AFFINE DELIGNE–LUSZTIG VARIETIES AT INFINITE LEVEL
(PRELIMINARY VERSION)

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CONTENTS

1. Introduction 2
1.1. Outline of the paper 4
Acknowledgements 4

Part 1. Two analogues of Deligne–Lusztig varieties for $p$-adic groups 5
2. Affine Deligne–Lusztig varieties at infinite level 5
2.1. Preliminaries 5
2.2. Deligne–Lusztig sets/varieties 7
2.3. Affine Deligne–Lusztig varieties and covers 8
2.4. Scheme structure 9
3. Case $G = \text{GL}_n$, $b$ basic, $w$ Coxeter 12
3.1. Notation 12
3.2. Basic $\sigma$-conjucacy classes, Isocrystals 12
3.3. The admissible subset of $V_b$ and $F$-cyclic lattices 12
3.4. Comparison and uniformization of DLV and ADLV 13
3.5. Connected components 17

Part 2. Cohomology 22
4. Definitions 22
5. Deligne–Lusztig varieties for unipotent groups 27
5.1. Combinatorial set-up 27
5.2. Cohomology of $Z_h^{(1)}$ 31
6. Deligne–Lusztig varieties for reductive groups over finite rings 35
6.1. Cohomology of $Z_h$ 36
6.2. Cohomology of $X_h$ 37
7. Affine Deligne–Lusztig varieties at finite level 41
7.1. Homology of Deligne–Lusztig varieties at infinite level 42

Part 3. Character formulae 43
8. On very regular elements 43
References 43
1. Introduction

In their fundamental paper [DL76], Deligne and Lusztig gave a powerful geometric approach to the construction of representations of finite reductive groups. To a reductive group $G$ over a finite field $\mathbb{F}_q$ and a maximal $\mathbb{F}_q$-torus $T \subseteq G$, they attached a Deligne-Lusztig variety. This is a locally closed subset of the flag manifold of $G$, consisting of all Borel subgroups of $G$ lying in a fixed relative position (depending on $T$) to their Frobenius translate. This variety has a cover, whose cohomology with constant $\ell$-adic coefficients realizes a natural correspondence between characters of $T(\mathbb{F}_q)$ (and (irreducible) $\overline{\mathbb{Q}}_\ell$-representations of $G(\mathbb{F}_q)$).

There are two possible ways of generalizing this construction to reductive groups over local fields: one can consider subsets cut out by Deligne–Lusztig conditions of either a semi-infinite flag manifold in the sense of Feigin-Frenkel [FF90], or in the affine flag manifold. The first approach was pioneered by Lusztig in [Lus79]. Later it was studied in detail in [Boy12, Cha15, Cha16a]. The second approach is based on Rapoport’s affine Deligne-Lusztig varieties from [Rap05]. It was introduced and then studied in detail in the case of $\text{GL}_2$ by the second named author in [Iva15a, Iva15b].

The goals of the present paper are to show that both of these constructions

- give basically the same varieties for all inner forms of $\text{GL}_n$ and unramified minisotropic tori,
- realize the unramified supercuspidal part of the local Langlands correspondence (i.e., for all $L$-parameters which factor through $L$-dual groups of unramified minisotropic tori) for $\text{GL}_n$ and are compatible with the Jacquet-Langlands correspondence for its inner forms.

The first goal is achieved by computing both sides and defining an explicit isomorphism between Lusztig’s semi-infinite construction and an inverse limit of coverings of affine Deligne-Lusztig varieties. This in particular defines a natural pro- (perfect) scheme structure on the case of all inner forms of $\text{GL}_n$, along with a further cohomological method from [Iva15a].

