{Sh}_K\times_E\bar{\mathbb{Q}},\mathcal{L}_{\epsilon,K})$$

where ϵ is a representation of G and $\mathcal{L}_{\epsilon,K}$ is the corresponding sheaf.

1.4.2. *Langlands-Kottwitz method.* To push things forward, we need an integral model, so that we can talk about special fiber and relate the points in the special fiber with the motives (the Langlands-Rapoport conjecture). This is almost a theorem now, due to Kisin (2010), Kisin-Shin-Zhu (2022).

1.4.3. *Local Shimura variety.* This is introduced in 2014. In the work of Rapoport-Zink in 1996, they parametrize deformations of p-divisible groups by formal scheme. The generic space of this formal scheme, one get adic space which later. In the work of Rapoport-Viehmann, given $(G/\mathbb{Q}_p,b,\mu)$ where $b\in G(\breve{\mathbb{Q}}_p)$, $\mu:\mathbb{G}_m\to G$. $G(\mathbb{Q}_p)\times G_b(\mathbb{Q}_p)$ acts on $\{\mathrm{Sh}_{K_p}\}$. So we have the action of $G(\mathbb{Q}_p)\times G_b(\mathbb{Q}_p)\times I_E$ acts on $\varinjlim_K H^*(\mathrm{Sh}_{K_p}\times_{\breve{E}}\breve{\bar{E}},\bar{\mathbb{Q}}_l)$ (Kottwitz conjecture).

2. Global Shimura Varieties - Calvin Yost-Wolff

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*This section is edited by Calvin Yost-Wolff

2.1. **Hermitian symmetric domains.**

2.1.1. *Basic definitions.*

Example 2.1. Let $\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid \mathrm{im}(z) = y > 0\}$. This has the following structures:
- complex manifold
- Riemmanian form (given by Petersson inner form) with negative curvature
- transitive SL_2-action
- i the unique point fixed by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Definition. A *Hermitian symmetric domain* (HSD) is a mainfold with the following structures:
1. complex structure;
2. Riemmanian form with negative curvature;
3. an transitive action of adjoint real algebraic group;
4. an isometric involution (fixes exactly one point);
5. non-compact type;

Let X be a connected HSD, then
$$X = G(\mathbb{R})^+/K_p$$
for some real adjoint Lie group G and some stabilizer K_p of p by (3). It follows from (4) and (5) that involution acts on complexified tangent space at p as Cartan involutions: $\theta : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ such that $\mathfrak{g}_{\mathbb{C}}^{\theta}$ are a compact real form of $\mathfrak{g}_{\mathbb{C}}$. From this we can deduce that G is reductive and K_p is compact.

2.1.2. *Classification.* For $p \in X$ we can associate $u_p : U_1 = \{z \in \mathbb{C} \mid |z| = 1\} \to G(\mathbb{R})$ such that u_p fixes p and acts by z at the tangent space at p.

Theorem 2.1 (Classification of HSD). *All non-compact connected HSD are of the form (G, X) where G is a real adjoint reductive Lie group, X is a $G(\mathbb{R})^+$-conjugacy class of $u_p : U_1 \to G(\mathbb{R})$ such that*
1. *on $\mathrm{Lie}_{\mathbb{C}}(G)$, characters of $\mathrm{Ad}_{U_1,\mathbb{C}}$ are $z, 1, z^{-1}$;*
2. *$\mathrm{Ad}(u_p(-1))$ is a Cartan involution;*
3. *$u_p(-1)$ does not project to 1 in any factor of G.*

2.1.3. *Hodge structure.*

Definition. Let V be a real vector space. $V_{\mathbb{C}} = \oplus_{p+q=n} V^{p,q}$ is called a *Hodge structure* if $V_{p,q} \cong V_{q,p}$. A filtration given by $F^p V_{\mathbb{C}} = \oplus_{r \geq p} V^{r,s}$.

Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$. A Hodge structure can be identified with a representation $h : \mathbb{S} \to \mathrm{GL}(V)$ such that the Hodge structure on V is given by
$$h_{\mathbb{C}}(z_1, z_2)v = z_1^{-p} z_2^{-q} v, \quad \forall v \in V^{p,q}.$$

Definition. A *polarization* of V is a map $\psi : V \otimes V \to \mathbb{R}(-n)$ of $\mathbb{S}$-representations, where $\mathbb{S}_{\mathbb{C}} \cong \mathbb{C}^{\times} \times \mathbb{C}$ acts on $\mathbb{R}(-n)_{\mathbb{C}}$ by $z_1^{-n} z_2^{-n}$, such that $\psi_{\mathbb{C}}(h(i)u, \bar{v})$ is positive definite.

Consider all Hodge structures on V with
- specified dimensions of $V^{p,q}$;
- polarization $\psi : V \otimes V \to \mathbb{R}$;
- possible extra conditions (Hodge tensors);

These have a $\mathbb{C}$-structure inside a Flag manifold denoted by $S(d, t)$. Here $d : \{(p, q)\} \to \mathbb{Z}_{\geq 0}$ refers to the dimension conditions on $V^{p,q}$, and t is the polarization and the extra data.

Theorem 2.2. *If $S(d, t)$ satisfies Griffith's transversality (some condition on tangent space of $S(d, t)$ inside that of the flag manifold), then any connected component of $S(d, t)$ is a HSD. Conversely, any connected HSD is a connected component of some $S(d, t)$.*

Example 2.2. Let V be 2-dimensional, $d(1,0) = 1$, $d(0,1) = 1$, and $\psi = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. In this case, $S(d,t) \cong \mathcal{H}^- \subset \mathbb{P}^1(\mathbb{C})$ (correction: in the talk I said $\mathbb{H} \cup \mathbb{H}^-$ which is not true).

2.2. **Quotients by arithmetic subgroups.** Let (G,X) be a non-compact HSD, Γ an *arithmetic subgroup* (i.e., for any faithful representation $G \hookrightarrow \mathrm{GL}_n$, $\Gamma \cap G(\mathbb{R}) \cap \mathrm{GL}_n(\mathbb{Z})$ is of finite index in both Γ and $G(\mathbb{R}) \cap \mathrm{GL}_n(\mathbb{Z})$).

Theorem 2.3 (Baily-Borel)**.** *$\Gamma\backslash X$ has a unique structure of a quasi-projective algebraic variety if Γ is arithmetic and torsion-free.*

This idea behind this theorem is to show the Baily-Borel compactification X^{BB} has automorphic forms for the discrete subgroup Γ which separate the points X^{BB} and then embed X into projective space with these forms and apply Chow's theorem. The uniqueness comes from

Theorem 2.4. *(Borel, Kwack, Kobayashi) Any holomorphic map $f : (\mathbb{D}\backslash\{0\})^r \times \mathbb{D}^s \to X$ extends to a holomorphic map $f : (\mathbb{D}\backslash\{0\})^r \times \mathbb{D}^s \to X^{BB}$.*

2.3. **Shimura varieties.**

2.3.1. *Definition of Shimura varieties.*

Definition. A *connected Shimura datum* is (G,X) where G is a semisimple algebraic group over $\mathbb{Q}$, $X = G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class of homomorphism $u : U_1 \to G^{\mathrm{ad}}$ such that

(1) $\mathrm{Ad}(u)$ acts by $z, 1, z^{-1}$ on $\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{C}})$;
(2) $\mathrm{Ad}(u(-1))$ is a Cartan involution.
(3) G^{ad} has no $\mathbb{Q}$-factor H with $H(\mathbb{R})$ compact.

Definition. A *connected Shimura variety* is the inverse limit of varieties $\{\Gamma\backslash X\}_\Gamma$ (for Γ small enough and commensurate).

Each $\Gamma\backslash X$ is the quotient of a Hermitian symmetric domain. The $\Gamma\backslash X$ are defined over different fields.

Example 2.3. $\Gamma(N)\backslash\mathcal{H}$ is defined over $\mathbb{Q}(\zeta_n)$.

Definition. A *Shimura datum* is (G,X) where G is reductive algebraic group over $\mathbb{Q}$, $X = G(\mathbb{R})$-conjugacy class of homomorphism $h : \mathbb{S}^1 \to G_{\mathbb{R}}$ such that

(1) $\mathrm{Ad}(h)$ acts on $\mathrm{Lie}(G_{\mathbb{C}})$ with characters $z_1 z_2^{-1}, 1, z_1^{-1} z_2$;
(2) $\mathrm{Ad}(h(i))$ is a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$;
(3) G^{ad} has no $\mathbb{Q}$-factor H with $H(\mathbb{R})$ compact.

A *shimura variety* is $\{G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K\}_K$ where K is a compact subgroup of $G(\mathbb{A}_f)$.

$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K = \bigsqcup_i \Gamma_i\backslash X$ where $\Gamma_i = G(\mathbb{Q}) \cap g_i K g_i^{-1}$ and g_i runs over the representatives of $G(\mathbb{Q})$-K double cosets. By strong approximation, this is a finite union.

2.4. **Examples.**

Example 2.4 (Siegel modular variety)**.** We get this from the variation of Hodge structure $d(0,1) = n$, $d(1,0) = n$ and ψ a symplectic form. The group theoretical way to view this is to look at the group

$$\mathrm{GSp}(V,\psi) = \{g \in \mathrm{GL}(V) \mid \psi(gv, gu) = \lambda(g)\psi(v,u)\}.$$

acting on $\{X \in M_{n\times n}(\mathbb{C}) \mid Z^t = Z, \mathrm{im}(Z) > 0\}$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ+B)(CZ+D)^{-1}.$$

From the Hodge structure description, $GSp(\mathbb{Q})\backslash X \times GSp(\mathbb{A}_f)/K$ parameterizes Abelian varieties of dimension $dim(V)/2$ with a polarization and rigidity data (level structure).

Example 2.5 (Hodge type)**.** A *Hodge type* Shimura variety is one which embeds into a Seigel modular variety. We get these from a variation of Hodge structure $d(0,1) = n$, $d(1,0) = n$, ψ a symplectic form, and some Hodge tensors.

Example 2.6. Let B be a quaternion algebra over F, a totally ramified field, $G(\mathbb{Q}) = B^\times$ for some algebraic group over $\mathbb{Q}$. If

$$G(\mathbb{R}) \cong \mathbb{H}^\times \times \mathbb{H}^\times \times \cdots \times \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$$

have at least one $\mathrm{GL}_2(\mathbb{R})$, define

$$h(a+bi) = 1 \times 1 \times \cdots \times \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \times \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$$

When there is at least one copy of $\mathbb{H}$, (G,X) is not of Hodge type.

2.5. **Canonical models.**

2.5.1. *Canonical models of tori.* Since a torus is abelian, X is always a single point in a Shimura datum (T,X). Also

Proposition 2.1. *Each $T(\mathbb{Q})\backslash\{h\} \times T(\mathbb{A}_f)/K$ is a finite sets of points.*

We can now form a $\mathbb{Q}$ model of $Sh(T,X)$ by turning each point into a $\bar{\mathbb{Q}}$ point. Giving a model of $Sh(T,X)$ over a number field F amounts to giving an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ on the $\bar{\mathbb{Q}}$ points of $\{T(\mathbb{Q})\backslash\{h\} \times T(\mathbb{A}_f)/K\}_K$ commuting with the maps in the inverse limit. This gives an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ on the (topological) inverse limit of $\{T(\mathbb{Q})\backslash\{h\} \times T(\mathbb{A}_f)/K\}_K$, which is $T(\mathbb{Q})\backslash\{h\} \times T(\mathbb{A}_f)$, with a $T(\mathbb{A}_f)$ action on the right coming from the translation maps in the inverse system of the Shimura variety. Using CFT, giving our desired action is equivalent to giving a grouphom

$$\mathrm{Hom}(F^\times\backslash\mathbb{A}_F, T(\mathbb{Q})\backslash T(\mathbb{A}_f)).$$

Notice such a grouphom gives an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ on $T(\mathbb{Q}), \backslash T(\mathbb{A}_f)/K$ for all K.

Here is how we choose this group homomorphism: Define $\mu : \mathbb{G}_m \to \mathbb{S}_\mathbb{C} \to \mathbb{G}_\mathbb{C}$ as the composition of $z \mapsto (z,1)$ and $h_\mathbb{C}$. Choose F to the be the field of definition of μ. Then composing $\mu(\mathbb{A}_F)$ with the norm map $T(\mathbb{A}_F) \to T(\mathbb{A})$ and projecting onto the finite part is our desired grouphom.

2.5.2. *Canonical models of general Shimura varieties.*

Definition. Define $\mu : \mathrm{G}_m \to \mathbb{S}_\mathbb{C} \xrightarrow{h_\mathbb{C}} G_\mathbb{C}$ as the composition of $z \mapsto (z,1)$ and $h_\mathbb{C}$ for some $h \in X$. The *reflex field* of (G,X) is the field of definition of the conjugacy class of μ.

This is a finite extension of $\mathbb{Q}$: In particular

$$E(G,X) = \text{the fixed field of } \{\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) : \sigma(W\mu) = W\mu \text{ in the cocharacter lattice}\}.$$

Definition. The *canonical model* of $\mathrm{Sh}(G,X)$ is the unique model over $E(G,X)$ such that for all morphisms of Shimura data of the form $(T,\{h\}) \to (G,X)$ where T is a torus, the morphisms

$$\{(M(T,\{h\})_K)_\mathbb{C}\} \to \{(M(G,X)_K)_\mathbb{C}\}$$

are defined over $E(G,X) \cdot E(T,\{h\})$. The morphisms $(T,\{h\}) \to (G,X)$ are called *special points.*

Theorem 2.5. *There exists a unique canonical model of* $\mathrm{Sh}(G,X)$.

The uniqueness follows from the special points being very dense:

Lemma 2.1. *For any field extension L of $E(G,X)$, there exists a special point $(T,\{h\})$ of (G,X) such that $E(T,\{h\}) \cap L = E(G,X)$.*

3. Cohomology of Shimura Varieties - Alexander Bauman

3.1. **Modular curves.** Let f be a holomorphic form of weight 2 and level N. Then $f(z)dz \in H^0(X_1(N), \Omega')$ and $\bar{f}(z)d\bar{z} \in H^1(X_1(N), \mathcal{O})$. By Hodge decomposition, we can take their product to obtain an element in $H^1(X_1(N), \mathbb{C}) = H^0(X_1(N), \Omega') \oplus H^1(X_1(N), \mathcal{O})$.

More generally, let f be a cusp form of weight $k \geq 2$. Let $V_k = \mathrm{Sym}^{k-2}(\mathbb{C}^2)$ with the canonical GL_2-action. Then map

$$c_f : \gamma \mapsto \int_i^{\gamma i} f(z) \begin{pmatrix} z \\ 1 \end{pmatrix}^{k-2} dz$$

is an element in $H^1(\Gamma_1(N), V_k)$ (group cohomology) which can be identified with $H^1(Y_1(N), V_k)$. Moreover, since f is cuspidal, f also lies in the cohomology of the compactified modular curve $H^1(X_1(N), V_k)$.

The V_k's make sense l-adically, we can consider $H^1_{\text{et}}(Y_1(N)_{\bar{\mathbb{Q}}}, V_k)_f$ (the component on which Hecke operators like f) has a natural Galois action

$$\begin{array}{ccc} X_f & \xrightarrow{\varphi_2} & X_1(N) \\ \big\downarrow{\scriptstyle \varphi_1} & & \\ X_1(N) & & \end{array}$$

such that $a_{2,*}a_1^* c = a_p c$ where a_p is the p-th Fourier coefficients of f. One can show this Galois representation has the same L-function as f under the Langlands reciprocity.

Remark. The case $k = 1$ cannot be found in the cohomology of local systems. It is studied (and done) by Deligne-Serre.

3.2. **Beyond.** Let G be a connected reductive group over $\mathbb{Q}$ and $K \subseteq G(\mathbb{A}_f)$ a compact open subgroup, $K_\infty \subseteq G(\mathbb{R})$ a maximal compact. Consider the locally symmetric space

$$S_G(K_f) = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\infty K_f Z.$$

Let $\xi : G_{\mathbb{C}} \to \mathrm{GL}(V)$ be a finite-dimension algebraic representation. Then $V_\xi := G(\mathbb{Q}) \backslash V \times G(\mathbb{A}) / K_\infty \times K_f \times Z \twoheadrightarrow S_G(K_f)$ where $G(\mathbb{Q})$ acts on $V \times G(\mathbb{A})$ by $\xi \times m_l$ (left multiplication). We want to study

$$H^*(S_G, V_\xi) = \varinjlim_{K_f} H^*(S_G(K_f), V_\xi).$$

Notice that this has a natural $G(\mathbb{A}_f)$-action. For simplicity, suppose G^{ad} is anisotropic. Then $S_G(K_f)$ is compact.

