*

WEIL REPRESENTATION OF $\mathrm{SL}_2(\mathbb{F}_q)$

TALK GIVEN BY CALVIN YOST-WOLFF

Contents

Even though in this seminar we focus on SL_2, we will view SL_2 as Sp_2 in this talk since the more general result holds for Sp_{2n}.

The blueprint of this talk will be as follows:

$$\mathrm{Sp}_2 \rightrightarrows \mathrm{Sp}_4 \hookrightarrow \mathrm{Aut}(H(V))$$

$$\text{Irreps of } \mathrm{Sp}_2 \leftsquigarrow \text{Weil representation} \leftsquigarrow \text{Heisenberg representation}$$

More precisely, we will

(1) Define the Heisenberg group $H(V)$ and Heisenberg representations of it.
(2) Obtain the Weil representation of Sp_4 by an action on $\mathrm{Rep}(H(V))$, the category of representations of $H(V)$, with $\dim V = 4$.
(3) Define two embeddings of Sp_2 in Sp_4, and use Howe duality to decompose the Weil representation of Sp_4 with respect to these two embeddings to get irreducible representations of Sp_2.

1. Heisenberg Group and Heisenberg Represenations

Let k be the base field of $\operatorname{char} k \neq 2$.

Definition. A *symplectic vector space* over k is a vector space over k with a symplectic form ω. If $\dim V = 2n$, a subspace $W \subseteq V$ of dimension n is called *Lagrangian* if $\omega|_W = 0$.

If W is a Lagrangian, we have an identification

$$V = W \oplus W^\vee.$$

with $W = \mathrm{Span}_k(e_i)_{1\leq i\leq n}$, and $W^\vee = \mathrm{Span}_k(e_i^*)_{1\leq i\leq n}$, with e_i^* the dual basis of e_i. Choosing an identification $W \cong W^\vee$, we can write $V = W \oplus W$ and obtain a symmetric form B on W:

$$B(w_1, w_2) = \omega((w_1, 0), (0, w_2)).$$

Definition. The *Heisenberg group* $H(V)$ is a central extension of V by k, i.e., $H(V) = V \oplus k$, with group operations

$$(v_1, z_1) \cdot (v_2, z_2) = (v_1 + v_2, z_1 + z_2 - \frac{\omega(v_1, v_2)}{2}).$$

*Notes Taken by Guanjie Huang, who is the only person responsible for any possible mistake found in the notes

From now on, let $k = \mathbb{F}_q$, with q a p-power, $p \neq 2$.

Choose a Lagrangian splitting $V = W \oplus W$. Let $L^2(W)$ be the space of complex-valued functions on W with inner product $\langle f, g\rangle = \sum_{w\in W} f(w)\overline{g(w)}$. For each character $\psi : k \to \mathbb{C}^\times$ we can define the Fourier transform (related to ψ) by

$$\hat{f}_\psi(w) = \epsilon \cdot q^{-\frac{\dim W}{2}} \sum_{w\in W} f(u)\psi(-B(w,u)),$$

where $\epsilon = \pm 1$ depending on different W we choose (which will be described later).

Definition. For each nontrivial character $\psi : k \to \mathbb{C}^\times$, we can define the *Heisenberg representation* π_ψ of $H(V) = W \oplus W \oplus k$ on $L^2(W)$ as follows:

- The first W acts by translation, i.e., $\pi_\psi(w,0,0)f(w') = f(w' - w)$.
- The second W acts by translation on the Fourier transform, i.e., $\pi_\psi(0,w,0)f(w') = \psi(B(w',w))f(w')$.
- k acts via ψ, i.e., $\pi_\psi(0,0,t)f(w') = \psi(t)f(w')$.

Theorem 1.1 (Finite Stone-Von Neumann Theorem)**.** *$H(V)$ has two types of irreducible representations:*

(1) π_ψ for some non-trivial character, and
(2) representations with central characters trivial.

Remark. The sceond type of representations, can be viewed as irreducible representations of $H(V)/k \cong V$. Since V is abelian, we know they are 1-dimensional, and corresponds one-to-one with $\hat{V}$, hence also V.

Proof. The irreducibility of the second type is clear from the remark above. For irreducibility of π_ψ, see [Pra09, Theorem 3.1].

Now it remains to check that we exhaust all irreducible representations. On one hand, we know

$$|H(V)| = q^n \cdot q^n \cdot q = q^{2n+1}.$$

On the other hand,

	ψ non-trivial	ψ trivial
#	$q-1$	q^{2n}
dimension	q^{2n}	1

the sum of squares of dimensions is $(q-1)q^{2n} + q^{2n}1^2 = q^{2n+1}$. So from character theory of finite groups, we know the table above exhausts all possibilities of irreducible representations of $H(V)$. ☐

2. Weil Representation

$\mathrm{Sp}(V)$ acts on $H(V)$ by

$$g \cdot (v,z) = (gv, z), g \in \mathrm{Sp}(V), (v,z) \in H(V) = V \oplus k.$$

In particular, this action is trivial on the center. For a Heisenberg representation π_ψ and any $g \in \mathrm{Sp}(V)$, we can define a new representation ${}^g\pi_\psi$ on the same space by

$${}^g\pi_\psi(h) = \pi_\psi(g \cdot h), g \in \mathrm{Sp}(V), h \in H(V).$$

Since the action of $\mathrm{Sp}(V)$ is trivial on the center, we see that ${}^g\pi_\psi$ is an irreducible representation of $H(V)$ with central character ψ. So by Theorem 1.1, ${}^g\pi_\psi \cong \pi_\psi$. Choose $\rho(g) \in \mathrm{GL}(\pi_\psi)$ to realize this isomorphism. By Schur's Lemma, such $\rho(g)$ is unique up to scalar. Therefore we obtain a projective representation