To be more precise, let $K$ be a non-archimedian local field, let $\tilde{K} := \hat{K}^{nr}$ be the completion of the maximal unramified extension of $K$ and let $\sigma$ denote the Frobenius automorphism of $\tilde{K}/K$. For any algebro-geometric object $X$ over $K$, we write $\tilde{X} := X(\tilde{K})$ for the set of its $\tilde{K}$-points. Let $G$ be a connected reductive group over $K$. For $b \in \tilde{G}$, let $J_b$ be the $\sigma$-stabilizer of $b$, i.e.,

$$J_b(R) := \{ g \in G(R \otimes_K \tilde{K}) : g^{-1}b\sigma(g) = b \},$$

for any $K$-algebra $R$. Then $J_b$ is an inner form of a Levi subgroup of $G$ and if $b$ is basic, then $J_b$ is an inner form of $G$. Let $T$ be the centralizer of a maximal $\tilde{K}$-split torus in $G$. By Steinberg’s theorem, this is a maximal torus. For an element $w$ in the (finite) Weyl group of $G, T$, let $T_w$ be the form of $T$ by

$$T_w(R) := \{ g \in T(R \otimes_K \tilde{K}) : t^{-1}w\sigma(t) = \tilde{w} \},$$

for any $K$-algebra $R$, where $\tilde{w}$ is a lift of $w$ to $\tilde{G}$. We let the semi-infinite Deligne-Lusztig variety, $X^{(U)}_w(b)$, be the set of all Borel subgroups of $\tilde{G}$, which have relative position $w$ to their $b\sigma$-translate. It has a cover, $X^{(U)}_w(b)$, which coincides with Lusztig’s construction from [Lus79]. There is a natural $J_b(K) \times T_w(K)$-action on $X^{(U)}_w(b)$. On the other side, as in [Iva15a], for any bounded level subgroup $J \subseteq \tilde{G}$ and any $J$-double coset $wJ$, there is an
attached affine Deligne-Lusztig variety of level $J$, denoted $X^J_{\sigma}(b)$, acted on by $J_b(K)$. If $J$ is an Iwahori, then this coincides with Rapoport’s affine Deligne-Lusztig variety from [Rap05], and can be seen as the set of all Iwahori subgroups of $\tilde{G}$ having relative position $w_J$ to their $b\sigma$-translate.

Let now $G = GL_n$. Let $b \in \tilde{G}$ be basic, i.e., $J_b$ is an inner form of $G$. Let $w$ be a Coxeter element, such that $T_w(K) \cong L^\times$ for the degree $n$ unramified extension $L/K$. We fix two families of decreasing level subgroups of $\tilde{G}$ together with speical double cosets and build the inverse limits of the coresponding affine Deligne-Lusztig varieties. We obtain two affine Deligne-Lusztig varieties at infinite level, $X^\infty_b(b) \to X^\infty_w(b)$, both endowed with a $J_b(K)$-action. The involved map is a $T_w(\mathcal{O}_K)$-torsor. These are pro- (perfect) schemes over $\mathbb{F}_q$.

The main result of the first part of the paper is inspired by [DL76] §2.2. It gives an elegant description of $X^B_b(b)$, $X^{U}(b)$, $X^\infty_w(b)$ and $X^\infty_b(b)$ in terms of the isocrystal $V_b := (\tilde{K}^n, b\sigma)$. Namely, let $V^{\text{adm}}_b$ denote the set of all elements which do not lie in a proper sub-isocrystal.

**Theorem** (see Corollary 3.8). There is a commutative diagram with $J_b(K)$-equivariant maps:

$$
\begin{array}{cccc}
X^U_b(b) & \xrightarrow{\sim} & \{ x \in V^{\text{adm}}_b : \det(g_b(x)) \in K^\times \} & \xrightarrow{\sim} & X^\infty_b(b) \\
\downarrow T_w(K) & & & \downarrow T_w(\mathcal{O}_K) \\
X^B_b(b) & \xleftarrow{\sim} & V^{\text{adm}}_b / \tilde{K}^\times & \xrightarrow{\sim} & V^{\text{adm}}_b / \mathcal{O}^\times_b \xrightarrow{\sim} X^b_w(b),
\end{array}
$$

where $g_b(x)$ denotes the matrix with columns $(b\sigma)^i(x)$ for $0 \leq i \leq n - 1$. In particular, this endows $X^U_b(b)$, $X^B_b(b)$ with natural pro- (perfect) scheme over $\mathbb{F}_q$.