Theorem 3.1 (Matsushima's Formula)**.**

$$H^*(S_G, V_\xi) = \bigoplus_{\pi \in \mathcal{A}(G)} \pi^\infty \otimes H^*(\mathfrak{g}, K_\infty, \pi_\infty \otimes V_\xi)^{m(\pi)},$$

where $\mathcal{A}(G)$ is the set of automorphic representations of G, π^∞ (resp. π_∞) is the finite (resp. infinite) component of π (i.e., $\pi = \pi_\infty \otimes \pi^\infty$), $m(\pi)$ is the multiplicity of π as automorphic representation, and $H^*(\mathfrak{g}, K_\infty, -)$ is the $(\mathfrak{g}, K_\infty)$-cohomology (which depends only on the infinite component).

If $S_G(K_f)$ are Shimura varieties, then we can get Galois representations

Conjecture 3.1 (Buzzard-Gee). *Let $F/\mathbb{Q}$ be a number fiekd and $\iota_l : \bar{\mathbb{Q}}_l \to \mathbb{C}$ for any prime l. Then there exists a bijection*

$$\left\{\begin{array}{l}\textit{automorphic rep. of } \mathrm{GL}_m(\mathbb{A}_F)\\ \textit{algebraic at } \infty\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\textit{compatible systems of } G(\mathbb{Q})\\ \rho_l(\Pi) : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_m(\bar{\mathbb{Q}}_l)\\ \textit{unramified almost everywhere}\\ \textit{de Rham over } l\end{array}\right\}$$

such that $\mathrm{WD}(\rho_l(\Pi)|_{\mathrm{Gal}(\bar{F}_v/F_v)}) \cong \iota_l^{-1}\mathcal{L}_{F_v}(\Pi_v)$ *for all* $v \not| l$.

Remark. $\to$ is called constructing Galois representation and $\leftarrow$ is called modularity.

Theorem 3.2 (Harris-Lan-Taylor-Thorne, Scholze). $\to$ *exists when F is totally ramified or CM. Π has regular algebraic character.*

Remark. For modular curve this is weight $k \geq 2$.

Definition. An automorphic representation Π of $\mathrm{GL}_m(\mathbb{A}_F)$, with F CM, is *conjugate self-dual* if $\Pi^\vee \cong \Pi \circ c$, where c is the conjugation on F.

3.3. **Langlands-Kottwitz-Rapoport method.** Suppose (G, X) is a Shimura datum, G^{ad} is anisotropic, G has no endoscopy, $E = E(G, X)$ and $d = \dim X$. Then it follows from Theorem 3.1 that

$$H^d_{\mathrm{et}}(\mathrm{Sh}_{\bar{E}}, V_\xi) = \bigoplus_{\pi=\pi^\infty\otimes\pi_\infty, \xi(\pi)=\xi} \pi^\infty \otimes R_{\pi\mu}$$

where $R_{\pi,\mu}$ is the cocharacter depending on X, with an action of $\mathrm{Gal}(\bar{E}/E)$. If π has regular character, then there exists a unique $\xi(\pi)$ such that $G^*(\mathfrak{g}, K_\infty, V_\xi \otimes \pi_\infty) \neq 0$.

We expect $R_{\pi,\mu} = L_\mu \circ \rho_\pi|_{\mathrm{Gal}(\bar{E}/E)}$:

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\rho_\pi} {}^L G_E \xrightarrow{L_\mu} \mathrm{GL}(-).$$

where L_μ is the highest weight representation corresponding to μ (as cocharacter of G and so character of ${}^L G$).

The Langlands-Kottwitz-Rapoport method works as follows:

(1) Construct good integral models of Shimura variety.
(2) Study action of $G(\mathbb{A}^{\infty,p}) \times \mathrm{Frob}_p^{\mathbb{Z}}$.
(3) Relate to trace formula.

Say Π is a conjugate self-dual automorphic representation of $\mathrm{GL}(\mathbb{A}_F)$ with F CM and $F^+ \neq \mathbb{Q}$. Pick a unitary group U with signature $(1, m-1), (0, m), \cdots, (0, m)$ quasisplit at all finite places. Then Π transfers to U (Clozel-Labesse). Use Langland-kottwitz-Rapoport method on the transfer.

4. Integral Models of Shimura Varieties - Patrick Daniels

4.1. **Motivation.** Last time, Alex outlined the Langlands-kottwitz mothod. The idea is to study the action of $G(\mathbb{A}_f^p) \times \mathrm{Frob}_p^{\mathbb{Z}}$ on cohomology. Using Lefschetz trace formula, we can turn this into question about mod p points. The first step is to write down the integral models.

Remark. Different models might have different mod p points, we need the "right" integral models, or in other word we want the "canonical" integral models.

Let (G, X) be a Shimura datum, defined over the reflex field $\mathbb{E}$. Fix $K \subseteq G(\mathbb{A}_f)$ sufficiently small,

$$\mathrm{Sh}_K(G, X)$$

is a quasi-projective variety over $\mathbb{E}$. Fix a prime p, choose v of $\mathbb{E}$ dividing p. Let $E = \mathbb{E}_e$ and $\mathcal{O}_E = \mathcal{O}_{E_v}$. Suppose $K = K_p K^p$ where $K_p \subseteq \mathbb{Q}_p$ and $K^p \subseteq G(\mathbb{A}_f^p)$. The integral model is determined by the K_p part.

Fix K_p and define

$$\mathrm{Sh}_{K_p}(G,X) = \varprojlim_{K^p} \mathrm{Sh}_{K_pK^p}(G,X).$$

Definition. An *integral model* of $\mathrm{Sh}_{K_p}(G,X)$ is a collection of $\mathcal{O}_E$-schemes $(\mathcal{S}_{K^p})_{K^p}$ for every $K^p \subseteq G(\mathbb{A}_f^p)$ sufficeintly small, with

(1) a continuous action of $G(\mathbb{A}_f^p)$: $gK'^pg \subseteq K^p \Rightarrow [g] : \mathcal{S}_{K'^p} \to \mathcal{S}_{K^p}$
(2) $G(\mathbb{A}_f^p)$-equivariant isomorphisms:

$$\gamma_{K^p} : \mathcal{S}_{K^p} \otimes_{\mathcal{O}_E} E \cong \mathrm{Sh}_{K_pK^p}(G,X) \otimes_{\mathbb{E}} E$$

for every K^p.

Let $\mathcal{S}_{K_p} = \varprojlim_{K_p} \mathcal{S}_{K^p}$.

4.2. **Hyperspecial level structure.**

Definition. A subgroup $K_p \subseteq G(\mathbb{Q}_p)$ is *hyperspecial* if there is a reductive $\mathbb{Z}_p$-group scheme $\mathcal{G}$ such that

(1) $\mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = G \otimes_{\mathbb{Q}} \mathbb{Q}_p$, and
(2) $\mathcal{G}(\mathbb{Z}_p) = K_p$.

Such a subgroup exists in $G(\mathbb{Q}_p)$ iff $G_{\mathbb{Q}_p}$ is unramified (quasi-split and split over unramified extension).

Definition. An integral model $\mathcal{S}_{K_p}$ is *smooth* if

(1) each $\mathcal{S}_{K^p}$ is smooth over $\mathcal{O}_E$, and
(2) the transition morphisms $[g] : \mathcal{S}_{K'^p} \to \mathcal{S}_{K^p}$ are finite étale.

Conjecture 4.1 (Langlands)**.** *When K_p is hyperspeical, there should exist a smooth integral model for* $\mathrm{Sh}_{K_p}(G,X)$.

Definition. $\mathcal{S}_{K^p}$ satisfies the *extension property* if for every regular formally smooth $\mathcal{O}_E$-scheme $\mathcal{S}$, any morphism

$$\mathcal{S} \otimes_{\mathcal{O}_E} E \to \mathrm{Sh}_{K_p}(G,X)$$

extends uniquely to

$$\mathcal{S} \to \mathcal{S}_{K_p} = \varprojlim_{K^p} \mathcal{S}_{K^p}.$$

Lemma 4.1. *There exists at most one smooth integral model $\mathcal{S}_{K_p}$ with the extension property.*

Proof. Let $\mathcal{S}'_{K^p}$ be another one. Then $\mathcal{S}'_{K^p}$ is regular formally smooth, so

$$\gamma'_{K_p}\mathcal{S}'_{\mathcal{O}_E}E \cong \mathrm{Sh}_{K_p}(G,X)$$

extends to $\mathcal{S}'_{K^p} \to \mathcal{S}_{K^p}$. Similarly we get $\mathcal{S}_{K^p} \to \mathcal{S}'_{K^p}$. By uniqueness the composition is the identity. □

Definition. A smooth integral model satisfying the extension property is called a *canonical integral model.*

Example 4.1. Canonical integral models exist in the following cases:

(1) Siegel case (Mumford, Milne, Moonen)
(2) PEL case (Zink, Langlands-Rapoport, Kottwitz)
(3) Hodge and abelian type (Kisin for $p > 2$, Kim-Madapusi for $p = 2$).

Remark. This only makes sense for hyperspeical K_p. See recent work of Pappas-Rapoport.

4.3. **The Siegel case.** Let (V, ψ) be a symplectic sapce of dimension $2g$ over $\mathbb{Q}$, so

- V is a $2g$-dimensional $\mathbb{Q}$-vector space
- ψ is a nondegenerate alternating form.

For example, $\psi_{\text{std}} = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$.

Definition. The algebraic group GSp is defined by

$$\mathrm{GSp}(R) = \{g \in \mathrm{GL}(V \otimes_{\mathbb{Q}} R) \mid \psi(gv, gw) = c(g)\psi(v, w) \text{ for some } c(g) \in R^\times\}.$$

Let $S^\pm$ be the Siegel double space ($g \times g$ symmetric matrices over $\mathbb{C}$, positive or negative definite). Then $(\mathrm{GSp}, S^\pm)$ is a Shimura datum and $\mathrm{Sh}(\mathrm{GSp}, S^\pm)$ is *Siegel modular variety.* This has a moduli interpretation: for N with p not dividing N, define a functor $\mathcal{A}_{g,N}$ in $\mathbb{Z}_{(p)}$-schemes:

$$\mathcal{A}_{g,N}(S) = \{(A, \lambda, \eta) \mid (A, \lambda) \text{ is principle polarized abelian variety}, \eta \text{ is level } N \text{ structure}\}/\sim$$

where a level N structure is an isomorphism

$$\eta : A[N] \cong \underline{\mathbb{Z}/N\mathbb{Z}}_S^{2g}$$

s.t. there exists isomorphism $\varphi : \underline{\mathbb{Z}/N\mathbb{Z}}_S \to \mu_{N,S}$ such that

$$\varphi \circ \psi \circ (\eta \times \eta) = e^\lambda : A[N] \times A[N] \to \mu_N.$$

Choose a $\mathbb{Z}$-lattice $V(\mathbb{Z}) \subseteq V$ and define

$$K(N) = \{g \in \mathrm{GSp}(\mathbb{A}_f^p) \mid g \equiv 1 \mod NV(\mathbb{Z})\}.$$

Theorem 4.1. *$\mathcal{A}_{g,N}$ is representable by a smooth $\mathbb{Z}_{(p)}$-scheme and*

$$\mathcal{A}_{g,N} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \cong \mathrm{Sh}_{K(N)}(\mathrm{GSp}, S^\pm).$$

Theorem 4.2 (Milne-Moonen)**.** *$(\mathcal{A}_{g,N})_N$ is a canonical integral model for $\mathrm{Sh}_{K_p}(\mathrm{GSp}, S^\pm)$ where $K_p = \mathrm{GSp}(\mathbb{Z}_p)$.*

Sketch of proof. Let $\mathcal{S}$ be a regular formally smooth scheme over $\mathbb{Z}_p$ and let

$$\mathcal{S} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathrm{Sh}_{K_p}(\mathrm{GSp}, S^\pm)$$

be the map corresponding to projective system $(A, \lambda, \eta N)_N$. For any $l \neq p$, get a trivialization of Tate module $T_l(A)$. If $S = \operatorname{Spec} R$ is a DVR, it follows from Néron-Ogg-Shafarevich that A extends to R. $\square$

5. The Langlands-Rapoport Conjecture - Tasho Kaletha

5.1. **Basic idea.** We start with a Shimura datum (G, X). Take $K \subseteq G(\mathbb{A}_f)$ compact open and sufficiently small. We get quasi-projective smooth variety $\mathrm{Sh}_K(G, X)$ over the reflex field $\mathbb{E}$. When K is small and K_p is hyperspecial, we have integral models $\mathcal{S}_{K_p}$ over $\mathcal{O}_E$ ($E = \mathbb{E}_v$, $v \mid p$) of

$$\mathrm{Sh}_{K_p} = \varprojlim_{K^p} \mathrm{Sh}_K$$

which is canonical and smooth. We can look at the special fiber $\bar{\mathcal{S}}_{K_p} = \mathcal{S}_{K_p} \times_{\mathcal{O}_E} k$. On the one hand, the cohomology of the special fiber is related to the cohomology of the generic fiber, which is the Shimura variety, and tells us information about automorphic representation. On the other hand, they are related to the Hasse-Weil ζ-functions.

Therefore, we want to understand $\bar{\mathcal{S}}_{K_p}$ together with the $G(\mathbb{A}_f^p) \times \mathrm{Frob}_p$.

We can stop to think about what the Langlands-Rapoport conjecture lead us. The philosophy is that Shimura varieties parametrize motives. Last time, we have seen that in the Siegel case, Shimura varieties parametrize principle polarized abelian varieties.

Example 5.1 (Shimura varieties of Hodge type)**.** In this case, we have embedding $(G, X) \to (\mathrm{GSp}_{2g}, S^{\pm})$. Now we have a result of linear algebraic groups, saying that for any closed subgroup H of GL_n, we can find a bunch of tensors such that H is the stabilizer of these tensors. With this in mind, we can think of the Shimura varieties as moduli space of motives with G-structure.

5.2. **Motives.**

5.2.1. *The false category of motives.* We start with the category of smooth projective varieities over $\bar{\mathbb{F}}_p$. The morphisms (X, Y) consist of algebraic cycles on $X \times Y$ of $\deg X$, modulo equivalence relation (numerical equivalence). The Karoubi envelope is called the category of *effective motives.* $\mathbb{P}^1 = \{*\} \sqcup \mathbb{L}$, where $\mathbb{L}$ s called the Lefschetz motive. Formally adjoin the Tate motive, which is $\mathbb{T} = \mathbb{L}^{-1}$. Call the result cateogry $\dot{M}$. It is equipped with the following structures:

- $\otimes$, coming from Cartesian product of varieties;
- $\mathbb{Q}$-linear abelian category
- faithful exact $\otimes$-functors into different *realizaiton categories*
  - l-adic cohomology into $\mathbb{Q}_l$-vector spaces
  - crystalline cohomology, into isocrystals ($\breve{\mathbb{Q}}_p$-vector spaces equipped with Frobenius-semilinear automorphism)
  - Hodge structure functor into $\mathbb{C}$-vector spaces with $\mathbb{Z}$-grading, with semilinear automorphism α such that $\alpha^2 = (-1)^{\deg}$ on each graded pieces.
- $\dot{M}$ is semisimple with simple objects parametrized by the Weil numberes π, by $X \mapsto \pi$ if $\mathrm{End}(X)$ is division algebra with center $\mathbb{Q}(\pi)$

5.2.2. *Tannakian category.* Let G be an algebraic group over $\mathbb{Q}$, $\mathrm{Rep}(G)$ is a $\mathbb{Q}$-linear tensor category with an exact faithful $\mathbb{Q}$-linear functor $\omega : \mathrm{Rep}(G) \to \mathrm{Vect}_{\mathbb{Q}}$ given by forgetting the G-structure.

Given $\mathbb{Q}$-algebra R, consider $\mathrm{Aut}^{\otimes}(\omega_R) = \{\lambda_X \in \mathrm{End}(\omega(X) \otimes_{\mathbb{Q}} R), \text{functorial in } X, \lambda_{X \otimes Y} = \lambda_X \otimes \lambda_Y\}$.

