SUPERCUSPIDAL REPRESENTATIONS OF SL_2 OVER p-ADIC FIELD

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Contents

1. Notation and Background

1.1. **Notations.** Let F be a finite extension of $\mathbb{Q}_p$ ($p \neq 2$). Denote the residual field of F by k, a finite field of order q. Let the ring of integers of F be $\mathcal{O}_F$ and its maximal ideal $\mathfrak{p}$. $\{\mathfrak{p}^n\}_{n\geq 0}$ form a neighbourhood basis of F at 0, and under this topology F is locally compact and totally disconnected. Let ϖ be the uniformizer, and normalize the nontrivial discrete valuation ω on F so that $\omega(\varpi) = 1$. $F^\times/(F^\times)^2$ can be represented by $\{1, \varpi, \epsilon, \epsilon\varpi\}$ where ϵ is a fixed nonsuqare in $\mathcal{O}_F^\times$ (chosen to be -1 if $-1 \notin (F^\times)^2$).

Set

$$\mathfrak{u}_n = 1 + \mathfrak{p}^n$$

to be the neighbourhood basis at 1.

Remark. This is not the usual filtration subgroup used for $\mathcal{O}_F^\times$. It's only a neighbourhood basis.

We fix an additive quasi-character Ψ of F which is trivial on $\mathfrak{p}$ but nontrivial on $\mathcal{O}_F$.

For algebraic groups over F, we use the following notations:

objects	notations
algebraic groups	$\mathbb{G}, \mathbb{S}, \mathbb{T}$
groups of F-points	G, S, T
Lie algebras	$\mathfrak{g}, \mathfrak{s}, \mathfrak{t}$

Moreover, by embedding $\mathbb{G}$ in GL_n, for any linear algebraic group $\mathbb{G}$ over F, its group of F-points $\mathbb{G}(F)$ inherits a topology from F that makes $\mathbb{G}(F)$ totally disconnected and locally closed.

1.2. **Affine apartments, buildings and filtration subgroups.** Let $\mathbb{G}$ be a semisimple split group defined over F, and $\mathbb{S}$ a maximal torus of $\mathbb{G}$ split over F. Let $X_*(\mathbb{S})_F$ (resp. $X^*(\mathbb{S})_F$) be the group of F-rational cocharacters (resp. characters) of $\mathbb{S}$. By tensoring $\omega : F^\times \to \mathbb{Z}$ with $X_*(\mathbb{S})$, we get $\omega : \mathbb{S}(F) \to X_*(\mathbb{S})_F$, called the *valuation homomorphism* of $\mathbb{S}(F)$.

1.2.1. *Apartments.* Let $\Phi(\mathbb{G},\mathbb{S},F)$ be the root system associated to $(\mathbb{G},\mathbb{S})$ defined over F. For each $\alpha\in\Phi(\mathbb{G},\mathbb{S},F)$, the associated root group $\mathbb{U}_\alpha$ is F-isomorphic to the additive group $\mathbb{G}_a$.

Definition. A *Chevalley system* is a system of F-isomorphism $u_\alpha:\mathbb{G}_a\cong\mathbb{U}_\alpha$ satisfying the commutation relations of Chevalley.

Definition. A *valuation* of root datum is a collection $\varphi=(\varphi_\alpha)_{\alpha\in\Phi(\mathbb{G},\mathbb{S},F)}$ where $\varphi_\alpha:\mathbb{U}_\alpha(F)\to\mathbb{R}\cup\{\infty\}$ such that

(V0) For each $\alpha\in\Phi(\mathbb{G},\mathbb{S},F)$, $\varphi_a(\mathbb{U}_a(F))$ consists of at least 3 elements.

(V1) For each $\alpha\in\Phi(\mathbb{G},\mathbb{S},F)$ and $r\in\mathbb{R}$,

$$U_{\alpha,\varphi,r}:=\varphi_\alpha^{-1}([r,\infty))$$

is a subgroup of $\mathbb{U}_\alpha(F)$.

(V2) For every $n\in\mathrm{N}_{\mathbb{G}}(\mathbb{S})(F)$ that realizes the reflection along $\alpha\in\Phi(\mathbb{G},\mathbb{S},F)$, the function

$$u\mapsto\varphi_\alpha(u)-\varphi_{-\alpha}(n\cdot u\cdot n^{-1})$$

is constant $\mathbb{U}_\alpha(F^*)$ on $\mathbb{U}_\alpha(F)^*=\mathbb{U}_\alpha(F)\backslash\{1\}$.

