Math 612 Problem Set 1

(due Wed, Jan 26; hand-in at the start of class)

You are allowed to make the blanket assumption that $F = \mathbb{R}$ or $\mathbb{C}$, though many of the problems below hold for more general F.

1. Let $x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ be an ordered basis for $\mathfrak{sl}(2, F)$. Compute the matrices of $\operatorname{ad} x$, $\operatorname{ad} h$, and $\operatorname{ad} y$ relative to this basis.
2. Suppose that $x \in \mathfrak{gl}(n, F)$ has n distinct eigenvalues $a_1, \dots, a_n$ in F. Prove that the eigenvalues of $\operatorname{ad} x$ are precisely the n^2 scalars $a_i - a_j$ $(1 \le i, j \le n)$.
3. (a) Prove that the commutator of two derivations of an F-algebra is again a derivation.
   (b) Give an example of a Lie algebra L and two derivations δ, δ' of L such that $\delta \circ \delta'$ is *not* a derivation.
4. It may be helpful to use or compare to Section 1.2 of Humphreys.
   (a) (C_ℓ) Let $\dim V = 2\ell$ with basis $v_1, \dots, v_{2\ell}$ and let $\langle -, - \rangle \colon V \times V \to F$ be the skew-symmetric bilinear form defined by the matrix $\left(\begin{smallmatrix} 0 & 1_\ell \\ -1_\ell & 0 \end{smallmatrix}\right)$. The set of matrices in $\mathrm{GL}(V)$ which preserve $\langle -, - \rangle$ is the symplectic group $\mathrm{Sp}_{2\ell}(\mathbb{C})$. Calculate $\mathfrak{sp}_{2\ell}(\mathbb{C})$.
   (b) (B_ℓ) Let $\dim V = 2\ell + 1$ with basis $v_1, \dots, v_{2\ell+1}$ and let $\langle -, - \rangle \colon V \times V \to F$ be the symmetric bilinear form defined by the matrix $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_\ell \\ 0 & 1_\ell & 0 \end{smallmatrix}\right)$. The set of matrices in $\mathrm{GL}(V)$ which preserve $\langle -, - \rangle$ is the orthogonal group $\mathrm{O}_{2\ell+1}(\mathbb{C})$. Calculate $\mathfrak{o}_{2\ell+1}(\mathbb{C})$.
   (c) (D_ℓ) Let $\dim V = 2\ell$ with basis $v_1, \dots, v_{2\ell}$ and let $\langle -, - \rangle \colon V \times V \to F$ be the symmetric bilinear form defined by the matrix $\left(\begin{smallmatrix} 0 & 1_\ell \\ 1_\ell & 0 \end{smallmatrix}\right)$. The set of matrices in $\mathrm{GL}(V)$ which preserve $\langle -, - \rangle$ is the orthogonal group $\mathrm{O}_{2\ell}(\mathbb{C})$. Calculate $\mathfrak{o}_{2\ell}(\mathbb{C})$.
   (d) Construct a basis for $\mathfrak{o}_{2\ell}(\mathbb{C})$.
5. When $\operatorname{char} F = 0$, show that $[\mathfrak{sl}_n(F), \mathfrak{sl}_n(F)] = \mathfrak{sl}_n(F)$.
6. For small values of ℓ, isomorphisms occur among certain of the classical algebras. Please see Humphreys §1.2 for definitions of all these. Note that the definitions in §1.2 are exactly the same as listed in Problem 4, with the addition of A_ℓ, which doesn't appear in Problem 4. (A_ℓ for refers to the Lie algebra $\mathfrak{sl}_{\ell+1}(\mathbb{C})$.)

   ** Any attempt at 6(b) and 6(c) will count for full credit. Spoilers on the next page. **

   (a) Show that A_1, B_1, C_1 are all isomorphic, while D_1 is the one-dimensional Lie algebra.
   (b) Show that B_2 is isomorphic to C_2 and D_3 is isomorphic to A_3.
   (c) What can you say about D_2?
7. Let I be an ideal of L. Prove that each member of the derived series $D^n(I)$ or lower central series $C^n(I)$ of I is also an ideal of L.
8. Prove that L is solvable if and only if there exists a chain of subalgebras $L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_k = 0$ such that L_{i+1} is an ideal of L_i and such that each quotient L_i / L_{i+1} is abelian.
9. Show that if $L = [L, L]$ and $\dim L = 3$, then L is simple.

Regarding 6(b). I don't know how to construct this in a good way explicitly on basis elements, and I think a conceptual argument is quite tricky. Thank you to Teresa and Yutong for pointing this out to me and also for enduring an hour of trying to construct an explicit isomorphism with me.

A very nice write-up of these isomorphisms is Prof. Po-Lam Yung's note: `https://maths-people.anu.edu.au/~plyung/isom_LieAlg.pdf`. I learned the following arguments from his notes.

6b: B_2 **and** C_2**.** Let V be a 4-dimensional F-vector space and let $\omega\colon V \times V \to F$ be a nondegenerate alternating bilinear form. Now, since V is 4-dimensional, we have $\Lambda^4 V \cong F$. Via this isomorphism, we have a natural nondegenerate symmetric bilinear form $B(-,-)$ on $\Lambda^2 V$ given by the wedge product. This gives us an identification $\Lambda^2 V \cong \Lambda^2 V^*$. This means that we may view $\omega \in \Lambda^2 V^*$ as an element $x \in \Lambda^2 V$. Since $\mathfrak{sp}(V)$ preserves ω, it must also preserve x.

One can show that the induced action of $\mathfrak{sl}(V)$ on $\Lambda^2 V$ preserves $B(-,-)$, so of course $\mathfrak{sp}(V)$ must also. Hence $\mathfrak{sp}(V)$ must stabilize the orthogonal complement W of $\langle x \rangle$ in $\Lambda^2 V$. In other words, we have a Lie algebra homomorphism

$$\phi\colon\ \mathfrak{sp}(V) \to \mathfrak{o}(W).$$

We can show, using the same method as we did in class, that $\mathfrak{sp}(V)$ is simple. Since $\dim \mathfrak{sp}(V) = \dim \mathfrak{o}(W)$, then we know that either $\phi = 0$ or ϕ is an isomorphism. Since $\mathfrak{sp}(V)$ acts nontrivially on $\Lambda^2 V$, then $\phi \neq 0$.

6b: A_3 **and** D_3**.** One can use a similar trick to the above: Take V to again be a 4-dimensional F-vector space and consider the 6-dimensional $\Lambda^2 V$. Identifying $\Lambda^4 V \cong F$ gives rise to a nondegenerate symmetric bilinear form on $\Lambda^2 V$ which is preserved under the $\mathfrak{sl}(V)$ action on $\Lambda^2 V$. This gives a Lie algebra homomorphism

$$\phi\colon\ \mathfrak{sl}(V) \to \mathfrak{o}(\Lambda^2 V).$$

6c: D_2 **and what?** For 6(c), consider the 4-dimensional vector space $V = \mathfrak{gl}_2(F)$. Find a nondegenerate symmetric bilinear form on V which is preserved under left- and right-multiplication by $\mathfrak{sl}_2(F)$. What does this say?