The Drinfeld Curve and the Irreducible Representations of $\mathrm{SL}_2(\mathbb{F}_q)$

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1 $\mathrm{SL}_2(\mathbb{F}_q)$

For what follows, let q be a prime power. The purpose of this talk is to determine the set of characteristic 0 irreducible representations of $\mathrm{SL}_2(\mathbb{F}_q)$ which I will refer to as G.

$\mathrm{SL}_2(R)$ for any unital ring R, is the group of two by two matrices with coefficients in R and determinant 1. For $R = \mathbb{F}_q$, this group has size $(q^3 - q)$. When $q > 3$, the quotient of G by its center $(\pm I)$ is simple.

G has $q + 4$ conjugacy classes, which can be found in [1], so we will need to construct $q + 4$ irreducible representations.

It is often useful to think of G not merely as a finite group, but as the set of $\mathbb{F}_q$ points of the affine variety $\mathfrak{G} = \operatorname{Spec} \mathbb{F}_q[a, b, c, d]/(ac - bd - 1)$

Denote by $B \subset G$ the subgroup of upper triangluar matrices. The B stands for Borel. In general a Borel subgroup of an algebraic group is a *maximal (connected) solvable subgroup*, and one can check that B is indeed maximal solvable in G.

It will be important later to note that $G = B \sqcup BsB$ where s is the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Contained in B is the subgroup T of diagonal matrices. T stands for torus, because there is a containment $\iota : \mathbb{G}_m \subset \mathfrak{G}$ that identifies

$$t \mapsto \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$$

T as we have defined it is $\mathbb{G}_m(\mathbb{F}_q)$ for this subgroup. Also notice that $T \simeq \mathbb{F}_q^\times \simeq \mu_{q-1}$, a cyclic group

There is in another "special subgroup". $\mathbb{F}_{q^2}^\times$ acts on $\mathbb{F}_{q^2}$ in the obvious way. Considering the latter space as a two dimensional vector space over $\mathbb{F}_q$, this gives us an inclusion $\iota' : \mathbb{F}_{q^2} \hookrightarrow \mathrm{GL}_2(\mathbb{F}_q)$.

If we choose a $\mathbb{F}_q$ basis for $\mathbb{F}_{q^2}$ as 1 and α, where $\alpha^2 = d \in \mathbb{F}_q$, then the scalar $a + b\alpha$ is sent to the matrix $\begin{bmatrix} a & db \\ b & a \end{bmatrix}$. Note that $\det(\iota'(r)) = \mathrm{Norm}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(r)$.

Then within G, we can identify the subgroup T' of matrices $\begin{bmatrix} a & b \\ db & a \end{bmatrix}$ where (by the definition of G) $a^2 - bd^2 = 1$. We have an isomorphism

$$T' \simeq \{\text{Elements of } \mathbb{F}_{q^2} \text{ of norm } 1\} \simeq \mu_{q+1}$$

My notation makes it clear there is an analogy between T and T'. T' is the the $\mathbb{F}_q$ points of the scheme $\text{Spec}(\mathbb{F}_q[a,b]/(a^2 - db^2 - 1)$, included in $\mathfrak{G}$ in the manner described above.

$\text{Spec}(\mathbb{F}_q[a,b]/(a^2 - db^2 - 1)$ is *not* isomorphic to $\mathbb{G}_m$, as $\mathbb{F}_q$-schemes. However, if you basechange to $k = \overline{\mathbb{F}}_q$ (or even just to $\mathbb{F}_{q^2}$), the schemes $\text{Spec}(k[a,b]/(a^2 - db^2 - 1)$ and $\text{Spec}(k[t,t^{-1}])$ *are* isomorphic

The isomorphism is geometrically obvious (both schemes are genus 0 curves with two points removed) It can be written explicitly as

$$(a,b) \mapsto a - b\alpha$$

$$t \mapsto \left(\frac{1+t^2}{2t}, \frac{1-t^2}{2\alpha t} \right)$$

We use the word *torus* for any group scheme which is isomorphic to (some product of copies of) $\mathbb{G}_m$ when base changed to an algebraically closed field. So T and T' are both the $\mathbb{F}_q$ points of torii.

We say T is the *split torus*, while T' is *nonsplit*

2 Irreducible representations coming from T

The representation theory of T is easy, since $T \simeq \mu_{q-1}$. There are $q-1$ irreducible complex representations, all one dimensional.