(V3) For non-proportional roots $\alpha,\beta\in\Phi(\mathbb{G},\mathbb{S},F)$ and $r,s\in\mathbb{R}$, the commutator $[\mathbb{U}_{\alpha,\varphi,r},\mathbb{U}_{\beta,\varphi,s}]$ is contained in the group generated by $\mathbb{U}_{a\alpha+b\beta,ar+bs}$ for all $a,b\in\mathbb{Z}_{\geq0}$ such that $a\alpha+b\beta\in\Phi(\mathbb{G},\mathbb{S},F)$.

(V4) If $\alpha,2\alpha\in\Phi(\mathbb{G},\mathbb{S},F)$, then $\varphi=2\cdot\varphi|_{\mathbb{U}_{2\alpha}(F)}$.

(V5) For $\alpha\in\Phi(\mathbb{G},\mathbb{S},F)$ and $u\in\mathbb{U}_a(F)^*$, $\varphi_\alpha(u)=-\varphi_{-\alpha}(u')=-\varphi_{-\alpha}(u'')$ for the Tits element u',u'' of u.

In addition, the following axiom holds, we say that φ is *compatible* with ω:

(V6) For all $z\in Z_{\mathbb{G}}(\mathbb{S})(F)$,

$$\varphi_\alpha(z\cdot u\cdot z^{-1})=\varphi_\alpha(u)+\alpha(\omega_{Z_{\mathbb{G}}(\mathbb{S})}(z))$$

where $\omega_{Z_{\mathbb{G}}(\mathbb{S})}:Z_{\mathbb{G}}(\mathbb{S})(F)\to X_*(\mathbb{S})_F$ is the valuation homomorphism.

Define an action of $V(\mathbb{S})_F=X_*(\mathbb{S})_F\otimes_{\mathbb{Z}}\mathbb{R}$ on the set of valuations of root data: given $\varphi=(\varphi_\alpha)$, $v\in V(\mathbb{S})_F$, define $\varphi+v$ by

$$(\varphi+v)_\alpha(u)=\varphi_\alpha(u)+\langle a,v\rangle.$$

Definition. Two valuations φ_1 and φ_2 are *equipollent* if $\varphi_1=\varphi_2+V(\mathbb{S})_F$.

Define an action of $N_{\mathbb{G}}(\mathbb{S})(F)$ on the set of valuations of the root datum: given $n\in N_{\mathbb{G}}(\mathbb{S})(F)$, $\varphi=(\varphi_\alpha)$ valuation

$$(n\cdot\varphi)_\alpha(u)=\varphi_{n^{-1}\alpha}(n^{-1}\cdot u\cdot n),\quad u\in\mathbb{U}_\alpha(F).$$

From a Chevalley system $\{u_\alpha\}_{a\in\Phi(\mathbb{G},\mathbb{S},F)}$, we obtained a valuation of the root datum:

$$\varphi_a=\omega\circ u_a^{-1}.$$

Definition. The valuations obtained from a Chevalley system is called a *Chevalley valuation.*

Proposition 1.1. *All Chevalley valuations are equipollent and compatible wth ω.*

Definition. The *apartment* $\mathcal{A}=\mathcal{A}(\mathbb{G},\mathbb{S},F)$ is the equipollent class of all Chevalley valuations of $\Phi(\mathbb{G},\mathbb{S},F)$. It is an affine space under $V(\mathbb{S})_F=X_*(\mathbb{S})_F\otimes\mathbb{R}$.

We see that $N_{\mathbb{G}}(\mathbb{S})(F)$ acts on $A(\mathbb{G},\mathbb{S},F)$ by affine transformations, and $s\in\mathbb{S}(F)$ acts by $-\omega(s)=-(\alpha\mapsto\omega(\alpha(s)))\in V(\mathbb{S})_F$.

1.2.2. *Filtration subgroups.* Let $\tilde{\mathbb{R}} = \mathbb{R} \cup \mathbb{R}_+ \cup \{\infty\}$. We can endow $\tilde{\mathbb{R}}$ with an order by setting

$$\begin{aligned} \infty &> r, \quad \forall r \in \mathbb{R}; \\ r_+ &> r, \quad \forall r \in \mathbb{R}; \\ r_+ &< s, \quad \forall r < s. \end{aligned}$$

