RTG SEMINAR ON SHIMURA VARIETIES

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Contents

1. Hermitian symmetric domains

1.1. Basic definitions.

Example 1.1. Let $\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid \operatorname{im}(z) = y > 0\}$. This has the following structures:

- complex manifold
- Riemmanian form (given by Petersson inner form) with negative curvature
- transitive SL_2-action
- i the unique point fixed by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Definition. A *Hermitian symmetric domain* (HSD) is a mainfold with the following structures:

(1) complex structure;
(2) Riemmanian form with negative curvature;
(3) an transitive action of adjoint real algebraic group;
(4) an isometric involution (fixes exactly one point);
(5) non-compact type;

Let X be a connected HSD, then

$$X = G(\mathbb{R})^+/K_p$$

for some real adjoint Lie group G and some stabilizer K_p of p by (3). It follows from (4) and (5) that involution acts on complexified tangent space at p as Cartan involutions: $\theta : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ such that $\mathfrak{g}_{\mathbb{C}}^{\theta}$ are a compact real form of $\mathfrak{g}_{\mathbb{C}}$. From this we can deduce that G is reductive and K_p is compact.

1.2. **Classification.** For $p \in X$ we can associate $u_p : U_1 = \{z \in \mathbb{C} \mid |z| = 1\} \to G(\mathbb{R})$ such that u_p fixes p and acts by z at the tangent space at p.

Theorem 1.1 (Classification of HSD)**.** *All non-compact connected HSD are of the form (G, X) where G is a real adjoint reductive Lie group, X is a $G(\mathbb{R})^+$-conjugacy class of $u_p : U_1 \to G(\mathbb{R})$ such that*

(1) on $\mathrm{Lie}_{\mathbb{C}}(G)$, characters of $\mathrm{Ad}_{U_1,\mathbb{C}}$ are $z, 1, z^{-1}$;
(2) $\mathrm{Ad}(u_p(-1))$ is a Cartan involution;
(3) $u_p(-1)$ does not project to 1 in any factor of G.

1.3. **Hodge structure.**

Definition. Let V be a real vector space. $V_{\mathbb{C}} = \oplus_{p+q=n} V^{p,q}$ is called a *Hodge structure* if $V_{p,q} \cong V_{q,p}$. A filtration given by $F^p V_{\mathbb{C}} = \oplus_{r \geq p} V^{r,s}$.

Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$. A Hodge structure can be identified with a representation $h : \mathbb{S} \to \mathrm{GL}(V)$ such that the Hodge structure on V is given by

$$h_{\mathbb{C}}(z_1, z_2)v = z_1^{-p} z_2^{-q} v, \quad \forall v \in V^{p,q}.$$

Definition. A *polarization* of V is a map $\psi : V \otimes V \to \mathbb{R}(-n)$ of $\mathbb{S}$-representations, where $\mathbb{S}_{\mathbb{C}} \cong \mathbb{C}^\times \times \mathbb{C}$ acts on $\mathbb{R}(-n)_{\mathbb{C}}$ by $z_1^{-n} z_2^{-n}$, such that $\psi_{\mathbb{C}}(h(i)u, \bar{v})$ is positive definite.

Consider all Hodge structures on V with

- specified dimensions of $V^{p,q}$;
- polarization $\psi : V \otimes V \to \mathbb{R}$;
- possible extra conditions (Hodge tensors);

These have a $\mathbb{C}$-structure inside a Flag manifold denoted by $S(d,t)$. Here $d : \{(p,q)\} \to \mathbb{Z}_{\geq 0}$ refers to the dimension conditions on $V^{p,q}$, and t is the polarization and the extra data.

Theorem 1.2. *If $S(d,t)$ satisfies Griffith's transversality (some condition on tangent space of $S(d,t)$ inside that of the flag manifold), then any connected component of $S(d,t)$ is a HSD. Conversely, any connected HSD is a connected component of some $S(d,t)$.*

Example 1.2. Let V be 2-dimensional, $d(1,0) = 1$, $d(0,1) = 1$, and $\psi = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. In this case, $S(d,t) \cong \mathcal{H}^- \subset \mathbb{P}^1(\mathbb{C})$ (correction: in the talk I said $\mathbb{H} \cup \mathbb{H}^-$ which is not true).

