Math 612 Problem Set 2

(due Wed, Feb 9; hand-in at the start of class)

Unless otherwise stated, you are allowed to make the blanket assumption that $F = \mathbb{C}$, though many of the problems below hold for more general F.

1. Prove that L is solvable if and only if $[L, L]$ is nilpotent.
2. Let A be a finite-dimensional F-algebra.
   a. Prove that for any $f \in \mathrm{Der}(A)$, $x, y \in A$, and $a, b \in F$, we have
   $$(f - (a+b))^n(xy) = \sum_{i=0}^{n} \binom{n}{i} (f-a)^i(x) \cdot (f-b)^{n-i}(y).$$
   b. Prove that $\mathrm{Der}\, A$ contains the semisimple and nilpotent parts of all its elements.
3. [H4.1] Use Lie's theorem to prove that $L = \mathfrak{sl}_n(F)$ is semisimple as follows: Let R be the radical of L. Show that there is a change of basis such that $R \subset \mathfrak{b}(m, F)$. However, R must be closed under transposing of matrices (why?). So, R consists only of diagonal matrices and therefore must be 0 (why?).
4. [H4.3] Suppose $\mathrm{char}\, F = p > 0$. Let x be the permutation matrix in $\mathfrak{gl}(p, F)$ with nonzero entries in positions $(1, 2), \ldots, (p-1, p), (p, 1)$, and let $y = \mathrm{diag}(0, 1, \ldots, p-1)$. Show that the Lie subalgebra of $\mathfrak{sl}(p, F)$ generated by x and y is solvable, but that x and y have no common eigenvector. Conclude that Lie's theorem fails in positive characteristic.
5. [H4.5]
   a. Let $x, y \in \mathfrak{gl}(V)$ be commuting elements. Show that if x, y are semisimple (resp. nilpotent), then so is $x + y$ (resp. nilpotent).
   b. If $x, y \in \mathfrak{gl}(V)$ commute, prove that $(x + y)_s = x_s + y_s$, and $(x + y)_n = x_n + y_n$. Show by example that this can fail if x, y fail to commute.
6. [H5.1] Show that if L is nilpotent, then the Killing form of L is identically zero. Find a non-nilpotent Lie algebra whose Killing form is identically zero.
7. [H6.3] Show that if L is solvable, then every irreducible representation of L is one-dimensional.
8. [H6.6] Assume L is simple and that $\beta(\cdot, \cdot)$ and $\gamma(\cdot, \cdot)$ are two symmetric invariant bilinear forms on L. Show that if β and γ are non-degenerate, then they must be proportional. (Hint: Schur's lemma.)
9. [H6.7] Given that $\mathfrak{sl}_n(\mathbb{C})$ is simple, use the previous exercise to prove that its Killing form $\kappa(x, y)$ is $2n\, \mathrm{Tr}(xy)$.