The main objective in the second part of the paper is to study the cohomology of $X^U_b(b) = X^\infty_b(b)$. Using results of Viehmann [Vie08] on connected components of affine Deligne-Lusztig varieties, one deduces a decomposition of $X^\infty_b(b)$ into (scheme-theoretically) disjoint components. The group $J_b(K)$ acts transitively on the set of these components with stabilizers being maximal compact subgroups of $J_b(K)$. This reduces the computation of the cohomology of $X^\infty_b(b)$ to the computation for one such component $X$. It can be written as an inverse limit $X = \lim_{\to} X_h$ of finite dimensional varieties $X_h$, each endowed with an action of $G_h \times \mathcal{O}^\times_L / U_L^h$, where $G_h$ is a "level $h$"-quotient of the maximal compact subgroup of $J_b(K)$, $U^h_L$ are $h$-units of $L$ and $\mathcal{O}^\times_L / U^h_L$ is in a natural way a subgroup of $T_w(K) \cong L^\times$. We note that $X_1$ is a classical Deligne-Lusztig variety.

The idea in the description of the cohomology of $X_h$, is to decompose it into open/closed parts $X_h = Y_h \sqcup Z_h$. The cohomology of $Z_h$ contains the representations of positive level and is hard to compute. Therefore a variation of the involved techniques from [Cha15] [Cha16b] [Cha16a] involving Deligne-Lusztig constructions for unipotent groups is used. The main result concerning the cohomology of $Z_h$ is the following theorem.

**Theorem** (rough form; see Theorem 6.2). For each $\theta : \mathcal{O}^\times_L / U^h_L \to \mathbb{Q}^\times_L$, there exists an integer $r_\theta \leq 2(n - 1)(h - 1)$, such that

$$H^i_c(Z_h, \mathbb{Q}_L)[\theta] \neq 0 \iff i = r_\theta.$$

Moreover, $r_\theta$ can be determined explicitly in terms of the Howe factorization of $\theta$.

The scheme $Z_h$ decomposes into a finite disjoint union of copies of a subscheme, $Z^{(1)}_h$, which has a quite special property.
Corollary (see Theorem 5.17). The scheme $Z_h^{(1)}$ is a maximal variety in the sense of Boyarchenko–Weinstein [BW13], i.e., $H^i_c(Z_h^{(1)}, \mathbb{Q}_\ell) = 0$, unless $i$ or $n$ is even and the $n$-th power $\sigma^n$ of the Frobenius acts in its cohomology by the scalar $(-1)^{q^{ni}/2}$.

To compute the cohomology of $Y_h$ one uses a calculation of the fibers of $Y_h \to Y_{h-1}$, which turn out to be disjoint unions of affine lines, the behavior of which is governed by the determinant. Similar as in [Iva15a], this allows to show that the cohomology of $Y$ already comes from $Y_1$, i.e., accounts for representations of level 0. Putting these computations together, one obtains

Theorem (rough version of Theorem 6.1). For each $\theta: \mathcal{O}_L^\times / U_\ell^\times \to \mathbb{Q}_\ell^\times$, $H^i_c(X_h, \mathbb{Q}_\ell)[\theta]$ comes from $X_1$ if $i > 2(n-1)(h-1) + 1$ and $\theta$ is generic, and $H^i_c(Z_h, \mathbb{Q}_\ell)[\theta]$ if $i \leq 2(n-1)(h-1)$ and $\theta$ is generic, and $\mathcal{O}_L^\times / U_\ell^\times$ factors through the norm of $\theta$.