For each $(\varphi, *) \in A(\mathbb{G}, \mathbb{S}, F) \times \tilde{\mathbb{R}}_{\geq 0}$, we can define the *Moy-Prasad filtration subgroup* $\mathbb{G}(F)_{\varphi,*} = \langle U_{\alpha,\varphi,*}, \mathbb{S}(F)_* \rangle_{\alpha \in \Phi(\mathbb{G},\mathbb{S},F)}$ where

$$\begin{aligned} U_{\alpha,\varphi,r_+} &= \bigcup_{s>r} U_{\alpha,\varphi,s} \\ \mathbb{S}(F)_r &= \{s \in \mathbb{S}(F) \mid \omega(\chi(s) - 1) \geq r, \forall \chi \in X^*(\mathbb{S})_F\} \\ \mathbb{S}(F)_{r_+} &= \bigcup_{s>r} \mathbb{S}(F)_s. \end{aligned}$$

This form a decreasing filtration of $\mathbb{G}(F)$. These filtration behave well under conjugation in the sense that

$$\mathbb{G}(F)_{nx,r} = \mathbb{G}(F)^n_{x,r}, \quad n \in N_{\mathbb{G}}(\mathbb{S})(F).$$

Similarly, For each $(\varphi, *) \in A(\mathbb{G}, \mathbb{S}, F) \times \tilde{\mathbb{R}}$, we can define the $\mathcal{O}_F$-submodule $\mathfrak{g}(F)_{\varphi,*} \subseteq \mathfrak{g}(F)$.

1.2.3. *Buildings.*

Definition. The *building* $\mathcal{B}(\mathbb{G}, F)$ of $\mathbb{G}$ over F can be constructed by "gluing together" $\mathcal{A}(\mathbb{G}, \mathbb{S}, F)$ for various maximal F-split tori $\mathbb{S}$ of $\mathbb{G}$:

$$\mathcal{B}(\mathbb{G}, F) = \mathbb{G}(F) \times A(\mathbb{G}, \mathbb{S}, F)/\sim$$

where $(g, x) \sim (h, y)$ if and only if there exists $n \in N_{\mathbb{G}}(\mathbb{S}(F))$ such that $n \cdot x = y$ and $g^{-1}hn \in \mathbb{G}(F)_{x,0}$. We write $[g, x]$ for the equivalence class of (g, x) in the building.

Proposition 1.2. *$\mathbb{G}(F)$ acts on $\mathcal{B}(\mathbb{G}, F)$ on the first factor.*

(1) For any $g \in \mathbb{G}(F)$, $[g, A(\mathbb{G}, \mathbb{S}, F)] \cong A(\mathbb{G}, \mathbb{S}, F)$. These sets are called the apartments of $\mathcal{B}(\mathbb{G}, \mathbb{S}, F)$. There is a G-equivariant bijection

$$\begin{aligned} \{\text{apartments in } \mathcal{B}(\mathbb{G}, \mathbb{S}, F)\} &\cong \{\text{maximal split tori}\} \\ [g, A(\mathbb{G}, \mathbb{S}, F)] &\mapsto \mathbb{S}^g \end{aligned}$$

(2) By identifying $A(\mathbb{G}, \mathbb{S}, F)$ with $[e, A(\mathbb{G}, \mathbb{S}, F)]$, the stabilizer of $A(\mathbb{G}, \mathbb{S}, F)$ in $\mathbb{G}(F)$ is $N_{\mathbb{G}}(\mathbb{S})(F)$. The stabilizer of $x \in A(\mathbb{G}, \mathbb{S}, F)$ in $\mathbb{G}(F)$ is $\mathbb{G}(F)_{x,0}$.

Remark. We can define the Moy-Prasad filtration for all $y \in \mathcal{B}(\mathbb{G}, \mathbb{S}, F)$ by writting $y = gx$ for $g \in \mathbb{G}(F), x \in A(\mathbb{G}, \mathbb{S}, F)$ and

$$\mathbb{G}(F)_{gx,r} = \mathbb{G}(F)^g_{x,r}.$$

In particular it follows from Proposition 1.2(2) that for all $y \in \mathcal{B}(\mathbb{G}, \mathbb{S}, F)$, the stabilizer of y in $\mathbb{G}(F)$ is $\mathbb{G}(F)_{y,0}$.

For any Galois extension F'/F, we can view $\mathbb{G}$ semisimple split group define over F', and similarly define $\mathcal{B}(\mathbb{G}, F') := G(F') \times A(\mathbb{G}, \mathbb{S}, F')/\sim$. The Galois group $\mathrm{Gal}(F'/F)$ acts naturally on $\mathcal{B}(G, F')$ (on the first factor). Buildings behave functorially with respect to Galois extensions.

