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BOREL-WEIL-BOTT FOR $\mathrm{SL}_2(\mathbb{C})$

TALK GIVEN BY CALVIN YOST-WOLFF

Contents

The talk contains three parts

(1) Compute the representations of $\mathrm{SL}_2(\mathbb{C})$ in the cohomology of line bundles over $\mathbb{P}^1$.
(2) Motivate part (1) by showing we get all irreducible representations.
(3) Weyl character formula.

1. Representations of $\mathrm{SL}_2(\mathbb{C})$ from Cohomology of Line Bundles on $\mathbb{P}^1$

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$G = \mathrm{SL}_2(\mathbb{C})$ acts on $\mathbb{P}^1$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x : y] = [ax + by : cx + dy].$$

Given a G-equivarient line bundle $\pi : E \to \mathbb{P}^1$, a section $s : \mathbb{P}^1 \to E$ and an element $g \in \mathrm{SL}_2(\mathbb{C})$ acting on $\mathbb{P}^1$, we can form the following commutative diagram

$$\begin{tikzcd}
E \arrow[dr] & \mathbb{P}^1 \times E \arrow[l, "\rho_g"'] \arrow[r] \arrow[d, shift left] & E \arrow[d, shift left] \\
& \mathbb{P}^1 \arrow[u, dashed, bend left, "g^* s"] \arrow[r, "g\cdot"] \arrow[ul, dashed, bend left, "g\cdot s"] & \mathbb{P}^1 \arrow[u, dashed, bend left, "s"]
\end{tikzcd}$$

where

- g^*s is the pullback of s along $g\cdot$, i.e., $g^*s = g^*s(x) = (x, s(g \cdot x))$.
- ρ_g is induced by the G-equivarient action on E.
- $g \cdot s = \rho_g \circ g^*$.

We then have the following table of line bundles on $\mathbb{P}^1$ and corresponding actions of $\mathrm{SL}_2(\mathbb{C})$ on the cohomology groups.

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*Notes Taken by Guanjie Huang, who is the only person responsible for any possible mistake found in the notes

Invertible sheafs	Line bundles	Local sections	ρ_g	$g\cdot$	H^0	H^1
$\mathcal{O}_{\mathbb{P}^1}(-n)$	nth tautological bundle $\{([a:b],p) \mid p\in[a^n:b^n]\}$	degree $-n$ homogeneous rational functions $[a:b]\mapsto f(a,b)(a^n:b^n)$	$z\cdot g(a:b)^n$ $\mapsto z\cdot(a^n:b^n)$	$g\cdot S_f = S_{f\circ g}$	0	$\mathrm{Span}_{0<i<n}\langle x^{-i}y^{i-n}\rangle$ $\cong L_{n-2}$
$\mathcal{O}_{\mathbb{P}^1}(-1)$	tautological bundle $\{([a:b],p) \mid p\in[a:b]\}$	degree -1 homogeneous rational functions $[a:b]\mapsto f(a,b)(a:b)$	$z\cdot g(a:b)$ $\mapsto z\cdot(a:b)$	$g\cdot S_f = S_{f\circ g}$	0	0
$\mathcal{O}_{\mathbb{P}^1}(0)$	trivial bundle $\mathbb{P}^1\times\mathbb{C}$	degree 0 homogeneous rational functions $[a:b]\mapsto f(a,b)\cdot 1$	$z\cdot 1$ $\mapsto z\cdot 1$	$g\cdot S_f = S_{f\circ g}$	L_0	0
$\mathcal{O}_{\mathbb{P}^1}(n)$	dual of nth tau. bundle $\{([a:b],p) \mid p\in\mathrm{Hom}([a^n:b^n]),\mathbb{C}\}$	degree n homogeneous rational functions $[a:b]\mapsto$ $((r,s)\mapsto f(a,b)\frac{(r,s)}{(a^n,b^n)})$	$z\cdot(\frac{(r,s)}{g(a,b)^n})$ $\mapsto$ $z\cdot(\frac{(r,s)}{(a^n,b^n)})$	$g\cdot S_f = S_{f\circ g}$	L_n	0

- H^0 = global sections of line bundles
- L_n = degree n homogeneous polynomials in variables x and y.