2. Quotients by arithmetic subgroups

Let (G, X) be a non-compact HSD, Γ an *arithmetic subgroup* (i.e., for any faithful representation $G \hookrightarrow \mathrm{GL}_n$, $\Gamma \cap G(\mathbb{R}) \cap \mathrm{GL}_n(\mathbb{Z})$ is of finite index in both Γ and $G(\mathbb{R}) \cap \mathrm{GL}_n(\mathbb{Z})$).

Theorem 2.1 (Baily-Borel)**.** *$\Gamma \backslash X$ has a unique structure of a quasi-projective algebraic variety if Γ is arithmetic and torsion-free.*

This idea behind this theorem is to show the Baily-Borel compactification X^{BB} has automorphic forms for the discrete subgroup Γ which separate the points X^{BB} and then embed X into projective space with these forms and apply Chow;s theorem. The uniqueness comes from

Theorem 2.2. *(Borel, Kwack, Kobayashi) Any holomorphic map $f : (\mathbb{D}\backslash\{0\})^r \times \mathbb{D}^s \to X$ extends to a holomorphic map $f : (\mathbb{D}\backslash\{0\})^r \times \mathbb{D}^s \to X^{BB}$.*

3. Shimura varieties

3.1. **Definition of Shimura varieties.**

Definition. A *connected Shimura datum* is (G, X) where G is a semisimple algebraic group over $\mathbb{Q}$, $X = G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class of homomorphism $u : U_1 \to G^{\mathrm{ad}}$ such that

(1) $\mathrm{Ad}(u)$ acts by $z, 1, z^{-1}$ on $\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{C}})$;
(2) $\mathrm{Ad}(u(-1))$ is a Cartan involution.
(3) G^{ad} has no $\mathbb{Q}$-factor H with $H(\mathbb{R})$ compact.

Definition. A *connected Shimura variety* is the inverse limit of varieties $\{\Gamma\backslash X\}_\Gamma$ (for Γ small enough and commensurate).

Each $\Gamma\backslash X$ is the quotient of a Hermitian symmetric domain. The $\Gamma\backslash X$ are defined over different fields.

Example 3.1. $\Gamma(N)\backslash\mathcal{H}$ is defined over $\mathbb{Q}(\zeta_n)$.

Definition. A *Shimura datum* is (G, X) where G is reductive algebraic group over $\mathbb{Q}$, $X = G(\mathbb{R})$-conjugacy class of homomorphism $h : \mathbb{S} \to G_{\mathbb{R}}$ such that

(1) $\mathrm{Ad}(h)$ acts on $\mathrm{Lie}(G_{\mathbb{C}})$ with characters $z_1 z_2^{-1}, 1, z_1^{-1} z_2$;
(2) $\mathrm{Ad}(h(i))$ is a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$;
(3) G^{ad} has no $\mathbb{Q}$-factor H with $H(\mathbb{R})$ compact.

A *shimura variety* is $\{G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K\}_K$ where K is a compact subgroup of $G(\mathbb{A}_f)$.

$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K = \bigsqcup_i \Gamma_i\backslash X$ where $\Gamma_i = G(\mathbb{Q}) \cap g_i K g_i^{-1}$ and g_i runs over the representatives of $G(\mathbb{Q})$-K double cosets. By strong approximation, this is a finite union.

3.2. **Examples.**

Example 3.2 (Siegel modular variety)**.** We get this from the variation of Hodge structure $d(0, 1) = n$, $d(1, 0) = n$ and ψ a symplectic form. The group theoretical way to view this is to look at the group

$$\mathrm{GSp}(V, \psi) = \{g \in \mathrm{GL}(V) \mid \psi(gv, gu) = \lambda(g)\psi(v, u)\}.$$

acting on $\{X \in M_{n\times n}(\mathbb{C}) \mid Z^t = Z, \mathrm{im}(Z) > 0\}$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}.$$

From the Hodge structure description, $GSp(\mathbb{Q})\backslash X \times GSp(\mathbb{A}_f)/K$ parameterizes Abelian varieties of dimension $dim(V)/2$ with a polarization and rigidity data (level structure).