Proposition 1.3. *There is a unique system of injections*

$$\iota_{F_2F_1} : \mathcal{B}(\mathbb{G}, F_1) \hookrightarrow \mathcal{B}(\mathbb{G}, F_2) \quad (F_1 \subseteq F_2 \text{ are Galois extensions of } F)$$

with the following properties:

(1) the image of $\iota_{F_2F_1}$ is fixed by $\mathrm{Gal}(K_2/K_1)$;
(2) the restriction of $\iota_{F_2F_1}$ to any apartment is an affine mapping into an apartment of $\mathcal{B}(\mathbb{G}, F_2)$;

(3) $\iota_{F_2F_1}$ *is* $\mathbb{G}(F_1)$*-equivariant;*
(4) if $F_1 \subseteq F_2 \subseteq F_3$*, then* $\iota_{F_3F_1} = \iota_{F_3F_2} \circ \iota_{F_2F_1}$*.*

Moreover, if F'/F *is unramified (or even tamely unramified),* $\mathcal{B}(\mathbb{G}, F)$ *is the fixed point set of* $\mathrm{Gal}(F'/F)$ *in* $\mathcal{B}(\mathbb{G}, F')$*, and for any maximal* F'*-split* $\mathbb{T}$*,* $A(\mathbb{G}, \mathbb{T}, F') \cap \mathcal{B}(\mathbb{G}, F)$ *can be identified with* $A(\mathbb{G}, \mathbb{T}, F) \cong X_*(\mathbb{T})_F \otimes \mathbb{R}$*.*

Definition. The *depth* of an irreducible representation (π, V) of G is the least $r \in \mathbb{R}_{\geq 0}$ such that there exists $x \in \mathcal{B}(\mathbb{G}, F)$ for which V contains nonzero vectors invariant under G_{x,r_+}.

1.2.4. *Filtration subgroups in* SL_2. Now let $\mathbb{G} = \mathrm{SL}_2$, $\mathbb{B}$ the standard Borel subgroup of upper triangular matrices, $\mathbb{S}$ the standard maximal split torus of diagonal matrices Let G, B, S be the group of F-points of $\mathbb{G}, \mathbb{B}$ and $\mathbb{S}$.

We have $\Phi(\mathbb{G}, \mathbb{S}, F) = \{\pm\alpha\}$ with the two root subgroups

$$\mathbb{U}_\alpha = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$$
$$\mathbb{U}_{-\alpha} = \begin{pmatrix} 1 & \\ * & 1 \end{pmatrix}.$$

So $A(\mathbb{G}, \mathbb{S}, F)$ is an affine space under $X_*(\mathbb{S})_F \otimes \mathbb{R} \cong \mathbb{R}$. By identifying 0 in $A(\mathbb{G}, \mathbb{S}, F)$ and choosing coordinates so that $\alpha^\vee = 2$, we have $G_{0,0} = \mathrm{SL}_2(\mathcal{O}_F)$. This is a maximal compact subgroup of G which we denote by K. On the other hand,

$$G_{1,0} = \begin{pmatrix} \mathcal{O}_F & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathcal{O}_F \end{pmatrix} = K^\eta$$

where $\eta \in \mathrm{GL}_2(F)$ is given by

$$\eta = \begin{pmatrix} 1 & \\ & \varpi \end{pmatrix}.$$

Up to conjugacy, K and K^η are the only two maximal compact subgroups of G.

More generally, we can describe the Moy-Prasad filtration subgroups $G_{x,r}$ of G. It suffices to describe $G_{y,r}$ for $y \in A(\mathbb{G}, \mathbb{S}, F)$. For $r \in \tilde{\mathbb{R}}$, set

$$\lceil r \rceil = \begin{cases} \min\{n \in \mathbb{Z} \mid n \geq r\} & \text{if } r \in \mathbb{R} \\ \min\{n \in \mathbb{Z} \mid n > r\} & \text{if } r \in \mathbb{R}_+. \end{cases}$$

Then we have

$$U_{\alpha,y,r} = \begin{pmatrix} 1 & \mathfrak{p}^{\lceil r-y \rceil} \\ 0 & 1 \end{pmatrix}$$
$$U_{-\alpha,y,r} = \begin{pmatrix} 1 & 0 \\ \mathfrak{p}^{\lceil r+y \rceil} & 1 \end{pmatrix}$$
$$S_r = \begin{pmatrix} \mathfrak{u}_{\lceil r \rceil} & \\ & \mathfrak{u}_{\lceil r \rceil}. \end{pmatrix}$$
$$G_{y,r} = \begin{pmatrix} \mathfrak{u}_{\lceil r \rceil} & \mathfrak{p}^{\lceil r-y \rceil} \\ \mathfrak{p}^{\lceil r+y \rceil} & \mathfrak{u}_{\lceil r \rceil} \end{pmatrix}.$$

1.3. **Anisotropic tori.**

Definition. A torus $\mathbb{T}$ is *anisotropic* over F if $X_*(\mathbb{T})_F = \{1\}$.