The action of $\mathrm{SL}_2(\mathbb{C})$ on $H^1(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(-n)) = L_{n-2}$ can be seen in two different ways. To describe the SL_2-action, it suffices to describe $\begin{pmatrix}1&\beta\\&1\end{pmatrix}$, $\begin{pmatrix}\alpha&\\&\beta\end{pmatrix}$ and $\begin{pmatrix}1&\\ \beta&1\end{pmatrix}$.

(1) Serre duality.

By Serre duality, we have a non-canonical but functorial isomorphism

$$H^1(\mathbb{P}^1,\mathcal{O}(-n)) = H^0(\mathbb{P}^1,\underbrace{\omega_{\mathbb{P}^1}}_{\mathcal{O}_{\mathbb{P}^1}(-2)}\otimes\underbrace{\mathcal{O}_{\mathbb{P}^1}(-n)^\vee}_{\mathcal{O}_{\mathbb{P}^1}(n)})^\vee = L_{n-2}^\vee.$$

Let $e_{-i,-j}$ be the dual basis of $x^{i-1}y^{j-1}$, $i+j=n, i,j>0$. Now we wish to understand the action of $g=\begin{pmatrix}1&\beta\\&1\end{pmatrix}$ on this dual basis (clearly $\begin{pmatrix}1&\\ \beta&1\end{pmatrix}$ acts similarly and $\begin{pmatrix}\alpha&\\&\alpha^{-1}\end{pmatrix}$ acts by scalar). For nonnegative integer s,t such that $s+t=n-2$, we have

$$\begin{aligned} g\cdot e_{-i,-j}(x^ty^s) &= e_{-i,-j}(g^{-1}\cdot x^ty^s) = e_{-i,-j}((x-\beta y)^ty^s) \\ &= \sum_{m=0}^{t}\binom{t}{m}(-\beta)^{t-m}e_{-i,-j}(x^my^{t+s-m}) \\ &= \begin{cases}\binom{t}{i-1}(-\beta)^{t-i+1} & \text{if } i-1\le t\le n-2\\ 0 & \text{otherwise}\end{cases}. \end{aligned}$$

This shows that

$$\begin{aligned} g\cdot e_{-i,-j} &= \sum_{t=i-1}^{n-2}\binom{t}{i-1}(-\beta)^{t-i+1}e_{-t-1,t+1-n} \\ (\text{let } m=t+1-i) &= \sum_{m=0}^{j-1}\binom{i+m-1}{i-1}(-\beta)^m e_{i-m,m-j}. \end{aligned} \tag{1.1}$$

(2) Čech cohomology

- $\begin{pmatrix}\alpha&\\&\alpha^{-1}\end{pmatrix}\cdot x^{-i}y^{-j} = \alpha^{j-i}x^{-i}y^{-j}$. This can be checked using the usual Čech complex with two affine opens.
- To determine the action of $\begin{pmatrix}1&\beta\\&1\end{pmatrix}$ (and similarly $\begin{pmatrix}1&\\ \beta&1\end{pmatrix}$), we use the Čech complex with open cover by $U_1=\mathbb{P}_1-[1:0]$, $U_2=\mathbb{P}^1-[0:1]$ and $U_3=\mathbb{P}^1-[-\beta:1]$. So we have the following Čech complex:

$$0\to\Gamma(U_1)\oplus\Gamma(U_2)\oplus\Gamma(U_3)\to\Gamma(U_1\cap U_2)\oplus\Gamma(U_1\cap U_3)\oplus\Gamma(U_2\cap U_3)\to\Gamma(U_1\cap U_2\cap U_3)\to 0.$$

The action on local sections given by $g\cdot s_f = s_{f\circ g}$ yields $g\cdot[x^{-i}y^{-j}] = [(x+\beta y)^{-i}y^{-j}]$, where $[x]$ is the class of 1-cocycles represented by x. So we just need to find an element in

