Math 612 Problem Set 5

(due Wed, Mar 30; hand-in at start of class)

Unless otherwise stated, you are allowed to make the blanket assumption that $F = \mathbb{C}$, though many of the problems below hold for more general F.

1. [H11.6] Prove that an inclusion of one Dynkin diagram in another (e.g. E_6 in E_7 or E_7 in E_8) induces an inclusion of the corresponding root systems.
2. [H12.3] Let $\Phi \subset E$ satisfy (R1), (R3), (R4), but not (R2). Suppose that Φ is irreducible. Prove that Φ is the union of root systems of type B_n, C_n in E (if $\dim E = n > 1$), where the long roots of B_n are also the short roots of C_n. (This is called the non-reduced root system of type BC_n in the literature.)
3. [H12.5] In constructing C_ℓ, would it be correct to characterize Φ as the set of all vectors in I of squared length 2 or 4? Explain.
4. Prove that there exists a root system whose Dynkin diagram is E_6. Prove that there exists a root system whose Dynkin diagram is E_7. Prove that there exists a root system whose Dynkin diagram is E_8. (What order you decide to construct/prove this is up to you.)
5. Let Φ be a root system. Recall that for $\alpha, \beta \in \Phi$ if $(\alpha, \beta) > 0$, then $\alpha - \beta \in \Phi \cup \{0\}$ and if $(\alpha, \beta) < 0$ then $\alpha + \beta \in \Phi \cup \{0\}$. Prove the converse of these implications assuming that Φ is simply laced (all roots have the same length). Show by example that this converse fails if R is not simply laced.
6. [H11.4,12.7] Prove that the Weyl group of root system Φ is isomorphic to the direct product of the respective Weyl groups of its irreducible components. Describe $\operatorname{Aut}\Phi$ when Φ is not irreducible.
7. [H10.14] Prove that each point of E is W-conjugate to a point in the closure of the fundamental Weyl chamber relative to a base Δ. (Enlarge the partial order on E by defining $\mu \prec \lambda$ if and only if $\lambda - \mu$ is a nonnegative $\mathbb{R}$-linear combination of simple roots. If $\mu \in E$, choose $\sigma \in W$ for which $\lambda = \sigma\mu$ is maximal in this partial order.)
8. Prove Lemma A of §13.2: Each weight is conjugate under W to one and only one dominant weight. If λ is a dominant weight, then $\sigma\lambda \prec \lambda$ for all $\sigma \in W$, and if λ is a strongly dominant weight, then $\sigma\lambda = \lambda$ only when $\sigma = 1$.