Now let $\mathbb{G} = \mathrm{SL}_2$, all non-split tori are anisotropic and split over a quadratic extension F' of F. A torus is called *unramified* if F'/F is unramified, and *ramified* otherwise.

Let $\mathbb{T}$ be an anisotropic torus of $\mathbb{G}$. Since $p \neq 2$, the splitting field F' is always tamely unramified. Therefore by Proposition 1.3, we can view $\mathcal{B}(\mathbb{G}, F)$ as the set of $\mathrm{Gal}(F'/F)$-fixed points of $\mathcal{B}(\mathbb{G}, F')$, and $A(\mathbb{G}, \mathbb{T}, F) = A(\mathbb{G}, \mathbb{T}, F) \cap \mathcal{B}(\mathbb{G}, \mathbb{S}, F)$ consists of a single point, denoted by $\{y\}$. We can determine y in the following way.

Write $y = [g, x]$ for the point in the building corresponding to the equivalence class of (g, x). Since $\mathbb{T}$ is F'-split, we can write $\mathbb{T} = \mathbb{S}^g$, and therefore $A(\mathbb{G}, \mathbb{T}, F') = g \cdot A(\mathbb{G}, \mathbb{S}, F)$ for some $g \in \mathbb{G}(F')$. (here we identify $A(\mathbb{G}, \mathbb{S}, F)$ with $A(\mathbb{G}, \mathbb{S}, F')$, since $\mathbb{S}$ is F-split). Since $\sigma \in \mathrm{Gal}(F'/F)$ acts on $[g, x]$ by $\sigma \cdot [g, x] = [\sigma(g), x]$, so $[g, x]$ is Galois fixed if and only if $\sigma(g)g^{-1} \in \mathbb{G}(F')_x$. In this way we determine both g and x, hence $y = [g, x]$.

The following table shows the representatives of conjugacy classes of anisotropic tori in $\mathbb{G}$.

$\mathbb{T}$	splitting field F'	$A(\mathbb{G}, \mathbb{T}, F) = \{y\}$	ramification
$\mathbb{T}_{1,\epsilon}$	$F(\sqrt{\epsilon})$	$y = 0$	unramified
$\mathbb{T}_{\varpi^{-1},\epsilon\varpi} = \mathbb{T}_{1,\epsilon}^{\eta}$	$F(\sqrt{\epsilon})$	$y = 1$	unramified
$\mathbb{T}_{1,\varpi}$	$F(\sqrt{\varpi})$	$y = \frac{1}{2}$	ramified
$\mathbb{T}_{\epsilon,\epsilon^{-1}\varpi}$	$F(\sqrt{\varpi})$	$y = \frac{1}{2}$	ramified
$\mathbb{T}_{1,\epsilon\varpi}$	$F(\sqrt{\epsilon\varpi})$	$y = \frac{1}{2}$	ramified
$\mathbb{T}_{\epsilon,\varpi}$	$F(\sqrt{\epsilon\varpi})$	$y = \frac{1}{2}$	ramified

Here $\mathbb{T}_{\gamma_1,\gamma_2}$ is the torus consists of elements of the form

$$t(a, b) = \begin{pmatrix} a & b\gamma_1 \\ b\gamma_2 & a \end{pmatrix} \in \mathrm{SL}_2 .$$

When $-1 \notin (F^\times)^2$, then $\mathbb{T}_{1,\varpi} \cong \mathbb{T}_{\epsilon,\epsilon^{-1}\varpi}$ and $\mathbb{T}_{1,\epsilon\varpi} \cong \mathbb{T}_{\epsilon\varpi}$. Otherwise, there are no conjucy among them. For $\mathbb{T}_{\gamma_1,\gamma_2}$ in the above table, the element

$$g = \begin{pmatrix} 1 & -\frac{1}{2}\sqrt{\gamma_1\gamma_2^{-1}} \\ \sqrt{\gamma_1^{-1}\gamma_2} & \frac{1}{2} \end{pmatrix} \tag{1.1}$$

saitisfies $\mathbb{T}_{\gamma_1,\gamma_2} = \mathbb{S}^g$. So we can use $g^{-1}\sigma(g) \in \mathbb{G}(F')_{x,0}$ to determine the second coordinate of $y = [g, x]$. Moreover, it turns out tha both g^{-1} and $\sigma(g)$ lie in $\mathbb{G}(F')_{x,0}$, so $[g, x] = g \cdot [1, x] = [1, x] \in A(\mathbb{G}, \mathbb{S}, F)$.