$\mathrm{Span}\langle x^{-i}y^{-j}, i+j=n, i,j>0\rangle$ that is in the same class with $(x+\beta y)^{-i}y^{-j}$. The 1-cocycle $f=[(x+\beta y)^{-i}y^{-j}]$ can be thought of as $(f_{12}, f_{13}, f_{23}) \in \Gamma(U_1\cap U_2)\oplus\Gamma(U_1\cap U_3)\oplus\Gamma(U_2\cap U_3)$ such that $f_{12}-f_{13}+f_{23}=0$ on $\Gamma(U_1\cap U_2\cap U_3)$, $f_{13}=(x+\beta y)^{-i}y^{-j}$. So to find an element in $\Gamma(U_1\cap U_2)$ that represents the same cocycle class, we just need to find f_{12} that gives rise to the pole of $(x+\beta y)^{-i}y^{-j}$ at $[1:0]$, and cancelling out the poles at $[0:1]$ and $[-\beta:1]$ by an element $f_{23}\in\Gamma(U_2\cap U_3)$.

We do this by first expanding $(x+\beta y)^{-i}y^{-j}$ using local coordinate $z=\frac{y}{x}$ on $U_1\cap U_3$.

$$\begin{aligned}(x+\beta y)^{-i}y^{-j} &= x^{-i-j}(1+\beta z)^{-i}z^{-j}\\ &= x^{-i-j}\sum_{m=0}^{\infty}\binom{-i}{m}\beta^m z^{m-j}.\end{aligned}$$

So we know that meromorphic part is

$$x^{-i-j}\sum_{m=0}^{j-1}\binom{-i}{n}\beta^m z^{m-j}=\sum_{m=0}^{j-1}\binom{-i}{m}\beta^m y^{m-j}x^{-i-m}. \tag{1.2}$$

This is a section defined on $\Gamma(U_1\cap U_2)=\mathrm{Span}\langle x^{-i}y^{-j}, i,j>0, i+j=n\rangle$, and it is what we are looking for.

In fact, the coefficients in (1.2) are

$$\begin{aligned}\binom{-i}{m}\beta^m &= \frac{(-i)(-i-1)\cdots(-i-m+1)}{m!}\beta^m\\ &= \frac{i(i+1)\cdots(i+m-1)}{m!}(-\beta)^m\\ &= \binom{i+m-1}{m}(-\beta)^m=\binom{i+m-1}{i-1}(-\beta)^m,\end{aligned}$$

which are exactly the same as those appear in (1.1). So by identifying $e_{-i-m,m-j}$ with $x^{-i-m}y^{m-j}$, the actions described in (1.1) and (1.2) are also identified with each other as expected, since we are computing the same representation in two different ways.

2. Motivation

Given a subgroup $H\le G$ and a G-representation (π,V), we define the induced representation of (π,V) on G as

$$\mathrm{Ind}_H^G(\pi,V)=\{f:G\to V\mid f(gh)=\pi(h^{-1})f(g),\forall g\in G,h\in H\}$$

with G action on the left.

Remark. Here we require f to be algebraic (or holomorphic) in comparison to the finite group case where we take any function.

Now take $G=\mathrm{SL}_2(\mathbb{C})$, $H=\{\left(\begin{smallmatrix}* & *\\ & *\end{smallmatrix}\right)\}\le \mathrm{SL}_2(\mathbb{C})$, H acts on $V=\mathbb{C}$ by $\pi_n\begin{pmatrix}\alpha & \beta\\ & \alpha^{-1}\end{pmatrix}v=\alpha^n v$ (i.e., take a weight of the maximal torus T, and view it as a representation of H by pullback along $H\to H/U=T$, where U is the unipotent radical). We then have

$$\begin{aligned}\mathrm{Ind}_H^G(\pi_n,V) &= \{f:G\to\mathbb{C}\mid f(gh)=\pi(h^{-1})f(g)\}\\ &= \Gamma\Big(G/H\to\frac{G\times\mathbb{C}}{(gh,z)\sim\pi_n(h)z}\Big)\\ &= \Gamma(\mathcal{O}_{\mathbb{P}^1}(n)).\end{aligned}$$