Theorem 5.1 (Tannakian Reconstruction Theorem)**.** *The natural homomorphism $G \to \mathrm{Aut}^{\otimes}(\omega)$ is an isomorphism.*

The categories obtainable this way are *rigid* abelian category, which means that

- there is internal $\underline{\mathrm{Hom}}(X, Y) \in \mathcal{C}$ such that
$$\mathrm{Hom}(T, \underline{\mathrm{Hom}}(X, Y)) = \mathrm{Hom}(T \otimes X, Y);$$
- the natural map $\underline{\mathrm{Hom}}(X_1, Y_1) \times \underline{\mathrm{Hom}}(X_2, Y_2) \cong \underline{\mathrm{Hom}}(X_1 \otimes X_2, Y_1 \otimes T_2)$;
- every object is *reflexive*, i.e., $(X^{\vee})^{\vee} \cong X^{\vee}$, where $X^{\vee} = \underline{\mathrm{Hom}}(X, 1)$.

Definition. A rigid abelian $\mathbb{Q}$-linear tensor category that admits a $\mathbb{Q}$-linear exact faithful tensor functor ω is called a *neutral Tannakian category*, and ω is called a *fiber functor.*

Theorem 5.2. *There is a bijection between affine $\mathbb{Q}$-group scheme and pairs $(\mathcal{C}, \omega)$ of neutral Tannakian category and fiber functor.*

What if only $\mathcal{C}$ is given? Pick ω, we get G_ω. Given ω_1, ω_2, we have
$$\mathrm{Hom}^{\otimes}(\omega_1, \omega_2)$$
is a torsor for G_{ω_1} (G_{ω_2}) under fpqc topology.

Proposition 5.1. *Fix ω, then $\omega' \mapsto \mathrm{Hom}^{\otimes}(\omega, \omega')$ is a bijection between fiber functors and G_ω-torsor.*

Definition. A *Tannakian category* is a rigid abelian $\mathbb{Q}$-linear tensor category taht admits a fiber functor defined over an extension of $\mathbb{Q}$.

5.2.3. *Gerb.* The gerb of $\mathcal{C}$ is the category of all its fiber functors.

Example 5.2. If $(\mathcal{C}, \omega)$ is a neutral Tannakian cateogry, the gerb is $\mathrm{Tors}(G)$.

Theorem 5.3. *If $\mathcal{C}$ is a Tannakian cateogry, then the natural functor $\mathcal{C} \to \mathrm{Rep}(\mathrm{Fib}(\mathcal{C}))$ is an equivalence.*

If $\mathcal{G}$ is a gerb, then the natural functor

$$\mathcal{G} \to \mathrm{Fib}(\mathrm{Rep}(\mathcal{G}))$$

is an equivalence.

5.2.4. *The true category of motives.* If $\mathcal{C}$ is a $\mathbb{Q}$-linear tensor category, we have

$$\underline{\mathrm{Hom}}(X, X) = X^\vee \otimes X \xrightarrow{ev} 1.$$

This leads us to the *trace morphism*

$$\mathrm{Hom}(X, X) = \mathrm{Hom}(1 \otimes X, \otimes) = \mathrm{Hom}(1, \underline{\mathrm{Hom}}(X, X)) \xrightarrow{\mathrm{Hom}(1,\mathrm{ev})} \mathrm{Hom}(1, 1) = \mathbb{Q}.$$

Any tensor functor respects this. But on Vect,

$$\mathrm{tr}(\mathrm{id}) = \dim$$

while on $\dot{M}$, we have

$$\mathrm{tr}(\mathrm{id}) = \sum (-1)^i \dim H^i.$$

Therefore $\dot{M}$ fails to be a Tannakian category.

We can fix this by modify the morphism $X \otimes Y \to Y \otimes X$ on the graded pieces. It is then a theorem of Deligne that this makes it a Tannakian category.

5.2.5. *Galois gerb.* A *Galois gerb* is an extension

$$1 \to G(\bar{\mathbb{Q}}) \to \mathcal{G} \to \Gamma_{\mathbb{Q}} \to 1$$

where G is an affine algebraic $\bar{\mathbb{Q}}$-group, that splits over some finite extension K, which means that one can find section $\Gamma_{\mathbb{Q}} \supseteq \Gamma_K \to \mathcal{G}$

A *representation* of $\mathcal{G}$ is a homomorphism of extension

$$\mathcal{G} \to \mathcal{GL}(V) = \{(\phi, \sigma) \mid \sigma \in \Gamma_{\mathbb{Q}}, \phi : V \cong V^\sigma = V \otimes_\sigma \sigma\}.$$

5.2.6. *Pseudo-motivic Galois gerb.* Let $W(p^\infty)$ be the subgroup of $\mathbb{C}^\times$ generated by all p^∞-Weil numbers. Let P be the pro-torus with $X^*(P) = W(p^\infty)$. We want extension

$$1 \to P \to \mathcal{P} \to \Gamma_{\mathbb{Q}} \to 1,$$

i.e., a class in $H^2(\Gamma, P)$.

Lemma 5.1. $H^2(\Gamma, P) \hookrightarrow \prod_v H^2(\Gamma_v, P)$.

This means that a Galois gerb can be described as a compatible system of gerbs at each places. Consider the Galois gerb given such that under the embedding above, each place is given by:

- In the l-adic case, we have $\mathrm{Gal}(\bar{\mathbb{Q}}_p) = \mathcal{P}_l$, i.e., the trivial Galois gerbe
- In the crystalline case, we have the *Dieudonne gerb*

$$1 \to \mathbb{G} = \varprojlim \mathbb{G}_m \to D \to \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to 1$$

  where the morphisms between $\mathbb{G}_m$'s are power maps.
- At ∞ we have $W_{\mathbb{C}/\mathbb{R}} = \mathcal{P}_\infty$, where $W_{\mathbb{C}/\mathbb{R}}$ is Weil group

$$1 \to \mathbb{C}^\times \to W_{\mathbb{C}/\mathbb{R}} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1.$$

5.3. **The statement of the conjecture.**

Conjecture 5.1 (Langlands-Rapoport). *There is an bijection*

$$\mathcal{S}_{K_p}(\bar{k}) = \bigsqcup_{\varphi} I_\varphi(\mathbb{Q})\backslash X_p(\varphi) \times X^p(\varphi)$$

which is compatible with the $G(\mathbb{A}_f^p) \times \mathrm{Frob}_p$*-action.*

Now we explain the notation.

$\varphi : \mathcal{P} \to G(\bar{\mathbb{Q}}) \rtimes \Gamma_{\mathbb{Q}}$, is called *admissible motives with G structure.* Start with Shimura datum (G, X), $h : \mathbb{S} \to G_{\mathbb{R}} \to \mathbb{G}_{\mathrm{ab},\mathbb{R}}$. Base change to $\mathbb{C}$, we have

$$h_{\mathrm{ab}} : \mathbb{S}_{\mathbb{C}} = \mathbb{C}^\times \times \mathbb{C}^\times \to \mathbb{G}_{\mathrm{ab},\mathbb{C}}.$$

μ_{ab} is restriction of h_{ab} on $\mathbb{C}^\times \times \{1\}$, which gives us $G_{\mathrm{ab}} \rtimes \Gamma_{\mathbb{Q}} \to \mathcal{P}$.

For φ and the Dieudonne gerb $\mathcal{D}$, we have homomorphism

$$D \to \mathcal{P}_p^{\mathrm{un}} \xrightarrow{\varphi_p^{\mathrm{un}}} G(\breve{\mathbb{Q}}_p) \rtimes \Gamma_p^{\mathrm{un}}.$$

For some n, the composite factors through D_n,

$$\begin{aligned} X_l(\varphi) &= \{g \in G(\mathbb{Q}_l) \mid \mathrm{Ad}(g)\varphi_l = \mathrm{Std}_l\} \\ X_p(\varphi) &= \{g \in G(\breve{\mathbb{Q}}_p) \mid g^{-1} b\sigma(g) \in G(\breve{\mathbb{Z}}_p)\mu(p)G(\breve{\mathbb{Z}}_p)\} \\ X^p(\varphi) &= \prod_{l \neq p} X_l(\varphi). \end{aligned}$$

One the second line we have the notations (b, μ). b comes from The Dieudonne gerb at the crystalline. More precisely, let L_n be the subfield of $\mathbb{Q}_p^{\mathrm{un}}$ of degree n over $\mathbb{Q}_p$. Then $D = \varprojlim D_n$, where D_n has a section over $\Gamma_{L_n} \subseteq \Gamma_{\mathbb{Q}_p}$. There is a canonical element in D_n mapping to σ, and let (b, σ) denote its image in $G(\mathbb{Q}_p^{\mathrm{un}}) \rtimes \Gamma_p^{\mathrm{un}}$ under the map $\phi_p^{\mathrm{un}} : \mathcal{P}_p^{\mathrm{un}} \to (\mathbb{Q}_p^{\mathrm{un}}) \rtimes \Gamma_p^{\mathrm{un}}$. The image $b(\varphi)$ of b in $G(L)$ is well-defined up to σ-conjugacy, i.e., if $b(\varphi)'$ also arises in this way, then $b(\varphi)' = g^{-1} \cdot b(\varphi) \cdot \sigma g$. μ is the one from Definition 2.5.2, which is a cocharacter of G.

$I_\varphi(\mathbb{Q})$ is the $\mathbb{Q}$-points of the centralizers of φ in G.

6. Trace Formula for Cohomology of Shimura Varieties - Alexander Bertoloni Meli

6.1. **Motivation.** Fix $(G, X, p, \mathcal{G})$ where

- (G, X) is a Shimura datum;
- p is a prime such that $G_{\mathbb{Q}_p}$ is unramified;
- $\mathcal{G}$ is a reductive model of $G_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$;
- $K_p = \mathcal{G}(\mathbb{Z}_p)$;
- E reflex field of
- $\mathbb{F}_q$ residue field of lift of p to E;
- ξ an algebraic representation of G.

Our goal is to compute

$$T(\Phi^n \times f) = \sum_i (-1)^i \operatorname{tr}(\Phi^n \times f \mid H_c^i(\mathrm{Sh}, \xi)).$$

As Alex has introduced, the algebraic representation ξ gives rise to a sheaf on Shimura variety, and its l-adic cohomology ($l \neq p$) has an action of $\mathrm{Gal}(\bar{E}/E) \times G(\mathbb{A}_f)$. Here Φ is the geometric Frobenius, and $f = f^p 1_{K_p}$ where $f^p = \sum_g 1_{K^p g^{-1} K^p} \in \mathcal{H}(G(\mathbb{A}_f^p), K^p)$.

We want to relate $T(\Phi^n \times f)$ to $\underline{\mathrm{Sh}}_{K_p}(\bar{\mathbb{F}}_q)$. We have some base change maps

$$\mathrm{Sh}_{\bar{E}} \to \underline{\mathrm{Sh}} \leftrightarrow \underline{\mathrm{Sh}}_{\bar{\mathbb{F}}_q}.$$

By push-forward and pull-back, we get adjunction:

$$H^i_c(\mathrm{Sh}_{\bar{\mathbb{F}}_q}, \xi) \to H^i_c(\mathrm{Sh}_{\bar{E}}, \xi).$$

It turns out the this is an isomorphism (this is true if they are smooth and proper; unfortunately, these are not proper in general, but we still have the isomorphism). Therefore, we can instead compute the trace on cohomology of the special fiber,

$$T(\Phi^n \times f)_{\bar{S}} = \sum_i (-1)^i \operatorname{tr}(\Phi^n \times f \mid H^i_c(\mathrm{Sh}_{\bar{\mathbb{F}}_q}, \xi)).$$

using Lefschetz-Verdier trace formula.

We think of $f = 1_{K_p} 1_{K^p g^{-1} K^p}$, which acts on $\underline{\mathrm{Sh}}_{K,\mathbb{F}_q}$ by Hecke correspondence. So the twisted version $\Phi^n \times f$ acts by correspondence

$$\underline{\mathrm{Sh}}_{K,\mathbb{F}_q} \xleftarrow{\pi(g)} \underline{\mathrm{Sh}}_{K_p K^p_g} \xrightarrow{\Phi^n \circ \pi(1)} \underline{\mathrm{Sh}}_{K,\mathbb{F}_q}$$

where $K^p g = K^p \cap gKg^{-1}$.

Theorem 6.1 (GLV-trace formula)**.** *Assume $\Phi^n \times f$ acts by correspondence*

$$X \xleftarrow{c_1} C \xrightarrow{c_2} X.$$

Then we have

$$T(\Phi^n \times f)_{\bar{S}} = \sum_{y \in \mathrm{Fix}} \operatorname{tr}(\Phi^n \times f, \xi_{c(y)})$$

where $\mathrm{Fix} = \{y \in \mathrm{Sh}_{K_p}(\bar{\mathbb{F}}_q) : c_1(y) = c_2(y)\}$.

Therefore, we need a good understading of $\underline{\mathrm{Sh}}_{K_p}(\bar{\mathbb{F}}_q)$. In general we only have The Langlands Rapoport Conjecture 5.1. But we have some good understanding in the Siegel case.

6.2. **Siegel case.**

6.2.1. *Moduli interpretation.* Let $M_{g,N}$ be the moduli space of principal polarized abelian variety of dimension g with level N structure, where $p \nmid N$. This means that we have a triple (A, λ, φ) where

- $\lambda : A \cong A^\vee$;
- $\varphi : H^1(A_{\bar{F}_q}, \mathbb{Z}/N\mathbb{Z}) \cong V_{\mathbb{Z}/N\mathbb{Z}}$ such that φ takes $\langle -, - \rangle_\lambda$ to $\langle -, - \rangle$, where V is a $\mathbb{Z}$-module of rank $2g$.

To compute $M_{g,N}(\mathbb{F}_q)$, there are two steps:

(1) compute the size of fixed $\mathbb{Q}$-isogeny class;
(2) compute the number of isogeny classes.

Here $(A, \lambda, \varphi) \sim_{\mathbb{Q}} (A', \lambda', \varphi')$ means that there exists $f : A \to A'$ such that $\hat{f} \circ \lambda' \circ f = c\lambda$ for some $c \in \mathbb{Q}^\times$.

Proposition 6.1. *Fix A_0, giving a $\mathbb{Q}$-isogeny $f : A \to A_0$ this is same as giving a pair (λ^p, Λ_p) where we have lattices*

$$\Lambda^p \subseteq H^1(A_0, \mathbb{A}^p_f), \Lambda_p \subseteq H^1_{crys}(A_0/W) \otimes K$$

where W is the Witt vector of $\mathbb{F}_q$, and K is the fraction field of W, such that

- *Λ^p is fixed by $\Gamma = \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$;*
- *$f(\Lambda_p) \subseteq \Lambda_p$, $V(\Lambda_p) \subseteq \Lambda_p$.*

In summary, (A, λ, φ) that is $\mathbb{Q}$-isogenous to (A_0, λ_0) gives rise to an element of $Y^p \times Y_p$ where

$$Y^p = \{(\Lambda, \varphi_\Lambda) \mid \Lambda \subseteq H^1(A_0, \mathbb{A}^p_f), \varphi_\Lambda : \Lambda/N\Lambda \cong V_{\mathbb{Z}/N\mathbb{Z}}, \Lambda^\perp = c\Lambda, c \in \mathbb{A}^p_f\}$$
$$Y_p = \{\Lambda \subseteq W_p \mid \Lambda^\perp = c\Lambda, f(\Lambda) \subseteq \Lambda, V(\Lambda) \subseteq \Lambda, c \in K^\times\}.$$

We get a bijection

$$[M_{g,N}(\mathbb{F}_q)]_{(A_0,\lambda_0)} \leftrightarrow I(\mathbb{Q})\backslash Y^p \times Y_p.$$

But this is not good enough. We want a group-theoretical description of Y^p and Y_p, so that we can relate to the orbital integral.