Example 1.1. Let's take a look at $\mathbb{T} = \mathbb{T}_{1,\varpi}$, and try to compute $A(\mathbb{G}, \mathbb{T}, F) = \{y\} = \{[g, x]\}$. We can take

$$g = \begin{pmatrix} 1 & -\frac{1}{2}\varpi^{-\frac{1}{2}} \\ \varpi^{\frac{1}{2}} & \frac{1}{2} \end{pmatrix}$$
$$g^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\varpi^{-\frac{1}{2}} \\ -\varpi^{\frac{1}{2}} & 1 \end{pmatrix}$$
$$\sigma(g) = \begin{pmatrix} 1 & \frac{1}{2}\varpi^{-\frac{1}{2}} \\ -\varpi^{\frac{1}{2}} & \frac{1}{2} \end{pmatrix}$$
$$\sigma(g)g^{-1} = \begin{pmatrix} 0 & \varpi^{-\frac{1}{2}} \\ -\varpi^{\frac{1}{2}} & 0 \end{pmatrix} \in \mathbb{G}(F')_{x,0}.$$

So we see that $x = \frac{1}{2}$. Moreover, g also lies in $\mathbb{G}(F')_{\frac{1}{2},0}$. So $y = [g, \frac{1}{2}] = \frac{1}{2}$.

The filtration on $\mathbb{S}(F')$ induces one on $\mathbb{T}(F')$ by conjugation, and so on $T = \mathbb{T}(F)$ as the fixed points under Galois group. Since $g \in \mathbb{G}(F')_{y,0}$, and $T \subseteq G_{y,0}$, $T_r = T \cap G_{y,r}$. In particular, $T_0 = T$ and for each $r \in \tilde{\mathbb{R}}$, we have

$$T_r = \left\{ \begin{pmatrix} a & b\gamma_1 \\ b\gamma_2 & a \end{pmatrix} \mid a \in \mathfrak{u}_{\lceil r \rceil}, b\gamma_1 \in \mathfrak{p}^{\lceil r-y \rceil} \right\}.$$

In the case $T = T_{\gamma_1,\gamma_2}$, $\mathfrak{t} = \mathrm{Lie}(T)$ is the one-dimensional subalgebra of $\mathfrak{g}$ spanned by

$$X_T = \begin{pmatrix} 0 & \gamma_1 \\ \gamma_2 & 0 \end{pmatrix}$$

For $r \in \tilde{\mathbb{R}}$, the corresponding filtration submodule of $\mathfrak{t}$ is

$$\mathfrak{t}_r = \{aX_T \mid a \in F, a\gamma_1 \in \mathfrak{p}^{\lceil r-y \rceil}\}.$$

Definition. For $r \geq 0$, elements $X = aX_T \in \mathfrak{t}_{-r}$ satisfying $\omega(a\gamma_1) = -r - y$ are called *G-generic of depth r.*

Let $r > 0$, then there is a natural group isomorphism $e_r : \mathfrak{t}_r/\mathfrak{t}_{r_+} \to T_r/T_{r_+}$ called the *Moy-Prasad isomorphism.* If ϕ is a character of T of depth r, we say that ϕ is *G-generic* if there exists a G-generic element $\Gamma \in \mathfrak{t}_{-r}$ of depth r such that

$$\phi(e_r(X)) = \Psi(\mathrm{Tr}(\Gamma X)).$$

In our case, all positive-depth characters of T are G-generic.

1.4. **Representation theory preliminaries.**

1.4.1. *Supercuspidal representations.* Now let $\mathbb{G} = \mathrm{SL}_2$, $\mathbb{U} = \mathbb{U}_\alpha$.

Definition. A representation (π, V) of G is *smooth* if $\mathrm{Stab}_G(v)$ is open for all $v \in V$.