Let W be a finite dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. By Frobenius reciprocity we have the following natural isomorphism:

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G\pi_n,W)=\mathrm{Hom}_H(V,W|_H). \tag{2.1}$$

By highest weight theory, there exists nonzero $w \in W$ such that $\left(\begin{smallmatrix} 1 & \beta \\ & 1 \end{smallmatrix}\right) \cdot w = w$. So for some n,

$$\operatorname{Hom}_H(\pi_n, W|_H) \neq 0.$$

So by (2.1),

$$\operatorname{Hom}_G(\operatorname{Ind}_H^G \pi_n, W) \neq 0.$$

Since both $\operatorname{Ind}_H^G \pi_n \cong L_n$ and W are irreducible, we must have $W \cong \operatorname{Ind}_H^G \pi_n$. So we capture all irreducible representations as cohomology of line bundles on $\mathbb{P}^1$.

3. Weyl Character Formula

Why we go through all these trouble? One reason is that it tells us some representation theory fact using geometry.

The geometry we use here is the following theorem.

Theorem 3.1 (Atiyah-Bott fixed point formula)**.** *Let X be a compact manifold and $\pi : E \to X$ a line bundle on X. Let $f : X \to X$, $\rho_f : E \to E$ be an automorphism of (X, E) with isolated fixed points. Then we have*

$$\sum_{i \geq 0} (-1)^i \operatorname{Tr}(f, H^i(X, E)) = \sum_{\substack{p \in X \\ f(p) = p}} \frac{\operatorname{Tr}(\rho_f \circ f^* |_{\pi^{-1}(p)})}{\det(I - df_p)}. \tag{3.1}$$

Take $X = \mathbb{P}^1$, $f = \left(\begin{smallmatrix} \alpha & \\ & \alpha^{-1} \end{smallmatrix}\right)$, $\alpha \neq 1$. Define ρ_f previously with $E = \mathcal{O}_{\mathbb{P}^1}(n)$. There are two fixed points $[0:1]$ and $[1:0]$.

- At [0:1]
  - Tangents look like $[x:1]$ with $\left(\begin{smallmatrix} \alpha & \\ & \alpha^{-1} \end{smallmatrix}\right)$ acting by scaling up by α^2.
  - $\rho_{\left(\begin{smallmatrix} \alpha & \\ & \alpha^{-1} \end{smallmatrix}\right)} : z \cdot \frac{(r,s)}{(0,\alpha^{-n})} \mapsto z \cdot \frac{(r,s)}{(0,1)}$, i.e., acts on fibre by scaling down by α^{-n}.
- At [1:0]
  - Tangents look like $[1:x]$ with $\left(\begin{smallmatrix} \alpha & \\ & \alpha^{-1} \end{smallmatrix}\right)$ acting by scaling down by α^{-2}.
  - $\rho_{\left(\begin{smallmatrix} \alpha & \\ & \alpha^{-1} \end{smallmatrix}\right)} : z \cdot \frac{(r,s)}{(\alpha^n,0)} \mapsto z \cdot \frac{(r,s)}{(1,0)}$, i.e., acts on fibre by scaling up by α^n.

Therefore the right hand side of (3.1) becomes

$$\frac{\alpha^n}{1 - \alpha^{-2}} + \frac{\alpha^{-n}}{1 - \alpha^2}.$$

So it follows from (3.1) that

$$\operatorname{Tr}(\left(\begin{smallmatrix} \alpha & \\ & \alpha^{-1} \end{smallmatrix}\right), L_n) = \frac{\alpha^n}{1 - \alpha^{-2}} + \frac{\alpha^{-n}}{1 - \alpha^2} = \frac{\alpha^{n+1} - \alpha^{-n-1}}{\alpha - \alpha^{-1}}.$$

This is the Weyl character formula for $\mathrm{SL}_2(\mathbb{C})$.