We can produce, from these representations, representations of G as follows. Let ψ be a character of T. B contains a normal subgroup U, the unipotent matrices, and $B/U \simeq T$. Given the map $B \to T$, we can "pull back" irreducble representations of T to representations of B, which we will call $\widetilde{\psi}$.

These ψ produces some (in fact, most) of the irreducible representations of B.

We then define $R(\psi) := \text{Ind}_B^G(\widetilde{\psi})$ to get a representation of G. It remains to show that (most of) these representations are irreducible.

We want to understand $\langle R(\psi_1), R(\psi_2) \rangle_G$. By frobenius reciprocity this is equal to $\langle \psi_1, \text{Res}_B^G \text{Ind}_B^G \widetilde{\psi}_2 \rangle_H$. To compute the latter, we use Mackey Theory.

A full proof and short discussion of the theorem can be found in [3]. The result states that for H, K subgroups of a group L, and ρ a representation of H we have an isomorphism

$$\text{Res}_K^L \text{Ind}_H^L(\rho) \simeq \bigoplus_{s \in H \backslash L / K} \text{Ind}_{s^{-1}Hs \cap K}^K \rho^s$$

where ρ^s is the representation of the group $s^{-1}Hs$ with the same underlying vector space as ρ and action $\rho^s(g) \cdot v = \rho(sgs^{-1}) \cdot v$.

This helps us immensely! One first notes that $G = B \sqcup BsB$, as discussed earlier. So if we take the above formula with $H = K = B$, then $B \backslash G/B$ is a two element set, with representatives I and s. Furthermore $s^{-1}Bs \cap B = T$, so we can realize

$$Res^G_B R(\psi_2) \simeq \operatorname{Ind}_B^B \widetilde{\psi}_2 \oplus \operatorname{Ind}_T^B \psi_2^s$$

$\psi_2^s = \psi_2^{-1}$, since for $t \in T$, $sts^{-1} = t^{-1}$. So in conclusion

$$\langle R(\psi_1), R(\psi_2)\rangle_G = \langle \operatorname{Ind}_T^B \psi_1, \operatorname{Ind}_T^B \psi_2\rangle_B + \langle \operatorname{Ind}_T^B \psi_1, \operatorname{Ind}_T^B \psi_2^{-1}\rangle_B = \langle \psi_1, \psi_2\rangle_T + \langle \psi_1, \psi_2^{-1}\rangle_T$$

If $\psi \neq \psi^{-1}$, we see immediately that $\langle R(\psi), R(\psi)\rangle_G = 1$, so $R(\psi)$ is irreducible, and of degree $q + 1$[1]. We see furthermore that if $\psi_1 \neq \psi_2$ and $\psi_1 \neq \psi_2^{-1}$, then $R(\psi_1)$ shares no irreducible subrepresentations with $R(\psi_2)$.

If $\psi = \psi^{-1}$, then ψ is either $triv_T$ or the unique quadratic character. In this case $R(\psi)$ decomposes as a sum of two irreducible elements.

$R(triv_T)$ decomposes into two irreducible representations, $triv_G$, of degree 1, and a representation of degree q, called $\mathrm{ST_q}$, the Steinberg representation.

For ψ the quadratic character, $R(\psi)$ decomposes into two irreducible representations. One can see that these representations are conjugate under the action of $\mathrm{GL}_2(\mathbb{F}_q)$ via Clifford's Theorem.

We have thus produced $2 + 2 + \frac{q-3}{2}$ irreducible representations of G. There remain $\frac{q+3}{2}$ to construct.

As a final note, observe that the map $\psi \mapsto R(\psi)$ defines a functor from $\mathbb{C}[T]$ modules to $\mathbb{C}[G]$ modules. This functor admits a very nice characterization.

Define $\mathbb{C}[G/U]$ as the vector space with basis given by the left cosets of G/U. G has a left action on this vector space, and since T normalizes U, T can act on the right, making $\mathbb{C}[G/U]$ a $(\mathbb{C}[G], \mathbb{C}[T])$ bimodule.

We have a functor from $\mathbb{C}[T]$ modules to $\mathbb{C}[G]$ modules coming from $M \mapsto \mathbb{C}[G] \otimes_{\mathbb{C}[T]} M$. It can be seen that the process of "inflating" a T-representation to B and inducing to G is exactly given by this tensor.