Combined with results of Lusztig on the alternating sum of the $\ell$-adic cohomology $H^i_c(X_h, \mathbb{Q}_\ell)$, we obtain:

Corollary. For generic $\theta$, the representation $H^i_c(X_h, \mathbb{Q}_\ell)[\theta]$ is irreducible. Moreover, for generic $\theta \neq \theta'$, the representations $H^i_c(X_h, \mathbb{Q}_\ell)[\theta]$ and $H^i_c(X_h, \mathbb{Q}_\ell)[\theta']$ are non-isomorphic.

The idea now (as in [Cha15, Cha16b, Cha16a]) to use the cohomological computations to deduce trace formulas for the action of certain group elements. These traces are then enough to compare $\theta \mapsto H^i_c(X_h, \mathbb{Q}_\ell)[\theta]$ with the Langlands and Jacquet-Langlands correspondences.

1.1. Outline of the paper. The first part of the article is devoted to purely geometric properties of the Deligne–Lusztig constructions. In subsection 2 we define/recall both types of Deligne–Lusztig constructions for reductive groups over local fields and general facts about them. In Section 3 we provide the coverings of affine Deligne–Lusztig varieties with a scheme structure. In Section 4 we specialize to the case $G = GL_n$ and $b$ Coxeter and compare both constructions in this case. The main result here is Corollary 3.8.

The second part is devoted to the study of the cohomology of $X_h = Y_h \sqcup Z_h$ and the representations occurring there. In Section 5 some groups and varieties are defined. They are used in Section 6 to study the cohomology of $Z_h^{(1)}$ in terms of the set up of [Cha15, Cha16b, Cha16a]. We note that the notation in this section is different from the rest of the article, and is related to the notation in [Cha15, Cha16b, Cha16a] (in particular, $Z_h^{(1)}$ is denoted by $X$). In Section 6 the cohomology of $X_h$ is determined in detail: Section 6.1 uses results of Section 5 to determine the cohomology of $Z_h$ and Section 6.2 shows that the cohomology of $Y_h$ comes from the level one. Finally, a Mayer-Vietoris sequence for $X_h = Y_h \sqcup Z_h$ gives the desired result, Theorem 6.1. Finally, in Section 7 we use the results from Sections 5 and 6 to study the homology of the affine Deligne–Lusztig varieties of finite level.

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Part 1. Two analogues of Deligne–Lusztig varieties for $p$-adic groups

The aim of this part of the paper is to prove a comparison result relating two geometric constructions: affine Deligne–Lusztig varieties at infinite level and semi-infinite Deligne–Lusztig varieties.

2. Affine Deligne–Lusztig varieties at infinite level

2.1. Preliminaries.

2.1.1. Basic notations. Let $K$ be a non-archimedean local field with residue characteristic $p > 0$, and let $\tilde{K}$ denote the completion of a maximal unramified extension of $K$. We denote by $\mathcal{O}_K$, $k$, $p_K$ (resp. $\mathcal{O}$, $p$, $\tilde{k}$) the integers, the residue field and the maximal ideal of $K$ (resp. of $\tilde{K}$). Let further $q$ denote the cardinality of $k$. We write $\sigma$ for the Frobenius automorphism of $\tilde{K}$, which is the unique $K$-automorphism of $\tilde{K}$, lifting the $k$-automorphism $x \mapsto x^q$ of $\tilde{k}$. Finally, we denote by $\varpi$ a uniformizer of $K$ (and hence of $\tilde{K}$) and by $\text{ord} = \text{ord}_{\tilde{K}}$ the valuation of $\tilde{K}$, normalized such that $\text{ord}(\varpi) = 1$.

2.1.2. Group theoretic data. Let $G$ be a connected reductive group over $K$. Let $S$ be a maximal $\tilde{K}$-split torus in $G$. By [BT72, 5.1.12] it can be chosen to be defined over $K$. Let $T := \mathcal{Z}_G(S)$ (resp. $N := \mathcal{N}_G(S)$) be the centralizer (resp. normalizer) of $S$. By Steinberg’s theorem, $G$ is quasi-split, hence $T$ is a maximal torus. Let $B$ denote a $\tilde{K}$-rational Borel subgroup of $G$ containing $T$. Let $U$ be the unipotent radical of $B$, i.e., $B = TU$ and $U$ is defined over $\tilde{K}$.