$$\bar{w}_\psi : \mathrm{Sp}(V) \to \mathrm{PGL}(\pi_\psi).$$

3. Howe Duality

Let $\mathbb{V}$ be a 4-dimensional symplectic space over $k = \mathbb{F}_q$. We will introduce two ways to embed Sp_2 in $\mathrm{Sp}_4 = \mathrm{Sp}(\mathbb{V})$, corresponding to two decompositions

$$\mathbb{V} = W_i \oplus W_i = W_i \otimes V, i = 1, 2,$$

where V is a 2-dimensional symplectic vector space. We will explain what W_i's are later. The Heisenberg representation and Weil representation introduced above are now:

$$\bar{w}_\psi : \mathrm{Sp}(\mathbb{V}) \to \mathrm{PGL}(\pi_\psi),$$
$$\pi_\psi : H(\mathbb{V}) \to L^2(W_i).$$

Now assume we have such a decomposition, we have an embedding $\mathrm{Sp}(V) \hookrightarrow \mathrm{Sp}(\mathbb{V})$. The centralizer of $\mathrm{Sp}(V)$ in $\mathrm{Sp}(\mathbb{V})$ is $O(W_i)$.

We can embed $\mathrm{Sp}(V) \times O(W_i) \hookrightarrow \mathrm{Sp}(\mathbb{V})$. The projective representation $\bar{w}_\psi$ factors through a representation w_ψ when restricted on $\mathrm{Sp}(V) \times O(W_i) \hookrightarrow \mathrm{Sp}(\mathbb{V})$, i.e., we have the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Sp}(V) \times O(W_i) & \hookrightarrow & \mathrm{Sp}(\mathbb{V}) & \xrightarrow{\bar{w}_\psi} & \mathrm{PGL}(\pi_\psi) \\
 & \overset{w_\psi}{\searrow} & & \nearrow & \\
 & & \mathrm{GL}(\pi_\psi) & &
\end{array}$$

To describe w_ψ, we just need to describe its restriction on $\mathrm{Sp}(V) = \mathrm{SL}_2$ and $O(W_i)$.

- On $Sp(V)$,

$$w_\psi \begin{pmatrix} 1 & 0 \\ r & 0 \end{pmatrix} f(x) = \psi(B(rx, x)) f(x)$$
$$w_\psi \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} f(x) = f(\alpha^{-1} x)$$
$$w_\psi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f(x) = \hat{f}_\psi(x).$$

- On $O(W_i)$,

$$w_\psi(g) f(x) = f(g^{-1} x).$$

In either case, we have

$$\pi_\psi = \bigoplus_{\chi \in \widehat{O(W_i)}} W_i(\chi),$$
$$W_i(\chi) = \{f : W_i \to \mathbb{C} \mid f(g^{-1} w) = \chi(g) f(w), \forall g \in O(W_i), w \in W_i\}.$$

Remark. By decomposing W_i into $O(W_i)$-orbits, it's easy to see that this summand is direct. To see $W_i(\chi)'s$ spans π_ψ, we can count dimension using the table of explicit description of W_i's given below.

Each $W_i(\chi)$ is a representation of $\mathrm{Sp}(V) = \mathrm{SL}_2$.

The following table decribes W_i's and the SL_2-representations arising from $W_i(\chi)$.

	$W_1 = k \oplus k$	$W_2 = \mathbb{F}_{q^2}$
ϵ	1	-1
B	$B((x_1, y_1), (x_2, y_2)) = x_1 y_2 + x_2 y_1$	trace map
$O(W_i)$	$\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \right\} \cong \mathbb{F}_q^\times$	norm 1 elements $\cong (\mathbb{F}_{q^2}^\times)^{q-1} \cong \frac{(q-1)\mathbb{Z}}{(q^2-1)} \cong \frac{\mathbb{Z}}{(q+1)}$
$\widehat{O(W_i)}$	$\frac{\mathbb{Z}}{(q-1)}$	$\frac{\mathbb{Z}}{(q+1)}$
$O(W_i)$-orbits	$\{(0,0)\}, (0, k^\times), (k^\times, 0)$, $\{(z_1, z_2) \mid z_1 z_2 = z \in k^\times\}$	classfied by norm
$\#O(W_i)$-orbits	$q + 2$	q
$\dim W(\chi)$	$= \begin{cases} q+1 & \chi \neq 1 \\ q+2 & \chi = 1 \end{cases}$	$= \begin{cases} q-1 & \chi \neq 1 \\ q & \chi = 1 \end{cases}$
$W(\chi), \chi = 1$	contains trivial rep (functions fixed by Fourier transform), Steinberg (irreducible of dimension q)	$\cong \bigoplus_{\alpha \in \hat{k}} k_\alpha$
$W(\chi), \chi = \mathrm{sgn}$	splits into two irreducible representations	not irreducible
$W(\chi)$ otherwise	$\cong \mathrm{Ind}_B^{\mathrm{SL}_2}(\chi)$, irreducible, $W(\chi) \cong W(\chi^{-1})$	cuspidal, irreducible, $W(\chi) \cong W(\chi^{-1})$ $W(\chi) \cong \bigoplus_{\alpha \in \hat{k}, \alpha \neq 1} k_\alpha$

References

[Pra09] Amritanshu Prasad. On character values and decomposition of the weil representation associated to a finite abelian group. *arXiv preprint arXiv:0903.1486*, 2009.