Math 612 Problem Set 3

(due Fri, Feb 25; hand-in by 11am **can slip under the door of my office**)

Unless otherwise stated, you are allowed to make the blanket assumption that $F = \mathbb{C}$, though many of the problems below hold for more general F.

1. (Semi-direct products)
   (a) Let L and M be Lie algebras and assume that we have a Lie algebra morphism $\eta\colon M \to \mathrm{Der}(L)$. Show that the vector space $L \oplus M$ equipped with the bracket
   $$[(l, m), (l', m')] = ([l, l']_L + \eta(m)(l') - \eta(m')(l), [m, m']_M)$$
   is a Lie algebra and $L \oplus \{0\}$ is an ideal.
   (b) Conversely, let X be a Lie algebra, $L \subset X$ an ideal, $M \subset X$ a subalgebra, such that $X = L \oplus M$ as a vector space. Show that $m \mapsto \mathrm{ad}(m)|_L$ is a Lie algebra morphism $\eta\colon M \to \mathrm{Der}(L)$ and that the bracket on X coincides with the bracket on $L \oplus M$ defined above.
   (c) Show that M is an ideal if and only if η is trivial.
2. Define the character of an $\mathfrak{sl}(2)$-module V to be the Laurent polynomial
   $$\chi_V(t) = \sum_i (\dim V_i) t^i,$$
   where V_i denotes the i-weight space of V.
   (a) What are the characters of the irreducible $\mathfrak{sl}(2)$-modules?
   (b) Show that $\chi_V(t) = \chi_W(t)$ if and only if $V \cong W$ for any $\mathfrak{sl}(2)$-modules V, W.
   (c) Show that $\chi_{V\otimes W}(t) = \chi_V(t)\chi_W(t)$ for any $\mathfrak{sl}(2)$-modules V, W.
   (d) [H7.6] Deduce a tensor product rule for $\mathfrak{sl}(2)$; i.e., how many copies of $\mathrm{Sym}^l(\mathrm{std})$ occur in the irreducible decomposition of $\mathrm{Sym}^m(\mathrm{std}) \otimes \mathrm{Sym}^n(\mathrm{std})$?
3. [J. Stembridge]
   (a) Deduce from Exercise 2 that the odd and even parts of every character $\chi_V = \sum_i a_i t^i$ are symmetric and unimodal; i.e., $a_i = a_{-i}$, $a_0 \geq a_2 \geq a_4 \geq \cdots$, and $a_1 \geq a_3 \geq a_5 \geq \cdots$.
   (b) Show that if $W = \mathrm{Sym}^m(\mathrm{Sym}^n(\mathrm{std}))$, then the coefficient of t^{2k-mn} in $\chi_W(t)$ is the number of partitions of k into at most m parts, each of size $\leq n$ (i.e., integer solutions of $n \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$ and $\sum \lambda_i = k$). Thus (a) implies that these numbers are symmetric and unimodal as a function of k.
   (c) Show that $\mathrm{Sym}^m(\mathrm{Sym}^n(\mathrm{std})) \cong \mathrm{Sym}^n(\mathrm{Sym}^m(\mathrm{std}))$ as representations of $\mathfrak{sl}(2)$. [For an arbitrary finite-dimensional L-module V, it is an old conjecture (still open in almost all cases?) that the L-module $\mathrm{Sym}^n(\mathrm{Sym}^m(V))$ contains the L-module $\mathrm{Sym}^m(\mathrm{Sym}^n(V))$ when $n > m$. This is called the Foulkes conjecture.]
   (d) (**Optional to prove**) In fact, $t^{mn}\chi_W(t)$ is the Gaussian polynomial $(t^2)_{m+n}/((t^2)_m(t^2)_n)$, where $(q)_k := (1-q)(1-q^2)\cdots(1-q^k)$. By (b), the Gaussian polynomial is symmetric and unimodal! (The unimodality is the interesting part of this.)
4. [H8.6] Determine the basis of $\mathfrak{sl}(n)$ that is dual (via the Killing form) to the "standard" basis $\{e_{ij} : 1 \leq i \neq j \leq n\} \cup \{e_{ii} - e_{i+1,i+1} : 1 \leq i < n\}$.

5. [H8.5] Prove that if L is semisimple and H is a maximal toral subalgebra, then H is its own normalizer in L.
6. [H8.7] Assume L is semisimple and $H \subset L$ is a maximal toral subalgebra.
   (a) Prove that $C_L(h)$ is reductive for all $h \in H$.
   (b) Prove that it is possible to choose h so that $C_L(h) = H$.
   (c) Characterize when this happens for $\mathfrak{sl}(n)$.