6.2.2. *Group theoretical description.* Fix

$$W^p \cong V \otimes \mathbb{A}_f^p, W_p \cong V \otimes K.$$

The Frobenius element in $\Gamma \rightsquigarrow \gamma \in G(\mathbb{A}_f^p)$, and $f \rightsquigarrow \delta\sigma$, where $\delta \in G(K)$, σ the lift of Φ in K. Then

$$\begin{aligned} Y^p &\cong \{g \in G(\mathbb{A}_f^p)/K_N^p \mid g^{-1}\gamma g \in K_N^p\} \\ Y_p &\cong \{g \in G(K)/G(W) \mid g^{-1}\delta\sigma(g) \in G(W)aG(W), a \in G(K), a(V \otimes W) \subseteq V \otimes W\}. \end{aligned}$$

So

$$\begin{aligned} \#[M_{g,N}(\mathbb{F}_q)]_{(A_0,\lambda_0)} &= \int_{I(\mathbb{Q})\backslash G(\mathbb{A}_f^p)\times G(K)} 1_{K^pN}(g^{-1}\gamma g) 1_{G(W)aG(W)}(g^{-1}\delta\sigma(g)) \\ &= \mathrm{Vol}(I(\mathbb{Q})\backslash I(\mathbb{A}_f)) \cdot \mathcal{O}_\gamma(1_{K_N}^p) \cdot \mathcal{TO}_\delta(1_{G(W)aG(W)}). \end{aligned}$$

The isogeny classes are parametrized by additional $\gamma_0 \in G(\mathbb{Q})/G(\bar{\mathbb{Q}})$-conjugacy, ellitpic in $G(\mathbb{R})$, and we have the Kottwitz triple $(\gamma_0;\gamma,\delta)$. $\alpha(\gamma_0;\gamma,\delta) \in K(I_0/Q)^D$.

$$T(\Phi^n, f) = \sum_{\gamma_0, \alpha(\gamma_0;\gamma,\delta)=1} \sum_{\gamma,\delta} c(\gamma_0;\gamma,\delta)\mathcal{O}_\gamma(f^p)\mathcal{TO}_\delta(\phi_n)\,\mathrm{tr}(\xi|\gamma_0).$$

We can simplify by adding the factor

$$|K(I_0/\mathbb{Q})|^{-1} \sum_{\kappa \in K(I_0/\mathbb{Q})} \langle \alpha(\gamma_0;\gamma,\delta), k\rangle = \begin{cases} 1 & \text{if } \alpha(\gamma_0;\gamma,\delta) = 1 \\ 0 & \text{otherwise} \end{cases},$$

so that we can take sum over all $(\gamma_0;\gamma,\delta)$:

$$\sum_{\gamma_0}\sum_{\gamma,\delta}\sum_{\kappa} \cdots$$

This is where the idea of endoscopy goes in:

$$(\gamma_0, \kappa)/\sim \leftrightarrow (H, s, \eta, \gamma_H)/\sim.$$

So we get stable orbital integral $\mathcal{SO}_{\gamma_H}(h)$ (because κ is related to the transfer factor) as trace of representation:

$$T(f \times \Phi^n) = \sum_H \sum_{\gamma_H} c(G,H)\mathcal{SO}^H_{\delta_H}(h).$$

7. The Kottwitz set $B(G)$ and the refined local and global conjectures - Andy Gordan

7.1. **Kottwitz set.**

7.1.1. *Notations.* Let F be a finite extension of $\mathbb{Q}_p$ and $L = \breve{F}$ be the completion of F^{un}. Let $\sigma \in \mathrm{Gal}(F^{\mathrm{un}}/F)$ be the p-th power Frobenius. Since L is the completion of F^{un}, we can also think of $\sigma \in \mathrm{Gal}(L/F)$.

Let G be a connected affine algebraic group defined over F.

7.1.2. *Basic definition.*

Definition. The *Kottwitz set* of G is

$$B(G) = \{b \in G(L)\}/b \sim gb\sigma(g)^{-1}.$$

Example 7.1. Assume V is a finite dimensional vector space over F and $G = \mathrm{GL}(V)$. From this we get (V_L, Φ_b) where $V_L = V \otimes_F L$ is a finite dimensional vector space over L and

$$\Phi_b = b \circ (\mathrm{id} \otimes \sigma) : V \otimes_F L \to V \otimes_F L$$

is a Frobenius semilinear map, in the sense that

$$\Phi_b(\alpha v) = \sigma(\alpha)\Phi_b(v), \alpha \in L, v \in V_L.$$

This kind of σ-L-space is called *isocrystal.* Two isocrystals are isomorphic are isomorphic if they fit into the following commutative diagram

$$\begin{tikzcd} W_1 \arrow[r, "T"] \arrow[d, "\Phi_1"] & W_2 \arrow[d, "\Phi_2"] \\ W_1 \arrow[r, "T"] & W_2 \end{tikzcd}.$$

(V_L, Φ_b) is isomorphic to $(V_L, \Phi_{b'})$ iff $b \sim b'$.

Theorem 7.1 (Dieudonne, Manin)**.** *The category of σ-L-spaces is abelian and semisimple with simple objects $E_{s/r}$, $(r, s) = 1$, giving by basis and Φ on the basis:*

$$\Phi : \langle v_1, \cdots, v_{r-1}, v_r\rangle \mapsto \langle v_2, v_3, \cdots, \pi^s v_1\rangle.$$

Then we can extend this to a σ-semilinear map.

7.1.3. *First invariance: Newton map.* Let $\mathbb{D}$ be the pro-algebraic torus with $X^*(\mathbb{D}) \cong \mathbb{Q}$. The category of σ-L-spaces is equivalent ot the category of $\mathbb{D}$-representations by Theorem 7.1. Therefore, we each $b \in B(\mathrm{GL}(V))$, we get $\nu_b \in \mathrm{Hom}_L(\mathbb{D}, \mathrm{GL}(V))$ up to conjugacy. This gives us an isomorphism:

$$\nu : B(\mathrm{GL}(V)) \to (X_*(\mathrm{GL}(V))_{\mathbb{Q}}/G)^{\sigma}.$$

For general G and $b \in G(L)$, by Tannakian formalism $\mathrm{Rep}(G) \to \mathrm{Rep}(\mathbb{D})$ we can obatin $\nu_b \in \mathrm{Hom}_L(\mathbb{D}, G)$

we can obtain a similar map (not necessarily isomorphism):

$$n : B(G) \to (\mathrm{Hom}_L(\mathbb{D}, G)/G)^{\sigma}$$

called the *Newton map.*

Let H be a quasi-split group defined over F. Let A be its maximal split torus over F and B be the Borel defined over F. Let $\Psi : G_{\bar{F}} \to H_{\bar{F}}$. Let $\bar{\mathcal{C}}$ be the closed positive chamber of $X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$. For $b \in B(G)$ and $\nu_b : \mathbb{D} \to G$, we can conjugate $\Psi \circ \nu_b : \mathbb{D} \to H$ to lie in $\bar{\mathcal{C}}$.

7.1.4. *Second invariance: Kottwitz map.* We have the *Kottwitz map*

$$\kappa : B(G) \to X^*(Z(\hat{G})^{\Gamma_F}).$$

Example 7.2. Let $G = T$ be a torus. Then by duality the RHS can be identified with the Γ-coinvariant of the cocharacter group of the dual torus $X_*(T)_\Gamma$. Under this identification, the Kottwitz map is given by

$$\begin{aligned} \kappa : B(T) &\to X_*(T)_\Gamma \\ [N_{E/E_0}\mu(\pi_E)] &\leftarrow\!\shortmid [\mu] \end{aligned}$$

where T splits over E and E_0 is the unramified part of E.

7.1.5. *Basic elements.*

Definition. An element $[b] \in B(G)$ is *basic* if ν_b has central image. Let $B(G)_{\mathrm{bas}}$ be the set of basic element in $B(G)$. The Kottwitz map gives an bijection

$$\kappa : B(G)_{\mathrm{bas}} \cong X^*(Z(\hat{G})^{\Gamma}).$$

For $b \in G(R)$, we can define a group scheme J_b over F with functor of points:

$$J_b(R) = \{g \in G(R \otimes L) \mid g = b\sigma(g)b^{-1}\}.$$

If $b \sim b'$, then J_b is F-isomorphic to $J_{b'}$. If b is basic, then J_b is an inner form of G.

Now assume G is quasi-split, with T a maximal torus in B, the Borel defined over F. For $\mu \in X_*(T)$, we get $\mu_1 \in X^*(Z(\hat{G})^{\Gamma})$. We can also take the Galois average

$$\mu_2 = [\Gamma : \Gamma_\mu]^{-1} \sum_{\tau \in \Gamma/\Gamma_u} \tau\mu \in \bar{\mathcal{C}}.$$

Define $B(G, \mu) = \{b \in B(G) \mid \kappa(b) = \mu_1, n(b) \leq \mu_2\}$.

Proposition 7.1. *(1) $B(G, \mu)$ is finite.*
(2) This set contains exactly one basic element.

Example 7.3. Let $F = \mathbb{Q}_p$. Consider $B(\mathrm{GL}_2, \mu)$. Let $\mu(a) = \left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)$. Let $b = \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)$ and $d = \left(\begin{smallmatrix} & p \\ 1 & 0 \end{smallmatrix}\right)$. Then $(V_L, \Phi_b) = E_0 \oplus E_1$. $(V_L, \Phi_d) = E_{\frac{1}{2}}$.

For GL_2, κ is just the valuation of the determinant.

7.2. **Refined LLC.** Local Langlands correspondence is a conjectural finite-to-one surjection

$$\{\text{tempered representations of } G(F)\} \to \{\varphi : W_F \times \mathrm{SL}_2 \to \hat{G} \rtimes \Gamma\}.$$

It turns out that for different inner forms of G, the L-group is canonically isomorphic. So it's Vogan's idea to consider the compound L-packet.

$$\bigsqcup_{b \in B(G)_{\mathrm{bas}}} \Pi_{\varphi}^{J_b}.$$

Consider $S_\varphi = \mathrm{Cent}(\varphi, \hat{G})$.

Conjecture 7.1 (Kaletha)**.** *We have the commutative diagram*

$$\begin{tikzcd}
\bigsqcup_{b \in B(G)_{bas}} \Pi_{\varphi}^{J_b} \arrow[r, leftrightarrow] \arrow[d] & \mathrm{Irr}(S_\varphi/(S_\varphi \cap \hat{G}_{der})^{\circ}) \arrow[d] \\
B(G)_{bas} \arrow[r, leftrightarrow, "\kappa"] & X^*(Z(\hat{G})^{\Gamma}).
\end{tikzcd}.$$

In other words, the $\Pi_{\varphi}^{J_b}$ is identified with the irreducible algebraic representations of S_φ which is trivial on $(S_\varphi \cap \hat{G}_{der})^{\circ}$ and restricts to character $\kappa(b)$ on $Z(\hat{G})^{\Gamma}$.

8. p-adic Uniformization and Rapoport-Zink Space - Kartik Prasanna

8.1. **Background.**

8.1.1. *History.* In 1962, Tate realize the p-adic uniformization of elliptic curves with multiplicative split reduction.

In 1970, Raynaud bring formal schemes into the story (which can be thought of the integral model of rigid analytic space).

In 1972, Mumford realized p-adic uniformization of Mumford curves and abelian varieties, which led to the degeneration of abeilan varieties.

In 1976, Cerednik showed that Shimura curves admit p-adic uniformization; Drinfeld gave a more explicit conceptual proof of Cerednik's result, by introducing the Drinfeld space.

In 1990s, Rapoport-Zink generalized Drinfeld's construction to other groups.

8.1.2. *Some applications.*

(1) Let E be a elliptic curve with multiplicative reduction at p. Then there exists a unique $q \in \mathbb{Z}_p$, $v_{p(q)} > 0$ such that
$$\mathbb{G}_m^{a^n}/q^{\mathbb{Z}} \cong E_{\mathbb{Q}_p}^{a_n}$$
as rigid analytic spaces. For finite extension $L/\mathbb{Q}_p$, one has Galois-equivariant $L^\times/q^{\mathbb{Z}} \cong E(L)$. Therefore one can study the Galois action on torsion of elliptic curves. We have non-split short exact sequence
$$0 \to \mathbb{Q}_l(1) \to V_l(E) \to \mathbb{Q}_l \to 0.$$
When $l \neq p$, we can compute the extension class and check its local Langlands correspondence. When $l = p$, one can compute various p-adic Hodge theoretic invariants, say $\mathcal{L}_p(E) = \frac{\log_p(q)}{\mathrm{ord}_p(q)}$ which shows up the study of p-adic L-functions.

(2) Cycles on Shimura varieties.

(3) BSD conjecture. We have Heegner points.
$$\overline{h/\Gamma} \cong X_0(N)(\mathbb{C}) \to E(\mathbb{C}).$$
Similarly, one should have
$$h_p/\Gamma_B = X_B(\mathbb{C}_p) \to E(\mathbb{C}_p).$$

8.2. **Rigid analytic space.** Let K be a non-archimedean field. This is hard to work with in geoemtry because for example, the unit ball
$$B[0,1] = \{|z| \leq 1\} = \{|z| < 1\} \sqcup \{|z| = 1\}$$
is disconnected. There are two ways to solve this problem:

- restrict the open sets, or
- restrict the coverings.

8.2.1. *G-topological spaces.* Let X be equipped with

- a collection of subsets $\mathcal{U}$ called open sets;
- for each $U \in \mathcal{U}$, $\mathrm{Cov}(U)$ a set of set theoretical covertins of U by elements of $\mathcal{U}$;
- subject to some axioms.

This gives the notion of a *G-topological space.*

8.2.2. *Rigid analytic space.*

Definition. A *rigid analytic space* is a set X with a G-topology with a sheaf $\mathcal{O}_X$ of K-algebras, $(X, \mathcal{O}_X)$ is a locally ringed space such that there exists a covering $\{U_i\}$ of X, with $(U_i, \mathcal{O}_{U_i}$ isomorphic an affinoid space.

An *affinoid* $\mathrm{Sp}(A)$ where A is a quotient of the Tate algebra
$$T_n(K) = k\langle x_1, \cdots, x_n\rangle = \{\sum_{a_J} x^J \mid a_J \in K, |a_J| \to 0 \text{ as } |J| \to \infty\},$$

and $\mathrm{Sp}(A)$ is the set $M(A)$ of maximal ideals of A. An *affinoid subdomain* is a subset $U \subseteq M(A)$ is the image of $M(A') \to M(A)$ of some universal map among $\varphi : A \to B$ such that $M(\varphi)$ lands in U. An *admissible cover* of U is a set-theoretical covering $\{V_i\}$ by admissible opens such that the covering $\{M(\phi)^{-1}(V_i)\}$ of $M(B)$ has a refinement by finitetly many affinoid subdomains.

Example 8.1. Take $A = T_n(K)$, thought of as the closed ball $B[0,1]$ (since X_i can be sent to any number with absolute value less than or equal to one). We have

$$B[0,1] = B(0,1) \sqcup \partial B[0,1]$$

as admissible opens, but this is not an admissible covering.

There is a functor $\mathcal{X} \to \mathcal{X}^{\mathrm{an}}$ sending locally finite K-scheme to its rigid analytification, there for we have, for example, $\mathbb{A}_K^{\mathrm{an}}$ and $\mathbb{P}_K^{\mathrm{an}}$.

8.2.3. *Integral models.* Let R be a valuation ring of K and k the residue field. A *topological finitely-presented (tfp) R-algebra* is one of the form

$$\mathcal{A} = R\{x_1, \cdots, x_n\}/\mathcal{I}$$

where $\mathcal{R}\{x_1, \cdots, x_n\} = \{\sum_J x^J \in R[\![x_1, \cdots, x_n]\!] \mid |a_J| \to 0 \text{ as } |J| \to 0\}$ and $\mathcal{I}$ is a finitely generated ideal in $R\{x_1, \cdots, x_n\}$. If $\mathcal{A}$ is R-flat, then we call it *admissible.*

If $\mathcal{A} = R\{x_1, \cdots, x_n\}$, then $A \otimes_R K = K\langle x_1, \cdots, x_n\rangle$ and $\mathcal{A} \otimes_R k \cong k[x_1, \cdots, x_n]$.

If $\mathfrak{m}$ is a maximal ideal in R, we can associate a formal scheme

$$\mathrm{Spf}(\mathcal{A}) = \{\mathrm{Spec}(A/\mathfrak{m}A), \mathcal{O}_A\}$$

which is called a *tfp affine formal scheme.*

Raynaud globalized this construction. By a *tfp formal scheme* we mean a gadget which is locally isomorphic to tfp affien form scheme. There is some functor

$$\{\text{tfp formal } R\text{-schemes}\} \to \{\text{rigid analytic } K\text{-spaces}\}$$

constructed by Raynaud. He also showed that rigid analytic K-spaces with good properties can be obtained in this way.

8.3. **Drinfeld upper half plane.** Let K be a non-archimedean local field, with ring of integers R, uniformizer π, residue characteristic p and $|R/\pi| = q$.

8.3.1. *Bruhat-Tits tree for* $\mathrm{PGL}_2(K)$. The Bruhat-Tits tree I for $\mathrm{PGL}_2(K)$ is a graph, with

- vertices corresponding to homothety classes of lattices $M \subseteq K^2$
- an edge between M and M' if $M \supsetneq M' \supsetneq \pi M$.