Example 3.3 (Hodge type)**.** A *Hodge type* Shimura variety is one which embeds into a Seigel modular variety. We get these from a variation of Hodge structure $d(0, 1) = n$, $d(1, 0) = n$, ψ a symplectic form, and some Hodge tensors.

Example 3.4. Let B be a quaternion algebra over F, a totally ramified field, $G(\mathbb{Q}) = B^\times$ for some algebraic group over $\mathbb{Q}$. If

$$G(\mathbb{R}) \cong \mathbb{H}^\times \times \mathbb{H}^\times \times \cdots \times \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$$

have at least one $\mathrm{GL}_2(\mathbb{R})$, define

$$h(a + bi) = 1 \times 1 \times \cdots \times \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \times \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$$

When there is at least one copy of $\mathbb{H}$, (G, X) is not of Hodge type.

4. Canonical models

4.1. **Canonical models of tori.**

Since a torus is abelian, X is always a single point in a Shimura datum (T, X). Also

Proposition 4.1. *Each $T(\mathbb{Q})\backslash\{h\} \times T(\mathbb{A}_f)/K$ is a finite sets of points.*

We can now form a $\mathbb{Q}$ model of $Sh(T,X)$ by turning each point into a $\bar{\mathbb{Q}}$ point. Giving a model of $Sh(T,X)$ over a number field F amounts to giving an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ on the $\bar{\mathbb{Q}}$ points of $\{T(\mathbb{Q})\backslash\{h\}\times T(\mathbb{A}_f)/K\}_K$ commuting with the maps in the inverse limit. This gives an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ on the (topological) inverse limit of $\{T(\mathbb{Q})\backslash\{h\}\times T(\mathbb{A}_f)/K\}_K$, which is $T(\mathbb{Q})\backslash\{h\}\times T(\mathbb{A}_f)$, with a $T(\mathbb{A}_f)$ action on the right coming from the translation maps in the inverse system of the Shimura variety. Using CFT, giving our desired action is equivalent to giving a grouphom

$$\mathrm{Hom}(F^\times\backslash\mathbb{A}_F, T(\mathbb{Q})\backslash T(\mathbb{A}_f)).$$

Notice such a grouphom gives an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ on $T(\mathbb{Q}),\backslash T(\mathbb{A}_f)/K$ for all K.

Here is how we choose this grouphom: Define $\mu : \mathrm{G}_m \to \mathbb{S}_\mathbb{C}$ as the composition of $z \mapsto (z,1)$ and $h_\mathbb{C}$. Choose F to the be the field of definition of μ. Then composing $\mu(\mathbb{A}_F)$ with the norm map $T(\mathbb{A}_F) \to T(\mathbb{A})$ and projecting onto the finite part is our desired grouphom.

4.2. **Canonical models of general Shimura varieties.**

Definition. Define $\mu : \mathrm{G}_m \to \mathbb{S}_\mathbb{C}$ as the composition of $z \mapsto (z,1)$ and $h_\mathbb{C}$ for some $h \in X$. The *reflex field* of (G,X) is the field of definition of the conjugacy class of μ.

This is a finite extension of $\mathbb{Q}$: In particular

$$E(G,X) = \text{the fixed field of } \{\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) : \sigma(W\mu) = W\mu \text{ in the cocharacter lattice}\}.$$

Definition. The *canonical model* of $\mathrm{Sh}(G,X)$ is the unique model over $E(G,X)$ such that for all morphisms of Shimura data of the form $(T,\{h\}) \to (G,X)$ where T is a torus, the morphisms

$$\{(M(T,\{h\})_K)_\mathbb{C}\} \to \{(M(G,X)_K)_\mathbb{C}\}$$

are defined over $E(G,X)\cdot E(T,\{h\})$. The morphisms $(T,\{h\}) \to (G,X)$ are called *special points*.

Theorem 4.1. *There exists a unique canonical model of* $\mathrm{Sh}(G,X)$.

The uniqueness follows from the special points being very dense:

Lemma 4.1. *For any field extension L of $E(G,X)$, there exists a special point $(T,\{h\})$ of (G,X) such that $E(T,\{h\}) \cap L = E(G,X)$.*