Definition. A smooth representation (π, V) of G is *supercuspidal* if the *Jacquet module* $V_U = V/V(U) = 0$, where

$$V(U) = \mathrm{span}_{\mathbb{C}}\{\pi(u)v - v \mid u \in U, v \in V\}.$$

Remark. This is the greatest quotient of V on which U acts trivially. By Frobenius reciprocity, (π, V) is supercuspidal iff it's NOT isomorphic to any quotient of a parabolic induced representation.

1.4.2. *Compact induction.*

Definition. Given a closed subgroup H of G, and a smooth representation (π, V) of H, the *compactly induced representation* $\text{c-Ind}_H^G \pi$ is given by the right action by G on the space of functions

$$\{f : G \to V \mid \forall h \in H, g \in G, f(hg) = \pi(h)f(g), f \text{ is smooth and compactly supported mod } H\}.$$

Theorem 1.1 (Mautner)**.** *If H is open and compact mod the center Z of G, and if* $\text{c-Ind}_H^G \pi$ *is irreducible, then it is supercuspidal.*

1.4.3. *Heisenberg groups and Weil representations.* Let V be a finite dimensional symplectic vector space over k. The *Heisenberg group* $H(V) = V \oplus k$ (as set) has center $(0, k) \cong k$. For each character $\psi : k \to \psi$, there is a unique irreducible representation (called the *Heisenberg representation*) w_ψ of $H(V)$ with central character ψ. It follows that w_ψ extends to a representation w_ψ of $\mathrm{Sp}(V) \ltimes H(V)$ (called the *Weil representation*).

2. Supercuspidal Representations of $\mathrm{SL}_2(F)$

In this section we present the classification of supercuspidal representations of $\mathrm{SL}_2(F)$ following the work of J.K. Yu [Yu01]. Yu introduced the notion of a *generic tamely ramified cuspidal G-datum.* For $G = \mathrm{SL}_2(F)$, these data fall into two kinds. The first consists data of so-called degree 0, and give rise to depth-zero supercuspidal representations. The second consists data of so-called degree 1, and give rise to positive-depth supercuspidal representations. In either case, these supercuspidal representations are constructed by compact induction.

2.1. **Depth-zero supercuspidal representations.** Let σ be a *cuspidal representation** of $\mathbb{G}(k) = \mathrm{SL}_2(k)$ and pull it back along the mod $\mathfrak{p}$ map $\mathbb{G}(\mathcal{O}_F) \to \mathbb{G}(k)$ to obtain a representation (also denoted σ) of $K = \mathrm{SL}_2(\mathcal{O}_F)$. Let σ^η be the corresponding representation of K^η. It is known that the compactly iuduced representations

$$\text{c-Ind}_K^G(\sigma) \text{ and } \text{c-Ind}_{K^\eta}^G(\sigma^\eta)$$

are irreducible, hence supercuspidal.

Proposition 2.1. *Up to equivalence, any depth-zero supercuspidal representation of G rises as $\text{c-Ind}_{G_{y,0}}^G \sigma$ for a unique pair $(\sigma, G_{y,0})$ of cuspidal representation σ of $\mathrm{SL}_2(k)$ and maximal compact subgroup $G_{y,0}$ wtih $y \in \{0,1\} \subseteq A(\mathbb{G}, \mathbb{S}, F)$.*

2.2. **Positive-depth supercuspidal representations.** To construct positive depth supercuspidal representations, we need cuspidal G-datum of degree 1, which is a quadruple $(\mathbb{T}, y, r, \phi)$ where

- $\mathbb{T}$ is an anisotropic torus.
- $y \in A(\mathbb{G}, \mathbb{T}, F)$ as described in the table.
- $r = 2s$ is a positive real number, subject to
  - $r \in \mathbb{Z}$, if $\mathbb{T}$ is unramified
  - $r \in \mathbb{Z} + \frac{1}{2}$, if $\mathbb{T}$ is ramified
- ϕ is a G-generic character of $\mathbb{T}$ of depth r.

The idea of the construction of a supercuspidal representation of depth r from such a quadruple $(\mathbb{T}, y, r, \phi)$ is to extend ϕ to a depth r representation ρ of $TG_{y,s}$, whose compact induction to G is irreducible (hence supercuspidal). Here is the detail.