3 The Drinfeld Curve and irreducible representations coming from T'

We'd like to reproduce this work for T', but there is no analog of B. Inside $\mathfrak{G}_{\overline{\mathbb{F}_q}}$, the scheme $\mathrm{Spec}(\overline{\mathbb{F}}_q[a,b]/(a^2 - bd^2)$ sits inside a connected solvable subgroup scheme (a Borel), but this

[1]We call such character *in general position*

scheme is not defined over $\mathbb{F}_q$, so we can't induce a representation from the group of its $\mathbb{F}_q$ points.

Instead we have to look somewhere (seemingly) completely different. Let X be the affine scheme cut out by the equation $xy^q - x^q y = 1$. Some basic facts:

- X is a smooth irreducible plane curve.
- There is an action of G on X given by restricting action of G on $\mathbb{A}^2$. To see this note that for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in G

$$\begin{aligned}
&(ax + by)(cx + dy)^q - (ax + by)^q(cx + dy) \\
&= acx^{q+1} + adxy^q + bcx^q y + bdy^{q+1} - acx^{q+1} - bcxy^q - adx^q y - bdy^{q+1} \\
&= (ad - bc)xy^q - (ad - bc)x^q y
\end{aligned}$$

- There is a *different* action of μ_{q+1} on X, coming from $\zeta \cdot (x, y) = (\zeta x, \zeta y)$.
- These two actions commute, giving X a $G \times \mu_{q+1}$ action
- X has many interesting quotients from this action:
  - $X/G \simeq \mathbb{A}^1$
  - $X/U \simeq \mathbb{A}^1 - \{0\}$
  - $X/\mu_{q+1} \simeq \mathbb{P}^1 - \mathbb{P}^1(\mathbb{F}_q)$

Why does this matter? We are close to having a situation siliar to before, where we used a (G, T) bimodule to induce representations. X of course doe not have a nice module structure, but we can use it to produce bimodules via cohomology.

I'm going to say very little about the actual constructions and definitions of etale cohomology. [4] is a great source for those wanting to learn more, [1] has an appendix with a clean exposition of the relevant properties.

Given a variety V over a field k, and a choice of prime ℓ not dividing the characteristic of k, one can associate to V a sequence of $\mathbb{Q}_\ell$ vector spaces $H^i_c(V, \mathbb{Q}_\ell)$, called *the compactly supported etale cohomology with coefficients in* $\mathbb{Q}_\ell$, indexed by nonnegative i, with the following properties

- *Functorality* Given a morphism of varieties from $V \to W$ we get a map of vector spaces from $H^i_c(W, \mathbb{Q}_\ell) \to H^i_c(V, \mathbb{Q}_\ell)$. In particular, if a group G acts on a variety X, we also get a G action on each of the cohomology groups.
- *Vanishing* For $d > 2\dim(V)$, $H^d_c(V, \mathbb{Q}_\ell) = 0$. If V is affine, then $H^d_c(V, \mathbb{Q}_\ell) = 0$ for $d < \dim(V)$ as well.

- *Euler Characteristic* We can consider the virtual vector space $H_c^*(V, \mathbb{Q}_\ell) := \sum_{i \geq 0} (-1)^i H_c^i(V, \mathbb{Q}_\ell)$. Given an automorphism f of V, we define Tr_V^* as

  $$\mathrm{Tr}_V^*(f) = \sum_{i \geq 0} \mathrm{Tr}(\gamma, H_c^i(V, \mathbb{Q}_\ell))$$

  This value is actually rational and independent of choice of ℓ.

  If f has finite order, i.e. if f is the action of some element in a finite group G acting on V, f is in fact an integer. If V is defined over $\mathbb{F}_q$ and $f = F$ is the associated frobenius then $\mathrm{Tr}_V^*(F) = \#V(\mathbb{F}_q)$.

This technology will help us in the following way. Given the previously described group action, $H_c^*(V, \mathbb{Q}_\ell)$ is a virtual $(\mathbb{Q}_\ell[G], \mathbb{Q}_\ell[T'])$ bimodule, with the T' action being thought of as a right action. It will take the role of $\mathbb{C}[G/U]$ in the previous construction. Given a $\mathbb{Q}_\ell[T']$ module (an ℓ-adic T' representation) M, we produce a (virtual) $\mathbb{Q}_\ell[G]$ module representation by considering $H_c^*(V, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell[T']} M$.