Therefore M' that has an edge between M and M' are in bijection with lines in $M/\pi M$, or $\mathbb{P}^1(M/\pi M)$. Let $I_{\mathbb{R}}$ by the geometric realization of I. Then there is a bijection

$$\{\text{points in } I_{\mathbb{R}}\} \leftrightarrow \{\text{proportionality classes of norms in } K^2\}.$$

Let $s = [M]$, then $|\cdot|_M$ is the norm such that unit ball is M. For example

$$s = [M] = Re_1 \oplus Re_2, |a_1e_1 + a_2e_2|_s = \sup\{|a_1|, |a_2|\}$$
$$s' = [M'] = Re_1 \oplus \pi Re_2, |a_1e_1 + a_2e_2|_{s'} = \sup\{|a_1|, q|a_2|\}$$
$$(1-t)s + ts', |v|_t = \sup\{|a_1|, q^t|a_2|\}.$$

Let $\mathbb{C}_K$ be the completion of algebraic closure of K and the *Drinfeld upper half plane* is defined to be $\Omega^2 = \mathbb{P}^1(\mathbb{C}_K)\backslash\mathbb{P}^1(K)$. This is a rigid analytic space, which can by identified with homothety classes of K-linear maps $K^2 \to \mathbb{C}_K$. Therefore we have injective maps by pulling back the norm on $\mathbb{C}_K$ to K^2:

$$\Lambda : \Omega^2 \to I_{\mathbb{R}}.$$

For two adjacent vertices $s = [M]$ and $s' = [M']$, let's investigate $\Lambda^{-1}(s)$, $\lambda^{-1}(s')$ and $\lambda^{-1}([s, s'])$. Take a basis $\{e_1, e_2\}$ of K^2 such that $M \subseteq M'$, say as above. Identify z with the map $e_1 \mapsto z$, $z_2 \mapsto 1$. Therefore, we have the following identification

$$I^{-1}(s) = \{z \in \mathbb{P}^1(\mathbb{C}) \mid |az + b| = \sup\{|a|, |b|\}, \forall a, b \in K\}.$$

By setting $b = 0$, we immediately get $|z| = 1$. Moreover, if z is in the preimage of $\mathbb{F}_q^\times$ under the projection $\mathcal{O}_{\mathbb{C}_K} \twoheadrightarrow \bar{\mathbb{F}}_q$, then we can always find $|a| = |b| = 1$ while $az + b \in \mathfrak{p}_{\mathbb{C}_K}$ has norm less than 1. So we have to remove these cosets. In another word, $I^{-s} = \mathbb{P}^1(\mathbb{C}_K) \backslash (q+1)$ open disks, one of them being $|z| > 1$, and the rest being $k + \mathfrak{p}_{\mathbb{C}_K}$ centered at K-rational points, or equivalently $B(0, 1] \backslash q$ open disks centered at the K-rational points. Similarly, we can see that $\lambda^{-1}(s')$ is $B(0, 1/q] \backslash q$ open disks centered at the K-rational points. So we can see that the picture is actually symmetric (it seems to be asymmetric because our choice of coordinates). In fact, $\lambda^{-1}(s)$, $\lambda^{-1}(s')$ and $\lambda^{-1}([s, s'])$ are affinoid, and have natural structures as rigid analytic spaces over K. We can glue these constuction for all adjacent vertices s and s' to get "tube around the tree".

8.3.2. *Formal schemes.* Let $M \subseteq K^2 = V$ be associated to s. We can form

$$\mathbb{P}_s = \operatorname{Proj}(\operatorname{Sym}_R M).$$

Since $M \otimes_R K = V$, $\mathbb{P}_s \otimes_R K \cong \mathbb{P}(V)$.

Now we want to glue these schemes. For s and s', $M' \subseteq M$. There is a birational map $\mathbb{P}_s \to \mathbb{P}_{s'}$. Define $\mathbb{P}_{s,s'}$ to be the closure of the graph of this map in $\mathbb{P}_s \times \mathbb{P}_{s'}$. This can be identified with the blow up of $\mathbb{P}_s$ at the rational point in the special fiber corrsponding to s' (or vice versus). Let $\Omega_s = \mathbb{P}_s$ with the rational points in the speical fibered removed. $\hat{\Omega}_s$ is the formal completion of Ω_s along special fiber. Let $\Omega_{[s,s']} = \mathbb{P}_{[s,s']} - \{\text{rational points in the special fiber except the singular point}\}$. $\hat{\Omega}_{[s,s']}$ is the formal completion of $\Omega_{[s,s']}$. We can glue $\hat{\Omega}_{[s,s']}$ along $\hat{\Omega}_s$ to obatin $\hat{\Omega}^2$, which is the formal scheme whose generic fiber is Ω^2.

8.4. ***p*-adic uniformization.** Let us first start with the following classical result.

8.4.1. *Cerednik's theorem.*

Theorem 8.1. *Let $\mathcal{O}_D \subseteq D$ be a maximal order in D, a quaternion algbra defined over $\mathbb{Q}$ with discriminant pq. Then $X_D = \Gamma_{D,\infty} \backslash \mathfrak{h}$ is a compact curve over $\mathbb{Q}$, where Γ_D is the group of (reduced) norm one elements in $\mathcal{O}_D$ and $\Gamma_{D,\infty}$ is its image in $\mathrm{PGL}_2(\mathbb{R})$.*

For $\mathcal{O}_B \subseteq B$ definite definite quaternion algebra obtained by interchanging the invariants of D and ∞, with a fixed isomorphism $B \otimes \mathbb{Q}_p = M_2(\mathbb{Q}_{\shortmid})$. Let $\Gamma_{B,p} = \mathcal{O}_B[\frac{1}{p}]^\times / \mathbb{Z}[\frac{1}{p}]^\times \hookrightarrow \mathrm{PGL}_2(\mathbb{Q}_p)$ and $X_{B,p}^{\mathrm{an}} = \Gamma_{B,p} \backslash \Omega^2$ which is a rigid analytic space over $\mathbb{Q}_p$.

Theorem 8.2 (Cerednik)**.** *There exists an unramified extension $L/\mathbb{Q}_p$ such that $X_{B,p}^{an} \times L$ is algebraic, and isomorphic to $X_D \times_{\mathbb{Q}_p} L$.*

8.4.2. *Drinfeld's moduli interpretation.* Drinfeld showed $\hat{\Omega}^2$ is a moduli space for p-divisible groups. He considered $\Omega_K^d = \mathbb{P}^{d-r}(\mathbb{C}) \backslash \mathbb{P}^1(K)$, this is a rigid analytic space. Let D_0 be a central division algebra over K with invariant $1/d$. Let $\mathcal{O}_{D_0}$ be its maximal order.

Definition. A *special formal $\mathcal{O}_{D_0}$-module* is a pair $(\mathbb{X}, \iota)$ consisting of a d-dimensional form group X over S and a ring homomorphism

$$\iota : \mathcal{O}_{D_0} \to \operatorname{End}_S(X)$$

satisfying some conditions.

There exists unique such up to $\mathcal{O}_{D_0}$-isogeny. Fixing one of them, say $\mathbb{X}$. Then we have $\mathrm{End}_{\mathcal{O}_{D_0}}(\mathbb{X}) \otimes \mathbb{Q} \cong M_d(K)$.

Let $E = \hat{K}^{\mathrm{un}}$, $\mathrm{Nilp}_{\mathcal{O}_E} = \{\mathcal{O}_E\text{-schemes on which } \pi \text{ is locally nilpotent}\}$. The functor $S \mapsto (X, \rho)$ where X is a speical formal $\mathcal{O}_{D_0}$-module of height d^2, and $\rho : \mathbb{X} \times_k \bar{S} \to X \times_S \bar{S}$ is a quasi-isogeny. This functor is representable by $\hat{\Omega}^d \times \mathbb{Z}$ (which has an action of GL_d).

8.5. **Rapoport-Zink spaces.** For (B, V, b, μ), $b \in G(E)$, there is $\breve{M}$ on which $J_b(E)$ has an action. This has a p-adic uniformization when b is basic.

9. Local Shimura Varieties and the Kottwitz Conjecture - Guanjie Huang

9.1. **Motivation: an overview of global shimura varieties.** Recall that we studied the global Shimura varieties from the following three perspectives:

(1) Group theoretical construction.

We start with the Shimura datum $(G, X = \{h\})$ where G is a connected reductive group over $\mathbb{Q}$, and $\{h\}$ is a $G(\mathbb{R})$-conjugacy class of cocharacters $\mathbb{S}^1 \to G_{\mathbb{R}}$. In particular, $X = \{h\}$ has an action of $G(\mathbb{R})$. For each compact open subgroup K of $G(\mathbb{A}_f)$, one can define a symmetric space

$$\mathrm{Sh}(G, X)_K = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K.$$

It turns out that $\mathrm{Sh}(G, X)_K$ has the structure of quasi-projective algebraic variety defined over not just the complex number but a number field called the *reflex field* $E = E(G, \{\mu\})$ (which is independent of the choice of K). Therefore, we get a tower

$$\mathrm{Sh}(G, K) = \{\mathrm{Sh}(G, X)_K\}_K$$

equipped with $G(\mathbb{A}_f) \times \Gamma_E$-action.

(2) Cohomology and realization of global Langlands correspondence.

As consequence, there is also a $G(\mathbb{A}_f) \times \Gamma_E$-action on cohomology groups. For every algebraic representation $\xi : G_{\mathbb{C}} \to \mathrm{GL}(V)$, one can assoicate a $\mathbb{C}$-local system $\mathcal{F}_\xi$ on $\mathrm{Sh}(G, X)_K$ as the sheaf of sections

$$G(\mathbb{Q}) \backslash V \times X \times G(\mathbb{A}_f)/K \to \mathrm{Sh}(G, X)_K.$$

The l-adic cohomology (fixing an isomorphism $\mathbb{C} \cong \bar{\mathbb{Q}}_l$)

$$H^*_{\mathrm{et}}(\mathrm{Sh}(G, X), \mathcal{F}_\xi) = \sum_i (-1)^i H^i_{\mathrm{et}}(\mathrm{Sh}(G, X), \mathcal{F}_\xi).$$

also has an action of $G(\mathbb{A}_f) \times \Gamma_E$. It is expected that, as $G(\mathbb{A}_f) \times \Gamma_E$-virtual representations over $\bar{\mathbb{Q}}_l$,

$$H^*(\mathrm{Sh}(G, X), \mathcal{F}_\xi) \cong \bigoplus_{\pi_f} \pi_f \boxtimes m(\pi_f)(r_\mu \circ \mathrm{GLC}(\pi)|_{\Gamma_E}),$$

where π_f ranges over admissible $\bar{\mathbb{Q}}_l$-representations of $G(\mathbb{A}_f)$ such that there exists an automorphic representation π of $G(\mathbb{A})$ such that

- $\pi_f \cong (\pi)_f$, and
- $\pi_\infty \in \Pi_\infty(\xi)$, the L-packet of ξ.

$m(\pi_f)$ is integer and r_μ is the highest weight representation associated to μ (cocharacter of G, hence character of ${}^L G$).

(3) Moduli interpretation.

In many cases (not all tho), Shimura varieties parametrize abelian varieties with extra structure.

9.1.1. *An overview of local Shimura varieties.* In this talk, we will talk about local Shimura varieties from these three aspects.

(1) In subsection 9.2, we will introduce the group-theoretic input called local Shimura datum, and conjectural properties of local Shimura varieties attached to them.
(2) In subsection 9.3, we will talk about the cohomology of local Shimura varieties and the conjectural realization of instances of refined LLC.
(3) In subsection 9.4, we will talk about the connection between local Shimura varieties and Rapoport-Zink spaces, which are moduli spaces of deformations of p-divisible groups.

9.2. **Local Shimura datum.** Let F be a finite extension of $\mathbb{Q}_p$ with absolute Galois group Γ. Let $\breve{F} = \hat{F}^{\mathrm{un}}$. Let σ be the Frobenius on $\breve{F}$. Also fix a prime number $l \neq p$.

Definition. A *local Shimura datum* is a collection $(G, [b], \{\mu\})$ where

- G is a reductive group over F (for simplicity let's assume G is quasisplit);
- μ is a (geometric conjugacy class of) minuscule cocharacter of G, that is, the induced action on $\mathrm{Lie}(G)$ has only weights in $\{-1, 0, 1\}$.
- $[b] \in B(G, \mu) = \{[b] \in B(G) \mid \kappa(b) = \mu^\natural, \nu(b) \leq \bar{\mu}_{\mathrm{dom}}\}$:

$$\begin{tikzcd}
{[b]} \arrow[rr, mapsto] \arrow[dd, mapsto] & & \nu([b]) & \\
& B(G) \arrow[r, "\mu"] \arrow[d, "\kappa"] & (X_*(T)_{\mathbb{Q}}/G)^{\Gamma} & \bar{\mu}_{\mathrm{dom}} \\
\kappa([b]) & X^*(Z(\hat{G})^{\Gamma}) & X_*(T) \arrow[l] \arrow[u] & \\
& \mu^\natural & & \mu \arrow[ll, mapsto] \arrow[uu, mapsto]
\end{tikzcd}$$

Associated to a local Shimura datum $(G, [b], \{\mu\})$, one has

- the *reflex field* $E = E(G, \mu)$ (the field of definition of $\{\mu\}$), $\breve{E} = E \cdot \breve{F}$, $C = \bar{\breve{E}}$.
- the algebraic group J_b
$$J_b(R) = \{g \in G(R \otimes_F \breve{F}) \mid b = gb\sigma(g)^{-1}\}.$$
b is *basic* if J_b is an inner form of G.

Example 9.1. Take $F = \mathbb{Q}_p$, $G = \mathrm{GL}_n$, $\mu = (\underbrace{1, \cdots, 1}_{d}, 0, \cdots, 0)$ for some $1 \leq d \leq n$, and $b \in \mathrm{GL}_n(F)$ any element with characteristic polynomial $\mathrm{char}(b) = X^n - p^d$.

Conjecture 9.1 (Rapoport-Viehmann)**.** *Given $(G, [b], \{\mu\})$ as above, there is a tower of smooth rigid analytic spaces over $\breve{E}$*
$$\mathcal{M} = \mathcal{M}(G, [b], \{\mu\}) = \{\mathcal{M}(G, [b], \{\mu\})_K\}_K$$
indexed by compect open subgroup $K \subseteq G(F)$, with the following properties.

(1) $J_b(F)$ acts on each $\mathcal{M}_K$, compatible with the structure morphism $\mathcal{M}_{K'} \to \mathcal{M}_K$ for each $K' \subseteq K$.
(2) $G(F)$ acts on the whole tower, taking $\mathcal{M}_K$ to $\mathcal{M}_{gKg^{-1}}$.
(3) The tower is equipped with a Weil descent datum $\mathcal{M} \xrightarrow{\sim} \mathcal{M} \times_{\breve{E}, \sigma} \breve{E}$ down to E. In particular, $\mathcal{M}$ is equipped with an action of W_E.

(4) There is a system of compatible rigid analytic étale period morphisms

$$\pi^K : \mathcal{M}_K \to \hat{\mathcal{F}}(g, b, \{\mu\})^{wa}$$

where $\mathcal{F}(g, b, \{\mu\})^{wa}$ is a rigid analytic space parametrizing $\mu \in \{\mu\}$ such that (b, μ) is weakly admissible.

Define the pro-object $\mathcal{M}_\infty = \varprojlim_K \mathcal{M}_K$.

Theorem 9.1. *Local Shimura varietiets exist with all expected properteis, as solution to a natural moduli problem determined by the datum $(G, [b], \{\mu\})$.*

9.3. **Cohomology and the Kottwitz Conjecture.**

9.3.1. *What cohomology groups to be studied.* Take the faith that there is some nice theory of l-adic compactly supported étale cohomology on rigid analytic spaces. Then the expected properties of local Shimura varieties imply that there is an action of $G(F) \times J_b(F) \times W_E$ on the (complex of) l-adic cohomology

$$R\Gamma_c(\mathcal{M}_{\infty,C}, \bar{\mathbb{Q}}_l).$$

However, this space is to big (not admissible) to be studied (as each of $\mathcal{M}_K$ is wildly non-compact). One reasonable solution is to look at the "derived isotypic part".

Let ρ be an irreducible admissible representation of $J_b(F)$ with coefficients in $\bar{\mathbb{Q}}_l$. Consider

$$H^i_c(G, [b], \mu)[\rho] = H^i(R\operatorname{Hom}_{J_b(F)}(R\Gamma_c(\mathcal{M}_{\infty,C}, \bar{\mathbb{Q}}_l(\frac{\dim \mathcal{M}}{2})), \rho)).$$

This is still a representation of $G(F) \times W_E$.