Denote by e the Moy-Prasad isomorphisms

$$\mathfrak{t}_{s_+}/\mathfrak{t}_{r_+} \cong T_{s_+}/T_{r_+} \text{ and } \mathfrak{g}_{y,s_+}/\mathfrak{g}_{y,r_+} \cong G_{y,s_+}/G_{y,r_+}.$$

Since ϕ has depth r, its restriction to T_{s_+} factors through T_{s_+}/T_{r_+}. Pulling back along e, we obtain a quasi-character of $\mathfrak{t}_{s_+}/\mathfrak{t}_{r_+}$. We know that such a quasi-character arises as $X \mapsto \Psi(\mathrm{Tr}(\Gamma X))$ for some $\Gamma \in \mathfrak{t}_{-r}$. Therefore we conclude that

$$\phi(t) = \Psi(\mathrm{Tr}(\Gamma e^{-1}(t))), \quad t \in T_{s_+}.$$

Using this formula, we can also define a character Ψ_Γ of G_{y,s_+} via

$$\Psi_\Gamma(g) = \Psi(\mathrm{Tr}(\Gamma e^{-1}(g))), \quad g \in G_{y,s_+}.$$

In paricular, ϕ and Ψ_Γ agree on $T \cap G_{y,s_+} \supset T_{s_+}$, together they give a unique well defined character $\hat{\phi}$ of TG_{y,s_+} by

$$\hat{\phi}(tg) = \phi(t)\Psi_\Gamma(g), \quad t \in T, g \in G_{y,s_+}.$$

In case $G_{y,s_+} = G_{y,s}$, we can take $\rho = \hat{\phi}$. This occurs when

- $\mathbb{T}$ is ramified, and
- $\mathbb{T}$ is unramified with r odd

since in these cases s is a fraction for which $\lceil s \rceil = \lceil s_+ \rceil$ and $\lceil s \pm y \rceil = \lceil (s_+) \pm y \rceil$.

Otherwise, i.e., when $\mathbb{T}$ is unramified and r is even, Yu defines the subgroups

$$J^1 = T_r G_{y,s}$$
$$J^1_+ = T_r G_{y,s_+}.$$

In particular one sees that $TJ^1 = TG_{y,s}$, so we just need to define a representation ρ for TJ^1 The quotient J^1/J^1_+ has a symplectic structure (coming from ϕ and Ψ_Γ). Some quotient J^1/N of J^1 is isomorphic to $H(J^1/J^1_+)$. Therefore $J_1/N \cong H(J^1/J^1_+)/(J^1/J^1_+) \cong k$. Moreover, the

*Andy talked about how to construct these representations via Deligne-Lusztig induction character $\Psi_\Gamma : G_{y,s+} \to \mathbb{C}$ induces a character $J_1/N \to \mathbb{N}$ (which is again denoted by Ψ_Γ). So by Stone-von Neumann theorem one can obtain the Weil representation of $\mathrm{Sp}(J^1/J^1_+) \ltimes J/N$ with central character Ψ_Γ. The group T acts on J^1/J^1_+ by conjugation, and this action is symplectic. Therefore we have a homomorphism $T \to \mathrm{Sp}(J^1/J^1_+)$. Pull the Weil representation back along $T \ltimes J^1 \to \mathrm{Sp}(J^1/J^1_+) \ltimes J^1/N$, we obtain a representation $\tilde{\phi}$ of $T \ltimes J^1$. Then

(1) $\mathrm{Res}_{1 \ltimes J^1_+} \tilde{\phi}$ is Ψ_Γ-isotypic (by definition of the Weil representation).
(2) $\mathrm{Res}_{T_{0_+} \ltimes 1} \tilde{\phi}$ is 1-isotypic (the action is given by conjugation, so just need to check $[T_{0_+}, J^1] \subseteq J^1_+$).

Now we define ρ by $\rho(tj) = \phi(t)\tilde{\phi}(t,j)$ for $(t,j) \in T \ltimes J^1$. To check that this is well defined, let $tj = 1$, then $t, j = t^{-1} \in T \cap J^1 \supseteq T_r$. So

$$\rho(tj) = \phi(t) \cdot \underbrace{\tilde{\phi}(t,1)}_{=1} \cdot \underbrace{\tilde{\phi}(1,t^{-1})}_{=\Psi_\Gamma(t^{-1}) = \phi(t^{-1})} = 1.$$

Theorem 2.1. *Let $\rho = \rho(\mathbb{T}, y, r, \phi)$ be as above. The compactly induction representation*

$$\text{c-Ind}_{TG_{y,s}}^{G} \rho$$

is a supercuspidal representation of G of depth r, and all positive-depth supercuspidal representations of G arise in this way. Moreover, two such representations are equivalent if and only if the pairs $(\mathbb{T}, \phi)$ occuring in the data are G-conjugate.

References

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