Representations of $\mathrm{SL}(2,\mathbb{R})$

UMich Representation Theory Learning Seminar 2021

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This talk primarily follows sections II.1 and V.3 in *Non-Abelian Harmonic Analysis* by Roger Howe and Eng Chye Tan.

Indecomposable, Quasisimple, h-Multiplicity Free Modules

In the following, we denote by $\mathfrak{sl}(2)$ the Lie algebra of $\mathrm{SL}(2,\mathbb{R})$, i.e. the space of 2×2 real matrices with zero trace. There is a basis of $\mathfrak{sl}(2)$ given by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The *Casimir element* is $\mathcal{C} := h^2 + 2(e^+e^- + e^-e^+)$. It generates the center of the universal enveloping algebra of $\mathfrak{sl}(2)$, i.e. $Z(\mathcal{U}((2))) = \mathbb{C}[\mathcal{C}]$.

Definition. A representation V of $\mathfrak{sl}(2)$ is

(a) *Decomposable* if it can be written as a direct sum of two nontrivial subrepresentations. Otherwise it is *indecomposable.*

(b) *Quasi-simple* if $\mathcal{C}$ acts via a multiple of the identity on V.

(c) *h-multiplicity free* if the h-eigenspaces are of dimension at most 1.

Goal: To classify the indecomposable, quasi-simple, h-multiplicity free $\mathfrak{sl}(2)$-modules.

Definition. Let V be an $\mathfrak{sl}(2)$ module and $v_0 \in V$ be an h-eigenvector of eigenvalue λ. Then for each $k \in \mathbb{Z}_+$ define $v_k = (e^+)^k v_0$ and $v_{-k} = (e^-)^k v_0$.

Theorem. (Proposition 1.1.4 in Howe-Tan.) For $k \in \mathbb{Z}_+$, v_k is either zero or an h eigenvector of eigenvalue $\lambda + 2k$ and v_{-k} is either zero or an eigenvector of eigenvalue $\lambda - 2k$.

If v_0 is also an eigenvector for $\mathcal{C}$ with eigenvalue μ then the set of nonzero v_k forms a basis for the $\mathfrak{sl}(2)$-module V_0 generated by v_0. Also, for all $k \in \mathbb{Z}$ we have

$$e^+e^-v_k = \frac{\mu - (\lambda + 2k - 1)^2 + 1}{4} v_k := s_1(k)v_k$$
$$e^-e^+v_k = \frac{\mu - (\lambda + 2k)^2 - 2(\lambda + 2k)}{4} v_k := s_2(k)v_k$$

Note in particular that $s_1(k) = s_2(k-1)$ for all k.

Theorem. Let V be an indecomposable, quasisimple, h-multiplicity free $\mathfrak{sl}(2)$-module. Then $V = V_0$ for some V_0 as above.

Idea: V is completely classified by which v_k are killed by e^+ and e^-.
Recall that

$$e^+e^-v_k = \frac{\mu - (\lambda + 2k - 1)^2 + 1}{4} v_k$$

Hence there are at most two k such that $v_k \in \ker(e^+e^-)$, and so $\dim\ker(e^+e^-) \leq 2$. Also note that $\dim\ker(e^+e^-) = \dim\ker(e^-e^+)$ since $s_1(k) = s_2(k-1)$. Thus we have three cases:

Case A: If $\dim\ker(e^+e^-) = \dim\ker(e^-e^+) = 0$. In this case, there are no e^+ or e^--null vectors, and so all v_k are non-zero. We denote this as

$$(\circ) = \cdots \circ \circ \circ \cdots \circ \circ \circ \cdots$$

where the dots represent h eigenvectors, arranged by increasing h eigenvalues, from left to right.

Case B: If $\dim\ker(e^+e^-) = \dim\ker(e^-e^+) = 1$. One option in this case is that $\dim\ker(e^+) = 1$ and $\dim\ker(e^-) = 0$. Then there are two isomorphism classes:

$$(\circ]\circ) = \cdots \circ \circ \circ \cdots \circ \circ]\, \circ \cdots$$
$$(\circ]) = \cdots \circ \circ \circ \circ \cdots \circ]$$

where the right bracket "]" indicates that the h eigenvector to the immediate left of the bracket is killed by e^+. The other option is that $\dim\ker(e^+) = 0$ and $\dim\ker(e^-) = 1$. There are again two isomorphism classes:

$$(\circ[\circ) = \cdots \circ\, [\circ \circ \cdots \circ \circ \circ \circ \cdots$$
$$([\circ) = [\circ \circ \cdots \circ \circ \circ \circ \cdots$$

where the left bracket "[" indicates that the h eigenvector to the immediate right of the bracket is killed by e^-.