Theorem 9.2 (Fargues-Scholze)**.** *If ρ is admissible, then $H^i_c(G, [b], \{\mu\})[\rho]$ is an admissible $G(F)$-representation, and equals to 0 unless $0 \le i \le 2 \dim \mathcal{M}$.*

9.3.2. *Isocrystal LLC.* A Langlands parameter $\varphi : W_F \times \mathrm{SL}_2(\bar{\mathbb{Q}}_l) \to {}^LG(\bar{\mathbb{Q}}_l)$ is *supercuspidal* if φ is trivial on $\mathrm{SL}_2(\bar{\mathbb{Q}}_l)$ and $S_\varphi = \mathrm{Cent}(\varphi, \hat{G})$ has its connected component contained in $Z(G)^\Gamma$.

The isocrytal form of local Langlands conjecture predicts a bijection

$$\begin{array}{ccc} \bigsqcup_{b \in B(G)_{\mathrm{bas}}} \Pi_\varphi(J_b) & \overset{\iota_\varphi}{\longleftrightarrow} & \mathrm{Irr}(S_\varphi) \\ \downarrow & & \downarrow \\ B(G)_{\mathrm{bas}} & \overset{\kappa}{\longleftrightarrow} & X^*(Z(\hat{G})^\Gamma). \end{array}.$$

In another word, there should be a natural bijection

$$\iota_\varphi : \Pi_\varphi(J_b) \leftrightarrow \mathrm{Irr}(S_\varphi, \kappa(b)).$$

9.3.3. *The Kottwitz conjecture.* Fix a local Shimura datum $(G, [b], \{\mu\})$ with b basic. For $\rho \in \mathrm{Irr}_{\bar{\mathbb{Q}}_l}(J_b(F))$, consider the following virtual $G(F) \times W_E$-representation with coefficients in $\bar{\mathbb{Q}}_l$

$$H(G, [b], \{\mu\})[\rho] = \sum_{i \ge 0} (-1)^i H^i_c(G, [b], \mu)[\rho].$$

The Kottwitz conjecture is a description of this virtual representation using in terms of isocrystal LLC. Before we give the statement of the conjecture, let's make two observations:

- $\mu \in X_*(T)$ can also be thought of as a character of $\hat{G}$. From a recipe of Kottwitz, we can get the "highest weight representation" $r_\mu : {}^LG_E \to \mathrm{GL}_m$. For any L-parameter φ, the composition $r_\mu \circ \varphi|_{W_E}$ is a representation of $W_E \times S_\varphi$ (by definition of S_φ).
- If φ is supercuspidal, given any $\pi \in \Pi_\varphi(G)$, $\rho \in \Pi_\varphi(J_b)$, can extract an algebraic representation $\delta_{\pi,\rho} = \iota(\pi) \otimes \iota(\rho)^\vee$ of S_φ that measures the "relative positions" of π and ρ.

Conjecture 9.2 (Kottwitz conjecture)**.** *Fix a basic local Shimura datum $(G, [b], \{\mu\})$ and a supercuspidal L-parameter $\varphi : W_F \times \mathrm{SL}_2(\bar{\mathbb{Q}}_l) \to {}^L G(\bar{\mathbb{Q}}_l)$. Let $\rho \in \Pi_\varphi(J_b)$ be any element. Then*

$$H(G, [b], \mu)[\rho] = (-1)^{\dim \mathcal{M}} \sum_{\pi \in \Pi_\varphi(G)} \pi \boxtimes \mathrm{Hom}_{S_\varphi}(\delta_{\pi,\rho}, r_\mu \circ \varphi|_{W_E})$$

as virtual $G(F) \times W_E$ representations.

Example 9.2. Let $F = \mathbb{Q}_p$, $G = \mathrm{GL}_n$ and $\varphi : W_F \times \mathrm{SL}_2(\bar{\mathbb{Q}}_l) \to \mathrm{GL}_n(\bar{\mathbb{Q}}_l)$ be a supercuspidal L-parameter. Then φ is irreducible. In this case $S_\varphi = Z(\hat{G})^\Gamma = \mathbb{G}_m$. Hence the isocrystal LLC predicits

$$\begin{array}{ccc} \bigsqcup_{b \in B(G)_{\mathrm{bas}}} \Pi_\varphi(J_b) & \xleftrightarrow{\iota_\varphi} & \mathrm{Irr}(S_\varphi) = \mathbb{Z} \\ \downarrow\wr & & \downarrow\wr \\ B(G)_{\mathrm{bas}} & \xleftrightarrow{\kappa} & X^*(Z(\hat{G})^\Gamma) = \mathbb{Z}. \end{array} .$$

More concretely, $J_b = A^\times_{\kappa(b)}$, where $A_{\kappa(b)}$ is the central simple algebra of rank n^2 and Hasse invariant $\frac{d}{n}$ mod 1 (so J_b only depends on d mod n). Since every arrow is a bijection, each individual L-packet $\Pi_\varphi(J_b)$ is a singleton. The elements of these packets are "Jacquet-Langlands transfer" of each other.

Now Consider $(\mathrm{GL}_n, \mu = (1, \cdots, 1, 0, \cdots, 0), b)$ with $b \in \mathrm{GL}_n(F)$ with characteristic polynomial $\mathrm{char}(b) = X^n - p^d$. Then $J_b = A^\times_{\kappa(b)} = A^\times_d$ where A_d is a central simple algebra of rank n^2 and Hasse invariant $\frac{d}{n}$ over F. Now let's assume that $\Pi_\varphi(J_b) = \{\rho\}$ and $\Pi_\varphi(G) = \{\mathrm{JL}(\rho)\}$. Then Conjecture 9.2 predicts that

$$\begin{aligned} H(G, [b], \mu)[\rho] &= (-1)^{\dim \mathcal{M}} \sum_{\pi \in \Pi_\varphi(G)} \pi \boxtimes \mathrm{Hom}_{S_\varphi}(\delta_{\pi,\rho}, r_\mu \circ \varphi|_{W_E}) \\ &= (-1)^{\langle 2\rho_G, \mu \rangle} \mathrm{JL}(\rho) \boxtimes \mathrm{Hom}_{\mathbb{G}_m}((x \mapsto x^d), r_\mu \circ \varphi) \\ &= (-1)^{d(n-d)} \mathrm{JL}(\rho) \boxtimes \mathrm{Hom}_{\mathbb{G}_m}((x \mapsto x^d), \wedge^d(\varphi)) \\ &= (-1)^{d(n-d)} \mathrm{JL}(\rho) \boxtimes \wedge^d(\varphi). \end{aligned}$$

9.3.4. *Known results.* The full Conjecture 9.2 is known in the following cases:

- The Lubin-Tate and Drinfeld towers (Harris-Taylor);
- "Unramifield EL" type (Fargues, Shin);
- Some unitary "unramified PEL type" (Bertoloni-Meli-Nguyen).

These works rely crucially on global method (comparison with cohomology of global Shimura varieties).

However, if we forget about the W_E-action, purely local method can be adapted.

Conjecture 9.3 (Kottwitz conjecture, weakened form)**.** *Fix a basic local Shimura datum $(G, [b], \{\mu\})$ and a supercuspidal L-parameter $\varphi : W_F \times \mathrm{SL}_2(\bar{\mathbb{Q}}_l) \to {}^L G(\bar{\mathbb{Q}}_l)$. Let $\rho \in \Pi_\varphi(J_b)$ be any element. Then*

$$H(G, [b], \{\mu\})[\rho] = (-1)^{\dim \mathcal{M}} \sum_{\Pi_\varphi(G)} [\dim \mathrm{Hom}_{S_\varphi}(\delta_{\pi,\mu}, r_\mu)]\pi$$

as virtual $G(F)$-representation.

Theorem 9.3 (Hansen-Kaletha-Weinstein)**.** *Without assuming μ being miniscule and G being (a B-inner form of) quasi-split, Conjecture 9.3 holds up to some non-supercuspidal error term.*

9.4. **Connection with Rapoport-Zink spaces.**

9.4.1. *Integral RZ datum and RZ spaces.* A *simple rational RZ datum* in the EL case id a tuple $\mathcal{D}$ of the form $\mathcal{D} = (B, V, \{\mu\}, [b])$ where

- B is a central division algebra over $\mathbb{Q}_p$;
- V is a finite-dimensional B-module;
- $\{\mu\}$ is a geometric conjugacy class of miniscule cocharacters of $G = \mathrm{GL}_B(V)$. Let $E = E(G, \{\mu\})$, $\mathcal{O} = \mathcal{O}_{\breve{E}}$ and $\mathbb{F}$ be the residue field of $\mathcal{O}$.
- $[b] \in A(G, \{\mu\}) = \{[b] \in B(G) \mid \nu([b]) \leq \bar{\mu}_{\mathrm{dom}}\}$.

This data gives a filtration on $V_{\breve{\mathbb{Q}}_p}$, and the triple

$$(V_{\breve{\mathbb{Q}}_p}, b \circ (\mathrm{id} \otimes \sigma), \mathrm{Fil})$$

is the filtered isocrystal attached to a p-divisible group.

A *simple integral RZ datum* in the EL case consists of $(\mathcal{D}, \mathcal{O}_B, \Lambda)$ where

- $\mathcal{O}_B$ is a maximal order in B;
- Λ is an $\mathcal{O}_B$-stable lattice in V.

This induces an integral model $\mathcal{G} = \mathrm{GL}_{\mathcal{O}_B}(\Lambda)$ of G over $\mathbb{Z}_p$.

Let $\mathbb{X}$ be a p-divisible group over $\mathbb{F}$ whose associated isocrystal is isomorphic to $(V_{\breve{\mathbb{Q}}_p}, b \circ (\mathrm{id} \otimes \sigma))$. Let $\mathrm{Nilp}_{\mathcal{O}_{\breve{E}}}$ be the category of $\mathcal{O}_{\breve{E}}$-schemes on which π_E is locally nilpotent. Let $\mathrm{RZ}_{\mathcal{D}_{\mathbb{Z}_p}} : \mathrm{Nilp}_{\mathcal{O}_{\breve{E}}} \to \mathrm{Sets}$ sending S to the set of isomorphism classes of (X, ι, ρ) where

- X is a p-divisible group over S;
- $\iota : \mathcal{O}_B \to \mathrm{End}(X)$ is an action on $\mathcal{O}_B$ on X satisfying the *Kottwitz condition*:
$$\mathrm{char}(\iota(b); \mathrm{Lie}(X)) = \mathrm{char}(b; V_0), \quad \forall b \in \mathcal{O}_B,$$
where V_0 is the weight space given by μ.
- $\rho : X \times_S \bar{S} \to \mathbb{X} \times_{\bar{\mathbb{F}}_p} \bar{S}$ is an $\mathcal{O}_B$-linear quasi-isogeny, where $\bar{S}$ is the closed subscheme of S defined by $p\mathcal{O}_S$.

Theorem 9.4. *The functor $\mathrm{RZ}_{\mathcal{D}_{\mathbb{Z}_p}}$ is representable by a formal scheme, locally formally of finite type over* $\mathrm{Spf}(\mathcal{O})$.

Remark. Here "locally formally of finite type" means the map on the special fiber is locally of finite type.

Note that the group $J_b(\mathbb{Q}_p)$ acts on $\mathcal{M}_{\mathcal{D}_{\mathbb{Z}_p}}$ by post composing with ρ,

$$g : (X, \iota, \lambda) \mapsto (X, \iota, g \circ \rho), \quad g \in J_b(\mathbb{Q}_p).$$

9.4.2. *Passing to the generic fiber.* Fix an integral RZ datum $\mathcal{D}_{\mathbb{Z}_p}$, we obtain the formal scheme $\mathcal{M} = \mathcal{M}_{\mathcal{D}_{\mathbb{Z}_p}}$. Its generic fiber $\mathcal{M}^{\mathrm{rig}}$ is a rigid analytic space over $\breve{E}$. The following conjecture reprsents the connection between the theory of local Shimura varieties and Rapoport-Zink spaces.

Conjecture 9.4. *Let $\mathcal{D}_{\mathbb{Z}_p}$ be an integral RZ datum such that $\mathcal{G}$ is a parahoric group scheme. Then up to isomorphism, the generic fiber $\mathcal{M}^{rig}$ together with its action $J_b(\mathbb{Q}_p)$ depends on $\mathcal{D}_{\mathbb{Z}_p}$ via the quadruple $(G, [b], \{\mu\}, \mathcal{G})$. Moreover, $\mathcal{M}^{rig} \neq \emptyset$ iff $[b] \in B(G, \{\mu\})$.*

Remark. The "only if" part is known.

9.4.3. *The tower of rigid analytic space.* Fix an integral RZ datum. We can then associate to any compact open subgroup $K \subseteq G(\mathbb{Q}_p)$ the rigid analytic space $\mathcal{M}_K^{\mathrm{rig}}$ classifying K-level structure of $T_p(X^{\mathrm{univ}})$, which is a finite étale covering of $\mathcal{M}^{\mathrm{rig}}$. We obtain in this way a tower of rigid analytic spaces $\{\mathcal{M}_K^{\mathrm{rig}}\}$.

Proposition 9.1. *The tower of rigid anlytic spaces $\mathcal{M}_K^{rig}$ has the following properties.*

(1) $J_b(\mathbb{Q}_p)$ acts on each $\mathcal{M}_K^{rig}$, compatible with the structure morphism $\mathcal{M}_{K'}^{rig} \to \mathcal{M}_K^{rig}$ for each $K' \subseteq K$.

(2) *$G(F)$ acts on the whole tower, taking $\mathcal{M}_K^{rig}$ to $\mathcal{M}_{gKg^{-1}}^{rig}$.*
(3) *The tower is equipped with a Weil descent datum $\mathcal{M}^{rig} \xrightarrow{\sim} \mathcal{M}^{rig} \times_{\breve{E},\sigma} \breve{E}$ down to E. In particular, $\mathcal{M}^{rig}$ is equipped with an action of W_E.*
(4) *There is a system of compatible rigid analytic étale period morphisms.*

Comparing Proposition 9.1 with Conjecture 9.1, we should think of Rapoport-Zink spaces as integral models for local Shimura varieties.

Example 9.3. Let $\mathbb{X}$ be a p-divisible group of dimension d and height n over $\bar{\mathbb{F}}_p$. Define a functor $\mathrm{RZ}_{\mathbb{X}} : \mathrm{Nilp}_{\breve{\mathbb{Z}}_p} \to \mathrm{Set}$ by

$$A \mapsto \begin{Bmatrix} p\text{-divisible group } X \text{ over } A \text{ with a quasi-isogeny} \\ \rho : X \times_A (A/p) \to \mathbb{X} \times_{\bar{\mathbb{F}}_p} (A/p) \end{Bmatrix} / \cong .$$

Then Theorem 9.4 implies that $\mathrm{RZ}_{\mathbb{X}}$ is represented by a locally formally of finite type formal scheme $\mathcal{M}_{\mathbb{X}}$ over $\mathrm{Spf}(\breve{\mathbb{Z}}_p)$. We can take its rigid analytic generic fiber $\mathcal{M}_{\mathbb{X}}^{\mathrm{rig}}$ over $\breve{\mathbb{Q}}_p$. Say the isocrystal of $\mathbb{X}$ is $(\breve{\mathbb{Q}}_p^{\oplus n}, b\sigma)$ where $b \in \mathrm{GL}_n(\breve{\mathbb{Q}}_p)$.

We can also get a finite étale coverings $\mathcal{M}_K^{\mathrm{rig}} \to \mathcal{M}^{\mathrm{rig}}$ for each K a compact open subgroups of GL_n classifying level structures of $T_p(X^{\mathrm{univ}})$, the p-adic Tate module of the universal p-divisible group on $\mathcal{M}_{\mathbb{X}}$. More precisely, a K-level structure is a mod K class of trivialization

$$\eta : \mathbb{Z}_p^{\oplus n} \cong T_p(X^{\mathrm{univ}}).$$

Therefore, we have a tower of rigid analytic spaces $\{\mathcal{M}_K^{\mathrm{rig}}\}_K$ indexed by compact open subgroup $K \subseteq \mathrm{GL}_n(\mathbb{Q}_p)$ equipped with

(1) an action of $\mathrm{GL}_n(\mathbb{Q}_p)$, $g : M_K^{\mathrm{rig}} \to M_{gKg^{-1}}^{\mathrm{rig}}$;
(2) an action of $\mathrm{QIsog}(\mathbb{X}) \cong J_b(\mathbb{Q}_p)$ acts on each M_K^{rig}.
(3) an action of $W_{\mathbb{Q}_p}$, arising from a Weil descent datum induced by Frobenius morphism $\mathbb{X} \to \mathbb{X} \times_{\bar{\mathbb{F}}_p,\sigma} \bar{\mathbb{F}}_p$.

