Math 612 Problem Set 4

(due Wed, Mar 16; hand-in at start of class)

Unless otherwise stated, you are allowed to make the blanket assumption that $F = \mathbb{C}$, though many of the problems below hold for more general F.

In the exercises below, Φ is a root system embedded in a real Euclidean space E with base Δ and Weyl group W.

1. [H, Lemma 9.1] Suppose all reflections σ_α ($\alpha \in \Phi$) leave Φ invariant. Prove that if $\sigma \in \mathrm{GL}(E)$ leaves Φ invariant, fixes pointwise a hyperplane P of E, and sends some nonzero $\alpha \in \Phi$ to its negative, then $\sigma = \sigma_\alpha$ (and $P = P_\alpha$).
2. [H, Lemma 9.2] Prove that if $\sigma \in \mathrm{GL}(E)$ leaves Φ invariant, then $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$ for all $\alpha \in \Phi$, and $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$ for all $\alpha, \beta \in \Phi$.
3. [H9.3] In the table of possibilities for α, β (chart from class = Table in [H, §9.4]), show that the order of $\sigma_\alpha\sigma_\beta$ in W is (respectively) 2, 3, 4, 6 when $\theta = \pi/2$, $\pi/3$ (or $2\pi/3$), $\pi/4$ (or $3\pi/4$), $\pi/6$ (or $5\pi/6$).
4. [H9.4] Prove that the Weyl groups of A_2, B_2, G_2 are the dihedral groups of order $6, 8, 12$.
5. [H9.6] Prove that W is a normal subgroup of $\mathrm{Aut}\,\Phi$.
6. [H9.7] Prove that if E' is the linear span of some $\Delta' \subset \Delta$, then $\Phi' = \Phi \cap E'$ is a root system with base Δ'.
7. [H10.5-6] Prove that the map $w \mapsto (-1)^{\ell(w)}$ is a group homomorphism $W \to \{\pm 1\}$.
8. [H10.9]
   (a) Prove that there is a unique element $w_0 \in W$ of maximum length; moreover, $w_0^2 = 1$, $w_0\Phi^+ = \Phi^-$, and $\ell(w_0) = |\Phi^+|$.
   (b) Show that $-w_0$ is an automorphism of Φ that permutes Δ.