Case C: If $\dim\ker(e^+e^-) = \dim\ker(e^-e^+) = 2$. Here there are nine isomorphism classes:

$$
\begin{aligned}
(\circ[\circ]\circ) &= \cdots \circ \circ[\circ \circ \cdots \circ \circ] \circ \circ \cdots \\
(\circ[\circ]) &= \cdots \circ \circ[\circ \circ \cdots \circ \circ] \\
([\circ]\circ) &= \qquad\quad [\circ \circ \cdots \circ \circ] \circ \circ \cdots \\
([\circ]) &= \qquad\quad [\circ \circ \cdots \circ \circ] \\
(\circ]\circ]\circ) &= \cdots \circ \circ] \circ \circ \cdots \circ \circ] \circ \circ \cdots \\
(\circ]\circ]) &= \cdots \circ \circ] \circ \circ \cdots \circ \circ] \\
(\circ[\circ[\circ) &= \cdots \circ \circ[\circ \circ \cdots \circ \circ[\circ \circ \cdots \\
([\circ[\circ) &= \qquad\quad [\circ \circ \cdots \circ \circ[\circ \circ \cdots \\
(\circ] \circ [\circ) &= \cdots \circ \circ] \circ \circ \cdots \circ \circ[\circ \circ \cdots
\end{aligned}
$$

where the number of dots between a pair of brackets "[" and "]" is the same in all cases and is the dimension of the finite-dimensional piece in the composition series.

Vanishing of Matrix Coefficients

Definition. Let (ρ, V) be a representation of a Lie group G and let V^* be the dual space to V. Then for $\lambda \in V^*, v \in V, g \in G$ define

$$\phi_{\lambda,v}(g) := \lambda(\rho(g)v)$$

Then each $\phi_{\lambda,v}$ is called a *matrix coefficient* of ρ.

Matrix coefficients are interesting because:

- In appropriate coordinates, matrix coefficients become various classical special functions. (For example the standard context for understanding the Bessel functions is as matrix coefficients of the Euclidean groups.)
- The behavior of matrix coefficients at infinity is related to number theoretic questions, such as the Ramanujan conjecture, and to ergodic theory.

Definition. For $s \in \mathbb{C}$ define the spaces

$$
\begin{aligned}
S^{s,+} &= \{f \in C^\infty(\mathbb{R}^2 - \{0\}) \mid f(tx) = |t|^s f(x)\} \\
S^{s,-} &= \{f \in C^\infty(\mathbb{R}^2 - \{0\}) \mid f(tx) = |t|^s (\operatorname{sgn} t) f(x)\},
\end{aligned}
$$

where $\mathrm{SL}(2,\mathbb{R})$ acts on $S^{s,\pm}$ through its standard action on $\mathbb{R}^2$.

The $\mathrm{SL}(2,\mathbb{R})$ representations $S^{-1+it,+}(t \in \mathbb{R})$ and $S^{-1+it,-}(t \in \mathbb{R} - \{0\})$ are called the *principal series representations.* If $s \in (-2,0)$ then $S^{s,+}$ are called the *complementary series representations.* These are both *unitarizable* and in some cases *unitary* (see Theorem 1.3.1 in Howe-Tan). Define an $\mathrm{SL}(2,\mathbb{R})$-invariant pairing between $S^{-1-\overline{\alpha},\pm}$ and $S^{-1+\alpha,\pm}$ by for $f \in S^{-1-\overline{\alpha},\pm}, h \in S^{-1+\alpha,\pm}$:

$$(f,h) = \int_0^{2\pi} f(\cos\theta, \sin\theta)\overline{h(\cos\theta,\sin\theta)}\, d\theta$$

This identifies $S^{-1-\overline{\alpha},\pm}$ with the conjugate dual space $(S^{-1+\alpha,\pm})'$ (see example 1.2.12 in Howe-Tan), hence we have matrix coefficients

$$\phi_{f,h}(\tilde{a}) = (f, \rho(\tilde{a})h)$$

where ρ denotes the action of $\mathrm{SL}(2,\mathbb{R})$.

Definition. Define the semigroup:

$$A^+ = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a > 1 \right\} \subset \mathrm{SL}(2,\mathbb{R})$$

Theorem. (Proposition 3.1.5 in Howe-Tan.) For $f \in S^{-1-\overline{\alpha},\pm}, h \in S^{-1+\alpha,\pm}$, and $\tilde{a} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in A^+$,

$$|\phi_{f,h}(\tilde{a})| \le C\frac{\log a}{a}, \quad \text{if } \mathrm{Re}(\alpha) = 0, (\text{Think: princ ser reps})$$

and

$$|\phi_{f,h}(\tilde{a})| \le C\frac{1}{a^{1-\mathrm{Re}(\alpha)}}, \quad \text{if } 0 < \mathrm{Re}(\alpha) < 1. (\text{Think: compl ser reps})$$

where C is a constant dependent on f, h, and α.

Definition. A sequence in a space X *goes to infinity* if it has no limit point in X.

Definition. A complex-valued function f on a space X *vanishes at* ∞ if, for every sequence $\{x_n\}$ which goes to ∞, we have $\lim_{n\to\infty} f(x_n) = 0$.

Note: In A^+, $\tilde{a} \to \infty$ iff $a \to \infty$. Furthermore, to check that $\phi_{f,h}$ goes to infinity on $\mathrm{SL}(2,\mathbb{R})$, it is sufficient to check for sequences in A^+. Hence the matrix coefficients for complementary series representations and principal series representations vanish at infinity. In particular, coefficients for complementary series reps decay like $1/a^{1-\mathrm{Re}(\alpha)}$ and those for principal series reps decay like $1/a$. Furthermore:

Theorem. (Theorem 2.0.3 in Howe-Tan.) Let (ρ, V) be a unitary representation of $\mathrm{SL}(2,\mathbb{R})$ not containing the trivial representation. Then the matrix coefficients of (ρ, V) must vanish at ∞.