It turns out that $M_{\mathbb{X}}^{\mathrm{rig}}$ realizes the local Shimura variety attached to $(\mathrm{GL}_n, [b], \{\mu\})$ as in our main example of Drinfeld characters (with extra conditions on $\mathbb{Z}_p$-compatibility of ρ).

10. The Fargues-Fontaine curve and vector bundles - Serin Hong

10.1. **Recap on local Shimura varieties.** We start with local Shimura datum (G, b, μ). Given such a triple, conjecturally we can construct a tower $\{M(G, b, \mu)_K\}_K$ of rigid analytic spaces over $\breve{E}$. Its expected that its cohomology

$$H^*(M(G, b, \mu))$$

realizes LLC (Conjecture 9.2).

The prototype of local Shimura spaces are Rapoport-Zink spaces (RZ spaces). Let H_0 be a p-divisible grouup over $\bar{\mathbb{F}}_p$ of height n. For $R \in \mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$, we defined the Rapoport-Zink functor

$$\mathrm{RZ}_{H_0}(R) = \left\{ (H, \iota) \;\middle|\; \begin{matrix} H : \text{a } p\text{-divisible group over } R \\ \iota : H \otimes_R R/p \to H_0 \otimes_{\bar{\mathbb{F}}_p} R/p \text{ quasi-isogeny} \end{matrix} \right\}$$

RZ_{H_0} is represented by a formal scheme over $\breve{\mathbb{Z}}_p$.

Remark. RZ_{H_0} yields "integral model" for local Shimura varieties for GL_n. But RZ spaces exist for many "integral" local Shimura datum.

Theorem 10.1 (Scholze-Weinstein)**.** *Local Shimura varieties exist for all local Shimura data.*

So what's the insight of the constuction. In the RZ case, p-divisible groups are very "integral", in the sense that they are actually equivlaent to Dieudonné module over $\breve{\mathbb{Z}}_p$. If we invert p, we get isocrystals (which is a $\breve{\mathbb{Q}}_p$-vector space with Frobenius-semi-linear automorphism). Therefore we need some moduli of isocrystals with enough geometric structure. In turns out that this comes from the moduli of local Shtukas (defined in terms of vector bundles on Fargues-Fontaine curves).

10.2. **Construction of absolute Fargues-Fontaine curves.** Let $E/\mathbb{Q}_p$ be a finite extension with uniformizer π and residue field $\mathbb{F}_q$. Let C/E be a complete algebraically closed extension. Let $C^\flat = \varprojlim_{x\mapsto x^p} C$ be the *tilt* of C with pseudo-uniformizer ϖ (i.e., $0 < |\varpi| < 1$).

Remark. The construction is actually independent of the choice of π and ϖ.

Example 10.1. $E = \mathbb{Q}_p$ and $C = \mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$. In this case $C^\flat = \widehat{\bar{\mathbb{F}}_p((t))}$.

Let $\mathbb{A}_{\text{inf}} = \mathbb{A}_{\text{inf},E,C^\flat} = W(\mathcal{O}_{C^\flat}) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$. If E is unramified over $\mathbb{Q}_p$, then this is equal to $W(\mathcal{O}_{C^\flat})$.

Let x

$$\mathcal{Y} = \mathcal{Y}_{E,C^\flat} := \operatorname{Spa}(\mathbb{A}_{\text{inf}})\backslash\{\pi[\varpi] = 0\}$$

where $\operatorname{Spa}\mathbb{A}_{\text{inf}}$ is the set of continuous semi-norms of $\mathbb{A}_{\text{inf}}$. The q-Frobenius on $\mathcal{O}_{C^\flat}$ gives rise to a Frobenius automorphism φ on $\mathcal{Y}$. So we can further define

$$\mathcal{X} = \mathcal{X}_{E,C^\flat} = \mathcal{Y}/\varphi^{\mathbb{Z}}.$$

This is the *Fargues-Fontaine curve*. Both $\mathcal{X}$ and $\mathcal{Y}$ are adic spaces.

Example 10.2 (Points on $\mathcal{Y}$)**.** Let $E = \mathbb{Q}_p$.

(1) Classical points. Let $\mathbb{C}^\sharp$ be the untilt of $C^\flat$ (i.e., $(C^\sharp)^\flat \cong C^\flat$). The $\sharp$-map $C^\flat \to C^\sharp$ induces a surjection:

$$\theta_{C^\sharp}\mathbb{A}_{\text{inf}} \twoheadrightarrow \mathcal{O}_{C^\sharp}$$
$$\sum_{i\geq 0}[c_i]p^i \mapsto \sum_{i\geq 0} c_i^\sharp p^i.$$

So we get a norm on $\mathbb{A}_{\text{inf}}$ by

$$\nu_{C^\sharp}(f) = |\theta_{C^\sharp}(f)|_{C^\sharp}.$$

In particular we have

$$\nu_{C^\sharp}(p) = |p|_{C^\sharp} \neq 0$$
$$\nu_{C^\sharp}([\mu_{C^\sharp}]) = |[\varpi]^\sharp|_{C^\sharp} = |\varpi|_{C^\flat} \neq 0$$

Therefore we see that $\nu_{C^\sharp} \in \mathcal{Y}$.

(2) Gauss points. For $0 < \rho < 1$, we can define

$$\mu_\rho(\sum[c_i]p^i) = \sup_{i\geq 0}(|c_i|\rho^i).$$

Then we have

$$\nu_\rho(p) = \rho \neq 0$$
$$\nu_\rho([\varpi]) = |\varpi| \neq 0.$$

So we have $\nu_\rho \in \mathcal{Y}$.

Remark. As mentioned by Jared in this special RTG talks series, $\mathcal{Y}$ can be thought of as a punctured open unit disk.

(1) Let $\mathbb{D}^{\times}_{C^\flat}$ be the punctured open disk over $C^\flat$. Since
$$C = \mathrm{Spa}(C^b) \times \mathrm{Spa}(\mathbb{F}_p[\![t]\!]),$$
there exists a map $\mathbb{D}^{\times}_{C^\flat} \to \mathcal{Y}$ such that
$$\begin{gathered}\{\text{classical points on } \mathbb{D}_{C^\flat}\} \to \{\text{classical points on } \mathcal{Y}\} \\ (\text{ i.e., } \{X \in C^\flat \mid 0 < |x| < 1\} \to \{\text{untilts of } C^\flat\}/\cong) \\ \{\text{Guass points on } \mathbb{D}_{C^\flat}\} \to \{\text{Gauss points on } \mathcal{Y}\}\end{gathered}$$

(2) There exists a "radius function" $r : \mathcal{Y} \to (0,1)$ such that $r(\varphi(y)) = r(y)^{1/p}$. The action (a covering space action) of φ is properly discontinuous. So it makes sense to talk about $\mathcal{X} = \mathcal{Y}/\varphi^{\mathbb{Z}}$.

10.3. **Vector bundles on Fargues-Fontaine curve.** We start with a slogan.

Slogan: $\mathcal{X}$ is similar to $\mathbb{P}^1$.

Recall a vector bundle $\mathcal{V}$ on $\mathbb{P}^1$ decomposes as
$$\mathcal{V} = \bigoplus \mathcal{O}_{\mathbb{P}^1}(d_i), d_i \in \mathbb{Z}.$$

Definition. Let $\lambda = \frac{d}{r}$ be a reduced rational number (i.e., $(d, r) = 1$ and $r > 0$). Define
$$\mathcal{O}(\lambda)$$
to be the descent of $\mathcal{O}_{\mathcal{Y}}^{\oplus r}$ with φ-linear automorphism
$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ \pi^{-d} & & & & 0 \end{pmatrix}.$$

Theorem 10.2 (Fargues-Fontaine). *Any vector bundle $\mathcal{V}$ on $\mathcal{X}$ decompose as*
$$\mathcal{V} = \bigoplus \mathcal{O}(\lambda_i), \lambda_i \in \mathbb{Q}.$$

Corollary 10.1. *There exists a functorial bijection*
$$\begin{aligned}\{\textit{isocrystals over } \breve{E}\}/\cong &\to \{\textit{vector bundles on } \mathcal{X}\}/\cong \\ (V, \varphi_V) &\mapsto \textit{descent of } V \otimes \mathcal{Y} \textit{ with } \varphi_V \otimes \varphi\end{aligned}$$
which is compatible with rank, degree and dual.

Remark. This is not an equivalence of category since there are way more morphisms of vector bundles than isocrystals. For example, if two isocrystals have different slopes, there is no non-trivial morphism. But this is not the case for vector bundles. Therefore, it makes more sense to use vector bundles in the definition of local Shtukas.

Let G be a reductive group over E. A *G-isocrystal* over $\breve{E}$ is an exact $\otimes$-functor
$$\mathrm{Rep}_E(G) \to \mathrm{Isoc}_{\breve{E}}.$$
As pointed out in Andy's talk, there is a bijection
$$B(G) \leftrightarrow \{G\text{-isocrystals}\}/\cong.$$
In fact, a G-bundle (i.e., a G-torsor) on $\mathcal{X}$ is equivalent to an exact $\otimes$-functor $\mathrm{Rep}_E(G) \to \mathrm{Bun}(\mathcal{X})$.

Theorem 10.3 (Fargues)**.** *There exists a bijection*

$$B(G) \leftrightarrow \{G\text{-bundles on } \mathcal{X}\}/\cong$$
$$b \mapsto \text{descent of } G \times \mathcal{Y} \text{ with } b \otimes \varphi.$$

10.4. **Relative Fargues-Fontaine curve.** Let R be a perfectoid algebra over $C^\flat$ with pseudo-uniformizer ϖ_R. Let $S = \mathrm{Spa}(R, R^+)$ where R^+ is the ring of integral elements. Define

$$\mathcal{Y} = \mathcal{Y}_{E,S} = \mathrm{Spa}(W_{\mathcal{O}_E(R^+)}) \backslash \{\pi[\varpi_R] = 0\}.$$

Again $\mathcal{Y}$ has a Frobenius automorphism φ. Define

$$\mathcal{X} = \mathcal{X}_{E,S} = \mathcal{Y}/\varphi^{\mathbb{Z}}.$$

Remark. (1) Classification of vector bundles fails in the relative setting.
(2) $\mathcal{X}_{E,S}$ and $\mathcal{Y}_{E,S}$ are not obtained by a base change of $\mathcal{X}_{E,C^\flat}$ and $\mathcal{Y}_{E,C^\flat}$ (they don't even live over $C^\flat$). Still, if $S' \to S$ is defined over $C^\flat$, we will have $\mathcal{X}_{E,S'} \to \mathcal{X}_{E,S}$.

10.5. **Schematic curve and local class field theory.** Let

$$X^{\mathrm{alg}} = X^{\mathrm{alg}}_{E,C^\flat} = \mathrm{Proj}(\bigoplus_{n\geq 0} \mathcal{O}_{\mathcal{Y}}(\mathcal{Y})^{\varphi=\pi^n}) = \mathrm{Proj}(\bigoplus_{n\geq 0} (B^+_{\mathrm{cris}})^{\varphi=\pi^n}).$$

Remark. (1) $X^{\mathrm{alg}}_{E,C^\flat} = X_{\mathbb{Q}_p,C^\flat} \otimes_{\mathbb{Q}_p} E$.
(2) X^{alg} is geometrically simply connected, so $\pi_1(X^{\mathrm{alg}}) \cong \mathrm{Gal}(\bar{E}/E)$.

We have the following "GAGA" for adic curves

Theorem 10.4 (Kedlaya-Liu)**.** *There exists a map $\mathcal{X} \to X^{alg}$ of locally ringed spaces inducing* $\mathrm{Bun}(\mathcal{X}) \cong \mathrm{Bun}(X^{alg})$.

Theorem 10.5 (Fargues)**.** *Let F be a torsion* $\mathrm{Gal}(\bar{E}/E)$*-module. Then the map $s : X^{alg} \to$* $\mathrm{Spec}(E)$ *induces*

$$H^i_{et}(X^{alg}, s^*F) \cong H^i_{et}(\mathrm{Spec}(E), F).$$

Corollary 10.2. *The Tate duality comes from the Poincare duality for X^{alg}. We also obtain a geometric field of local class field theory*

Recall that local class field theory yields a reciprocity map

$$r_{\mathbb{Q}_p} : \mathbb{Q}_p^\times \hookrightarrow \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)^{\mathrm{ab}}$$

with image $W^{\mathrm{ab}}_{\mathbb{Q}_p}$.

Now lets sketch a proof. For any E finite extenison of $\mathbb{Q}_p$ of degree d. Consider

$$\mathrm{inv}_{E/\mathbb{Q}_p} : H^2(\mathrm{Gal}(E/\mathbb{Q}_p), E^\times) \xrightarrow{\sim} \mathbb{Z}/d\mathbb{Z}.$$

So we get

$$r_{E/\mathbb{Q}_p} : \mathbb{Q}_p^\times \twoheadrightarrow \mathbb{Q}_p^\times/N_{E/\mathbb{Q}_p}(E^\times) \cong \mathrm{Gal}(E/\mathbb{Q}_p)^{\mathrm{ab}}(\text{Tate duality}).$$

And we can take $r_{\mathbb{Q}_p}$ as the limit.

Theorem 10.6 (Fargues)**.** $H^2_{et}(X^{alg}, \mathbb{G}_m) = 0$.

Corollary 10.3. $H^2_{et}(X^{alg}, \mu_d) = \mathbb{Z}/d\mathbb{Z}$.

So we have still get (for unramified E and then for ramifield cases)

$$\mathrm{inv}_{E/\mathbb{Q}_p} : H^2(E/\mathbb{Q}_p) = H^2(\mathrm{Spec}(\mathbb{Q}_p), \mu_d) = H^2(X^{\mathrm{alg}}, \mu_d) = \mathbb{Z}/d\mathbb{Z}.$$

This gives a geometric proof of local class field theory.

11. Local Shtukas spaces - Robert Cass

11.1. **Function field.** Let C be a cruve over $\mathbb{F}_q$ with function field F. Let G be a split reductive group over F, with a fix choice $T \subseteq B \subseteq G$. Let $X_*(T)^+$ is the cone of dominant cocharacters.

11.1.1. *Modification of vector bundles.* Let R be a $\mathbb{F}_q$-algebra adn $C_R C \times_{\mathbb{S}_q} R$.

Definition. A *modification* of two G-bundles $\mathcal{E}_1, \mathcal{E}_2$ on C_R at a point $x : \operatorname{spec} R \to C$ is an isomorphism

$$\alpha : \mathcal{E}_1 | C_R - \Gamma_x \cong \mathcal{E}_2 | C_R - \Gamma_x$$

where $\Gamma_x \subseteq C_R$ is the graph of x.

For simplicity, assume $t \in \mathcal{O}_C$ is the uniformizing element. Let $\hat{D}_x$ be the formal neighbourhood of $\Gamma_x \subseteq C_R$. Set $\hat{D}_x^\circ = \hat{D}_x - \Gamma_x$. By our choice of t, we have $\mathcal{O}_{\hat{D}_x} = R[\![t]\!]$ and $\mathcal{O}_{\hat{D}_x^\circ} = R(\!(t)\!)$.

The data of a modification at x is equivalent to $\{\mathcal{E}_1|\hat{D}_x, \mathcal{E}_2|\hat{D}_x, \alpha'\}$ where $\alpha' : \mathcal{E}_1|\hat{D}_x^{circ} \cong \mathcal{E}_2|\hat{D}_x^\circ$. If $R = k = \bar{k}$, then $\mathcal{E}_i|\hat{D}_x$ are trivial. Then α' can be viewed as an element of

$$\operatorname{Aut}(G \times_{\mathbb{F}_q \times \hat{D}_x^\circ}) \cong G(R(\!(t)\!)).$$

Since we don't have canonical trivializations, we get

$$[\alpha']_k = G(k[\![t]\!]) \backslash G(k(\!(t)\!))/G(k[\![t]\!]) = \bigsqcup_{\mu \in X_*(T)^+} t^\mu.$$

We say α is *bounded* by $\mu \in X_*(T)^+$ if for any geometric point $\bar{y}$, $[\alpha']_{\bar{y}}$ is bounded by μ in the Bruhat order.