FOr the rest of this paper I will write this as $H_c^*(V, \mathbb{Q}_\ell) \otimes M$ Please note that this is a tensor product of right and left $\mathbb{Q}_\ell[T']$ modules, not a tensor product of T' representations.

As noted above $Tr_V^*(g)$ is an integer invariant of ℓ. So though the representations we will be producing technically depend on ℓ, their characters will not. Though our previous representations were over $\mathbb{C}$, the actual field is immaterial. In fact, ℓ won't really show up any more, so I will assume the choice and write $H_c^i(V)$

Here are two facts that will help us do computations.

1. For a group H acting on a variety V, $H_c^i(V)^H \simeq H_c^i(V/H)$. If $H \subset H'$ is a normal subgroup with H' extending the action of H, then this is an isomorphism of H/H' representations.

2. If V is irreducible, then $H_c^{2d}(V)$ is one dimensional, and the action of any group on V yields a finite action on the top cohomology

For θ a character of T', we define the *Deligne-Lusztig Induction of* θ as

$$R'(\theta) - \sum_{i \geq 0} (-1)^i H_c^i(X) \otimes V_\theta$$

Where V_θ is the one dimensional representation of T' coming from θ.

Notice that by dimensional vanishing, this is just $H_c^1(X) \otimes V_\theta - H_c^2(X) \otimes V_\theta$.

Furthermore, since X is connected, $H_c^2(X) \simeq V_{triv}$, so for nontrivial θ, $R'(\theta)$ is the (nonvirtual) representation $H_c^1(X) \otimes V_\theta$.

Looking first at the trivial case we have

$$\begin{aligned} R'(triv) &= H^1_c(X) \otimes V_{triv} - H^2_c(X) \otimes V_{triv} \\ &= H^1_c(X)^{T'} - V_{triv} \\ &= H^1_c(X/T') - V_{triv} \\ &= H^1_c(\mathbb{P}^1 - \mathbb{P}^1(\mathbb{F}_q)) - V_{triv} \end{aligned}$$

It is a not very hard exercise in etale cohomology to compute that $H^1_c(\mathbb{P}^1 - \mathbb{P}^1(\mathbb{F}_q))$ has dimension q. Slightly harder (and possibly in next week's talk??) is that, as a representation of G, this is St_q. I will omit the argument for this.

Now we can begin studying the induction of the nontrivial characters. There are a few main results that give us what we want. I will outline the proofs of some, but not all of them. My goal is to highlight how we can use the geometry of X to aid in representation theoretic computations.

Theorem 1. *For θ a character of T', $R'(\theta) = R'(\theta^{-1})$*

Proof. This is obvious if $\theta = triv$. For the others, it is sufficient to show that $H^1_c(X) \otimes V_\theta \simeq H^1_c(X) \otimes V_{\theta^{-1}}$.

Denote by $\widetilde{H^1_c(X)}$ the right T' module with underlying space $H^1_c(X)$ but with the normal action of T' twisted by inversion. If we show $\widetilde{H^1_c(X)} \simeq H^1_c(X)$, that would prove the proposition.

But the frobenius map $F : X \to X$ defines an automorphism of $H^1_c(X)$. For $\zeta \in T'$, $F \circ \zeta = \zeta^{-1} \circ F$, so F gives us the identification of the cohomology group and its twist. □

[TODO: generalize this to orbit of character under weil group?]

Theorem 2. *For θ a character of T', and ψ a character of T, $\langle R(\psi), R'(\theta) \rangle_G = 0$.*

Proof. This is true by direct computation for $\theta = triv$ (The only nontrivial case is $\langle R(triv), R'(triv) \rangle_G$, and the terms cancel)

Now consider nontrivial θ. It is enough to verify the result for $\langle R(\mathrm{reg}_T), R'(\theta) \rangle_G$. The left term is equivalent to

$$R(\mathrm{reg}_T) = \mathrm{Ind}_B^G \mathrm{Reg}_T = \mathrm{Ind}_B^G \mathrm{Ind}_U^B triv_U = \mathrm{Ind}_U^G triv_U$$

Then via Frobenius reciprocity, we see we are trying to compute $\langle triv_U, \mathrm{Res}_U^G R'(\theta) \rangle = \dim(H^1_c(X)^U \otimes V_\theta)$ as a vector space.