Let $\Phi := \Phi(G, S)$ denote the root system of $S$ in $G_{\tilde{K}}$. We see $0$ as a root with the root subgroup $T$. For a root $\alpha \in \Phi \cup \{0\}$, we denote by $U_{\alpha}$ the corresponding root subgroup.

For $X = G, B, T, U, \ldots$, we write $\tilde{X}$ for the set $X(\tilde{K})$ of $\tilde{K}$-rational points of $X$.

2.1.3. Iwahori subgroup, valuation filtrations and integral models. We let $I$ be an $\sigma$-stable Iwahori subgroup of $G$ (such one exists as $G$ is residually quasi-split over $K$). We fix a special vertex $x_0$ of the alcove corresponding to $I$ of the Bruhat-Tits building of $G$ over $\tilde{K}$. By [BT72, Sections 6.2, 6.4], attached to $x_0$ there are valuations of the root subgroups $U_{\alpha}$ for $\alpha \in \Phi$. Denote the corresponding filtration on $\tilde{U}_{\alpha}$ by $\tilde{U}_{\alpha,r}$ (for $r \in \tilde{\mathbb{R}}$, where $\tilde{\mathbb{R}}$ is the ordered monoid containing $\mathbb{R}$ as in [BT72, Section 6.4]) and fix an admissible schematic filtration on tori, which gives a filtration $\tilde{U}_{0,r} = \tilde{T}_r$ on $\tilde{T}$. As in [Yu02], there is a (uniquely determined) $\mathcal{O}$-model $\mathcal{U}_{\alpha,r}$ of $U_{\alpha} \otimes \tilde{K}$, such that $\mathcal{U}_{\alpha,r}(\mathcal{O}) = \tilde{U}_{\alpha,r}$. This model descends to $\mathcal{O}_K$, if $\alpha$ is fixed by $\sigma$.

Moreover, by [Yu02], attached to any concave function $f$ on $\Phi \cup \{0\}$ with values in $\tilde{\mathbb{R}}_{< \infty}$, there is a subgroup $\tilde{G}_f$ generated by $\tilde{U}_{a,f(a)}$ for $\alpha \in \Phi \cup \{0\}$, and $\tilde{G}_{f,K}$ possesses an unique integral model $\tilde{G}_{f,K}$, whose $\mathcal{O}$-rational points are $\tilde{G}_f$. The schematic closure of $U_{\alpha}$ in $\tilde{G}_f$ is $\mathcal{U}_{\alpha,r}$. Moreover, if $\tilde{G}_f$ is $\sigma$-stable, this model descends to a model of $G$ over $\mathcal{O}_K$. We denote by $f_{I}$ the concave function which has value $0$ on negative roots and $0$, and value $1$ on positive roots. Then $I = \tilde{G}_{f}$. If $f \geq f_{I}$, we write $I^f$ for $\tilde{G}_f$ (to indicate that we work with a level $I^f$ contained in $I$).

2.1.4. Loop groups. To have a common notation for equal and mixed characteristic cases, for a $k$-algebra $R$ define

$$\mathbb{W}(R) := \begin{cases} R \otimes_k \mathcal{O}_K & \text{if } \text{char}(K) > 0 \\ W(R) \otimes_{W(k)} \mathcal{O}_K & \text{if } \text{char}(K) = 0, \end{cases}$$
where \( W(k) \) denote the ring of Witt-vectors of \( k \). In the mixed characteristic case the ring \( \mathcal{W}(R) \) behaves well, only when \( R \) is a perfect \( k \)-algebra. For a scheme \( X \) over \( K \), let \( LX \) be the functor on \( k \)-algebras,

\[
LX: R \mapsto X(\mathcal{W}(R)[\sigma^{-1}]).
\]