11.1.2. *Period maps.* Suppose $\mathcal{E}_2$ is trivialized. Then we can view $[\alpha']$ as an element of

$$G(R(\!(t)\!))/G(R[\![t]\!]).$$

Definition. The *affine Grassmannian* Gr_G is the 'tale sheafification of the functor

$$\begin{aligned} \mathrm{AffSch}^{\mathrm{op}} &\to \mathrm{Set} \\ R &\mapsto G(R(\!(t)\!))/G(R[\![t]\!]). \end{aligned}$$

We have $\mathrm{Gr}_G \varinjlim_\mu \mathrm{Gr}_G^\mu$ is ind-projective. So for moduli space of modifications with $\mathcal{E}_2$ trivialized, we get a period map to Gr_G.

11.1.3. *Shtukas.* Let $F_R = \mathrm{id}_C \times \mathrm{Frob}_R : C_R \to C_R$. The moduli space of shtukas with one leg (bounded by μ) is

$$\mathrm{Sht}_\emptyset^{\leq \mu}(R) = \left\{ (x, \mathcal{E}, \alpha) \mid \begin{array}{c} x \in C(R) \\ \mathcal{E} \text{is a } G\text{-torsor on } C_R \\ \alpha : F_R^* \mathcal{E}|(C_R - \Gamma_x) \cong \mathcal{E}|(C_R - \Gamma_x) \end{array} \right\}.$$

If we also record a trivilization of $\mathcal{E}$ in a thickening $\hat{N}$ of an effective divisor $N \subseteq C$

$$G(\hat{N}) \curvearrowright \mathrm{Sht}_N^{\leq \mu} \to C - N.$$

Passing to limit over $\hat{N}$,

$$G(\mathbb{A}_C) \curvearrowright \mathrm{Sht}^{\leq \mu} \to \operatorname{Spec} F.$$

The étale cohomology gets action of $G(\mathbb{A}_c) \times \mathrm{Gal}(\bar{F}/F)$.

11.2. **p-adic geometry.** Let (G, b, μ) be a triple with G a reductive group over $\mathbb{Q}_p$, $b \in B(G)$ and μ a conjugacy class of cocharacters (not required to be miniscule).

Last time, we introduced the Fargues-Fontaine curve $\mathcal{Y}$ and $\mathcal{X} = \mathcal{Y}/\varphi^{\mathbb{Z}}$. By Theorem 10.3, we get a G-bundle $\mathcal{E}_b$ on $\mathcal{X}$. More generally for S perfectoid, we have $\mathcal{Y}_S$ and $\mathcal{X}_S$, effective Cartier divisors on $\mathcal{Y}_S$ parametrize untilt of $S^\flat$. In fact, $\mathcal{Y}_S$ only depends on $S^\flat$.

Let $k = \bar{\mathbb{F}}_p$. Let Perf_k be the category of perfectoid spaces over k ("glued out of perfect Banach algebras"). Let $K \subseteq G(\mathbb{Q}_p)$ be a compact open subgroup.

Definition. We define the *local shtuka*

$$\begin{aligned} \mathrm{Sht}_{G,b,\mu,K} : \mathrm{Perf}_K^{\mathrm{op}} &\to \mathrm{Set} \\ S &\mapsto \{(S^\sharp, \mathcal{E}, \alpha, \mathbb{P})\} \end{aligned}$$

where

- $S^\sharp$ is an untilt of S to E;
- $\mathcal{E}$ is a G-torsor on $\mathcal{X}_S$ which is trivial at all geometric points of S;
- $\alpha : \mathcal{E}|(\mathcal{X}_S - S^\sharp) \to \mathcal{E}_b|(\mathcal{X}_S - S^\sharp)$ is meromorphic and bounded by μ;
- $\mathbb{P}$ is a K-torsor on S.

Remark. Now we can see the proof of Theorem 9.1

(1) Frobenius was absorbed into $\mathcal{X}_S$.
(2) If we have $K' \subseteq K$, then we have finite étale map
$$\mathrm{Sht}_{G,b,\mu,K'} \to \mathrm{Sht}_{G,b,\mu,K}\,.$$
(3) since $\mathcal{E}_b|\mathcal{Y}_S$ is trivial, we have an étale period map
$$\mathrm{Sht}_{G,b,\mu,K} \to \mathrm{Gr}^{\leq\mu}_{G,\mathrm{Spd}\,\breve{E}} = B^+_{\mathrm{dR}},$$
the affine Grassmannian of for the disk $\mathcal{Y}$, which is a diamond. So $\mathrm{Sht}_{G,b,\mu,K}$ is étale over a diamond, and so it's also a diamond.
(4) Let $\mathrm{Sht}_{G,b,\mu} = \varprojlim \mathrm{Sht}_{G,b,\mu,K}$ parametrizes modifications from $\mathcal{E}_1$ to $\mathcal{E}_b$. So it gets an action of the automorphism groups of $\mathcal{E}_1$ and $\mathcal{E}_b$. Recall that $G(\mathbb{Q}_p) = \mathrm{Aut}(\mathcal{E}_1)$ and $J_b(\mathbb{Q}_p) \subseteq \mathrm{Aut}(\mathcal{E}_b)$, so we get an action of $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$ on $\mathrm{Sht}_{G,b,\mu}$.
(5) If μ is minuscule, then
$$\mathrm{Gr}^{\leq\mu}_{G,\mathrm{Spd}(\breve{E})} = (G/P_\mu)^\diamond.$$
the diamond of a smooth rigid analytic flag variety.

So to summarize, we get $\mathrm{Sht}_{G,b,\mu,K} = \mathcal{M}^\diamond_{G,b,\mu,K}$, the diamond of the local Shimura variety.

11.3. **Diamonds.** Over function field, we formed $C_R = C \times_{\mathbb{F}_q} R$ and also $C \times_{\mathbb{F}_q} C$. But $\mathrm{Spec}\,\mathbb{Z} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}$, and the same holds for $\mathbb{Z}_p$. Diamonds allow one to "forget" the structure map to $\mathrm{Spa}\,\mathbb{Z}_p$ and form absolute products over a deeper space.

Definition. A *diamond* is a functor $\mathrm{Perf}_k \to \mathrm{Set}$ given by a quotient of perfectoid space in characteristic p by pro-étale equivalence relation.

There is a functor

$$\begin{aligned} \{\text{analytic adic spaces over } \mathrm{Spa}\,\mathbb{Z}_p\} &\to \{\text{Diamonds}\} \\ X &\mapsto X^\diamond. \end{aligned}$$

Example 11.1. Let see how to make a diamond out of $\mathbb{Q}_p$. Recall that the $\widehat{\mathbb{Q}_p(\mu_{p^\infty})}$ has absolute Galois group $\mathbb{Z}_p^\times$. Define

$$\mathrm{Spd}\,\mathbb{Q}_p = \mathrm{Spa}(\mathbb{Q}_p)^\diamond = \mathrm{Spa}(\widehat{\mathbb{Q}_p(\mu_{p^\infty})}^\flat)/\mathbb{Z}_p^\times.$$

It turns out that

$$\mathrm{Spd}(\mathbb{Q}_p)(S) = \left\{ (S^\sharp, i) \mid \begin{array}{l} S^\sharp \text{ is a perfectoid space over } \mathbb{Q}_p \\ i : (S^\sharp)^\flat \cong S. \end{array} \right\}$$

Remark. If S is perfectoid over $\mathbb{Q}_p$, then $S^\diamond = S^\flat$.

For $S \in \mathrm{Perf}_k$,

$$\mathcal{Y}_S^\diamond = S \times \mathrm{Spd}\,\mathbb{Q}_p.$$

One is allowed to give meaning of $\mathrm{Spd}\,\mathbb{Q}_p \times \cdots \times \mathrm{Spd}\,\mathbb{Q}_p$.

12. Integral models of local Shimura varieties - Patrick Daniels

Let F be a p-adic field, and $\breve{F}$ is the completion of maximal unramified extension of F in $\bar{\mathbb{Q}}_p$.

12.1. **Motivation.** Let (G, b, μ) be a local Shimura datum, where

- G is a reductive group over $\mathbb{Q}_p$;
- $b \in G(\breve{\mathbb{Q}}_p)$;
- μ is a miniscule class of cocharacters.

Let E be the reflex field. Associated to (G, b, μ) is the tower of rigid analytic spaces over $\breve{E}$:

$$(M_K(G, b, \mu) \mid K \subseteq G(\mathbb{Q}_p) \text{ compact open}).$$

This (tower of) rigid analytic spaces is called *local Shimura vairiety.*

Recall that we have the étale period morphism

$$\pi_{\mathrm{GM}} : \mathrm{Sht}_K(G, b, \mu) \to \mathrm{Gr}_{G, \mathrm{Spd}\,\breve{E}, \leq \mu}$$

of diamonds. This morphism is defined for any μ, but when μ is miniscule,

$$\mathrm{Gr}_{G, \mathrm{Spd}\,\breve{E}, \leq \mu} = (G/P_\mu)^\diamond.$$

That being said,

$$\mathrm{Sht}_K(G, b, \mu) = M_K(G, b, \mu)^\diamond$$

for some rigid analytic space over $\breve{E}$.

12.2. **Classical story.** Start with EL or PEL local Shimura datum.

Example 12.1. GL_n is EL, GSp_{2n} is PEL.

Let $G = \mathrm{GL}(V)$ for finite dimensional $\mathbb{Q}_p$-vector space V with a fix lattice $\Gamma \subseteq V$. Taking $b \in G(\breve{\mathbb{Q}}_p)$, one can associate an isocrystal $(V \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p, b \circ (\mathrm{id} \otimes \sigma))$. By Dieudonné theory, together with Γ this gives rise to a p-divisible group X_b.

One can define a functor on p-nilpotent $\breve{\mathbb{Z}}_p$-algebra

$$\mathcal{M}(X_b) : R \mapsto \left\{ (X, \rho) \mid \begin{array}{l} X : \ p\text{-divisible group over } R; \\ \rho : X_{R/p} \to (X_b)_{R/p} \text{ quasi-isogeny} \end{array} \right\} / \cong .$$

Theorem 12.1 (Rapoport-Zink). *$\mathcal{M}(X_b)$ is representable by a formal scheme over* $\mathrm{Spf}\,\breve{\mathbb{Z}}_p$ *formally smooth and formally of finite type.*

To construct the local Shimura variety:

$$\mathrm{RZ}_{\mathrm{GL}(\Gamma)}(\mathrm{GL}(V), b, \mu) = \mathcal{M}(X_b)^{\mathrm{rig}}.$$

For $K \subseteq G(\mathbb{Q}_p)$, one can also define $\mathrm{RZ}_K(G, b, \mu)$ parametrizes trivilizations of $T_p(X^{\mathrm{univ}})$ mod K.

Question: Is there a similar story for local Shimura varieties in general? Namely, let $K \subseteq G(\mathbb{Q}_p)$ be hyperspecial, does there exist a formal scheme $\mathcal{M}$ over $\mathcal{O}_{\breve{E}}$ such that

$$\mathcal{M}^{rig} = M_K(G, b, \mu)?$$

Why should one be interetsted?

- Informs some geometry of speical fibers of global Shimura viarieties via Rapoport-Zink uniformizations.
- Arithmetic fundamental lemma (Wei Zhang) and Kudla-Rapoport conjectures, etc.

12.3. **Integral models of local Shtuka spaces.** Let's keep (G, b, μ) as before and G is unramified, there exists $\mathcal{G}$ over $\mathbb{Z}_p$ a reductive group scheme whose generic fiber is G.

Let $S = \mathrm{Spa}(R, R^+)$ be a perfectoid space of characteristic p. Define

$$\mathrm{Spa}\,\mathbb{Z}_p = \mathrm{Spa}(W(R^+), W(R^+)) \backslash \{[\varpi] = 0\}.$$

Then $S \times \mathrm{Spa}\,\mathbb{Z}_p$ has a action of Frobenius.

- Replace $X \times_{\mathrm{Spec}\,\mathbb{F}_q} S$ from the function field case.
- Nice analytic adic space.

Recall that that untilts $S^\sharp$ of S gives rise to closed Cartier divisor $S^\sharp \subseteq S \times \mathbb{Z}_p$.

Definition. Let $S = \mathrm{Spa}(R, R^+)$ be a perfectoid space of characteristic p, $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp,+})$. A *$\mathcal{G}$-shtuka* over S with one leg at $S^\sharp$ is a pair $(\mathcal{P}, \varphi)$ where

- $\mathcal{P}$ is a $\mathcal{G}$-torsor over $S \times \mathbb{Z}_p$;
- $\varphi : \mathrm{Frob}^* \mathcal{P}|S \times \mathbb{Z}_p - S^\sharp \to \mathcal{P}|S \times \mathbb{Z}_p - S^\sharp$ is a meromorphic modification away from $S^\sharp$.

One also has a notion of boundedness by μ for $\mathcal{G}$-shtukas.

Definition. The *integral model* $\mathcal{M}^{\mathrm{int}}(\mathcal{G}, b, \mu)$ for $\mathrm{Sht}_{\mathcal{G}(\mathbb{Z}_p)}(G, b, \mu)$ is the functor on perfectoid spaces of characteristic p

$$\mathcal{M}^{\mathrm{int}}(G, b, \mu) : S = \mathrm{Spa}(R, R^+) \to \{(S^\sharp, \mathcal{P}, \varphi, \iota_r)\}/\cong$$

where

- $S^\sharp$ untilt of S;
- $(\mathcal{P}, \varphi)$ is $\mathcal{G}$-shtuka over S with one leg at $S^\sharp$ bounded by μ;
- $\iota_r : \mathcal{P}|\mathcal{Y}_{[r,\infty)(\mathcal{S})} \cong \mathcal{G} \times \mathcal{Y}_{[r,\infty)}(S)$, $r \gg 0$ such that $(b \times \mathrm{Frob}) \circ \iota_r = \iota_r \circ \varphi$ where
$$\mathcal{Y}_{[r,\infty)}(S) = \{x \in \mathrm{Spa}(W(R+)) \mid 0 < |[\varpi]|_x < |p^r|_x\} \subseteq S \times \mathbb{Z}_p.$$

Remark. This is a v-sheaf, where v-topology is generated by all open covers of surjective maps of affinoids. Generic fiber is $\mathrm{Sht}_{\mathcal{G}(\mathbb{Z}_p)}(G, b, \mu)$.

12.4. **Representability.** Let X be a formal scheme over $\mathrm{Spf}\,\mathbb{Z}$, S a perfectoid space of characteristic p, define

$$X^\diamond(S) = \{(S^\sharp, f) \mid S^\sharp \text{ untilt}, f : S^\sharp \to X \text{ morphism of adic spaces}\}/\cong.$$

Proposition 12.1. *This defines a fully faithful functor*

$\{$*formal scheme flat normal locally formally of finite type*$\} \to \{$*v-sheaves over* $\mathrm{Spd}\,\mathcal{O}_E = (\mathrm{Spf}\,\mathcal{O}_E)^\diamond\}$.

Conjecture 12.1 (Scholze)**.** $\mathcal{M}^{\mathrm{int}}(\mathcal{G}, b, \mu)$ *is in the essential image of* $(-)^\diamond$.

Known cases:

$\mathcal{G}$	reductive over $\mathbb{Z}_p$	parahoric over $\mathbb{Z}_p$
EL or PEL	Rapoport-Zink, Scholze	Rapoport-Zink, Scholze
Hodge type	Kim, Howard-Pappas, Bultil-Pappas, Pappas-Rapoport	Hamacher-Kim, Pappas-Rapoport
abelian type	Pappas-Rapoport	Pappas-Rapoport
general	unknown	unknown

12.5. GL_n**-case.** Take $\mathcal{G} = \mathrm{GL}_n$ over $\mathbb{Z}_p$, $\mu = \{1, \cdots, 1, 0, \cdots, 0\}$ and $b \in B(G, \mu)$. Then

$$\mathcal{M}(X_b)^\diamond \cong \mathcal{M}^{\mathrm{int}}(G, b, \mu).$$

Let's sketch a proof. Let $S = \mathrm{Spa}(R, R^+)$ be a perfectoid space of characteristic p. A point in $\mathcal{M}(X_b)^\diamond(S)$ is a tuple

- $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp,+})$ is an untilt;
- X is a p-divisible group over $R^{\sharp,+}$;
- $p : X_{R^{\sharp,+}/\varpi} \to (X_b)_{R^{\sharp,+}/\varpi}$ is a quasi-isogeny.

The first piece in the tuple gives rise to an untilt. The second one gives rise to $(\mathcal{P}, \varphi)$ a $\mathcal{G}$-shtuka. The third one gives rise to $\iota_r : \mathcal{P}|\mathcal{Y}[r, \infty) \cong \mathcal{G}$.