But $H^1_c(X)^U \simeq H^1_c(X/U) = H^1_c(\mathbb{A} \backslash 0)$. This is (cohomology fact!) a one dimensional trivial representation of the group T', so $H^1_c(X)^U \otimes V_\theta = 0$. □

This shows that Deligne-Lusztig induction produces new characters compared to what we have done before. WHY ARE THEY CALLED CUSPIDAL?

To understand the dimension of the curpidal characters, we cite a theorem of [?] that for $\zeta \in T'$ nontrivial, $\mathrm{Tr}^*_X(\zeta) = \mathrm{Tr}^*_{X^\zeta}(1)$. But $X^\zeta = \emptyset$. This implies that, as a character of T', $H^*_c(X)$ is a multiple of the character of $\mathrm{reg}_{T'}$. This means for any two characters θ and θ' $\dim(H^*_c(X) \otimes V_\theta) = \dim(H^*_c(X) \otimes V_{\theta'})$. Since $\dim(H^*_c(X) \otimes V_{triv}) = \dim(\mathrm{St}_q) - \dim(triv) = q - 1$, this means $R'(\theta)$ is always degree $q - 1$.

Finally, we show an analog of Mackey theory, which will allow us to show that the representations we have induced are disntinct and (in all but one case) irreducible.

Theorem 3. *For θ_1 and θ_2 characters of T', we have*

$$\langle R'(\theta_1), R'(\theta_2) \rangle_G = \langle \theta_1, \theta_2 \rangle_{T'} + \langle \theta_1, \theta_2^{-1} \rangle_{T'}$$

Not that this immediately implies that for θ in general position, $R'(\theta)$ is irreducible. For θ' the unique nontrivial character of order 2, $R'(\theta')$ decomposes into two distinct, irreducible representations. Thus, from the characters of T', we have added $\frac{q-1}{2} + 2$ new ones. This, added to our previous $4 + \frac{q-3}{2}$, brings us to a total of $q+4$, which is the number we needed!

The proof of this theorem will be omitted. It involves understanding $H^*_c((Y \times Y)/G)$ as a $T' \times T'$ module, interpreting the inner product above as the dimension of a subspace of $H^*_c((Y \times Y)/G)$ where $T' \times T'$ acts in a specific way. For a full write up, see [1]

4 The Bigger Picture

Now, let G be any reductive algebraic group over an algebraically closed field of characteristic p, obtained by extension of scalars from a group G_0 over $\mathbb{F}_q$. So the frobenius $F : G \to G$ is well defined.

We choose a maximal, F-stable torus $T \subset G$ an F-stable Borel subgroup B containing T, and a unipotent radical $\subset B$.

Define $W = N(T_0)/T_0$ the Weyl group. Up to isomorphism, these schemes/groups are independent of choice of T. We will refer to an element $w \in W$ as if it were an element of G, by choosing an arbitrary lift.

Then define $Y(w) := \{gU \in G/U | g^{-1}F(g) \in UwU\}$

Define T^{wF} (a finite abelian group) as the subgroup of T fixed by $\mathrm{ad}(w) \circ F$.

In the case of SL_2, we choose T as the group of diagonal matrices, and U the group of upper triangular matrices with 1 along the diagonal.

The weyl group has two elements, the identity and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ In the first case $Y(w)$ turns out to be a finite set, in fact, this set is exactly G/U??

For the other case, we are solving

$$\begin{bmatrix} d & -a \\ b & -d \end{bmatrix} \begin{bmatrix} d^q & c^q \\ b^q & a^q \end{bmatrix} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} * & * \\ ac^q - ca^q & * \end{bmatrix} = \begin{bmatrix} -x & 1 - xy \\ -1 & -y \end{bmatrix}$$

So a and c have to satisfy exactly the equations of the Drinfeld curve (the other entries pose no additional requirements)

References

[1] Bonnafé, Cédric. Representations of SL2(Fq). 1st ed. 2011., Springer London : Imprint: Springer, 2011.

[2] Deligne, P., and G. Lusztig. “Representations of Reductive Groups Over Finite Fields.” Annals of Mathematics, vol. 103, no. 1, Annals of Mathematics, 1976, pp. 103–61, https://doi.org/10.2307/1971021.

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[4] Milne, James S. Etale Cohomology (PMS-33). Princeton University Press, 2016.