Math 612 Problem Set 6

(due Wed, Apr 13; hand-in at start of class)

Unless otherwise stated, you are allowed to make the blanket assumption that $F = \mathbb{C}$, though many of the problems below hold for more general F.

1. [H13.7] Let $\gamma_1, \ldots, \gamma_\ell$ be an obtuse basis of a Euclidean space E (i.e., all $(\gamma_i, \gamma_j) \leq 0$ for $i \neq j$). Prove that the dual basis is acute (i.e., all $(\gamma^*, \gamma_j^*) \geq 0$ for $i \neq j$). [Reduce to the case $\ell = 2$.]
2. Deduce from the previous exercise that every dominant integral weight has nonnegative rational Δ-coordinates.
3. [H21.2] Draw the weight diagram of $L(\lambda_1 + \lambda_2)$ for the Lie algebra of type B_2, where λ_1 and λ_2 are the fundamental dominant integral weights:
$$\lambda_1 = \alpha_1, \qquad \lambda_2 = \frac{1}{2}(\alpha_1 + 2\alpha_2).$$
4. [H21.6] Let $V = L(\lambda)$ for some $\lambda \in \Lambda^+$. Show that $V^* \cong V(-w_0\lambda)$ where w_0 is the unique member of W that sends Δ to $-\Delta$.
5. [H21.7] For any finite-dimensional L-module V, let $\Pi(V) = \{\nu \in \Lambda : \dim(V_\nu) > 0\}$ denote the set of weights occurring in V. Given $V = L(\lambda)$ and $W = L(\mu)$ with $\lambda, \mu \in \Lambda^+$, prove that $\Pi(V \otimes W) = \Pi(V) + \Pi(W)$ and determine the weight multiplicities of $V \otimes W$ in terms of those of V and W. Deduce that $V(\lambda + \mu)$ occurs exactly once as a summand of $V \otimes W$.
6. [H21.11] Let $L = \mathfrak{sl}(n, \mathbb{C})$ and let $V_k = \Lambda^k(\mathbb{C}^n)$ be the subspace of skew-symmetric tensors in $(\mathbb{C}^n)^{\otimes k}$. Show that each V_k is an irreducible L-module and determine its highest weight.
7. [H22.2] Prove that the universal Casimir element c_L belongs to the center of the universal enveloping algebra.