This is the loop space of \( X \). For a scheme \( \mathcal{X} \) over \( \mathcal{O}_K \), let \( L^+\mathcal{X} \) be the functor on \( k \)-algebras,

\[
L^+\mathcal{X}: R \mapsto \mathcal{X}(\mathcal{W}(R)).
\]

This is space of positive loops of \( \mathcal{X} \).

\[\text{2.1.5. Affine flag varieties and covers.}\]

Let \( f: \Phi \cup \{0\} \to \tilde{\mathbb{R}}_{<\infty} \) be a concave function, such that \( I^f \subseteq I \) is \( \sigma \)-stable. By [PR08, Theorem 1.4] in the equal characteristic case and [Zhu14 Theorem 1.5] in the mixed characteristic case, the fpqc-sheaf \( LG/L^+G_f \) on the category of \( k \)-algebras resp. perfect \( k \)-algebras is represented by an Ind-scheme of Ind-finite type resp. Ind-perfect algebraic space of Ind-perfect finite type. We denote it by \( \mathcal{F}^f \). In case \( f = f_1 \), we write \( \mathcal{F} \) for \( \mathcal{F}^f \). It is the affine flag manifold. We have \( \mathcal{F}^f(k) = \tilde{G}/I^f \). Whenever \( f \leq g \) are two concave functions, there is a natural projection \( \mathcal{F}^g \to \mathcal{F} \).

\[\text{2.1.6. Weyl groups.}\]

Let \( W_{\text{fin}} := N/T \) denote the (finite) Weyl group of \( S \) relative to \( \tilde{K} \). By [Bor91, Theorem 21.2], every connected component of \( N \) meets \( \tilde{G} \), hence \( W_{\text{fin}} = \tilde{N}/\tilde{T} \). In particular, the \( \text{Gal}_K \)-action on \( W_{\text{fin}} \) factors through a \( \text{Gal}(\tilde{K}/K) \)-action. Let \( \tilde{W} = \tilde{N}/(\tilde{T} \cap I) \) denote the extended affine Weyl group of \( S \). It sits in the exact sequence

\[
0 \to X_*(T)_{\text{Gal}_K} \to \tilde{W} \to W_{\text{fin}} \to 0.
\]

\[\text{2.1.7.} \sigma\text{-stabilizers.}\]

For \( b \in \tilde{G} \), we denote by \( J_b \) the \( \sigma \)-stabilizer of \( b \), which is the \( K \)-group defined by

\[
J_b(R) := \{ g \in G(R \otimes_K \tilde{K}) : g^{-1}b\sigma(g) = b \}
\]

for any \( K \)-algebra \( R \) (cf. [RZ96, 1.12]). Then \( J_b \) is an inner form of the centralizer of the Newton point \( b \) (which is a Levi subgroup of \( G \)). In particular, if \( b \) is basic, i.e., the Newton point of \( b \) is central, then \( J_b \) is an inner form of \( G \).

Let \( w \in W_{\text{fin}} \) and let \( \dot{w} \in \tilde{N} \) be a lift. We denote by \( T_w \) the \( \sigma \)-stabilizer of \( w \) in \( T \), which is the \( K \)-group defined by

\[
T_w(R) := \{ t \in T(R \otimes_K \tilde{K}) : t^{-1}\dot{w}\sigma(t) = \dot{w} \}.
\]

for any \( K \)-algebra \( R \). As \( T \) is commutative, the definition only depend on \( w \), and not on the choice of the lift \( \dot{w} \).

\[\text{2.1.8. Bruhat and Iwahori decomposition. Refinements.}\]

The \( \tilde{K} \)-rational Bruhat decomposition states that \( \tilde{G} = \bigsqcup_{w \in W_{\text{fin}}} \tilde{B}\dot{w}\tilde{B} \) (where \( \dot{w} \) is some lift of \( w \) to \( \tilde{G} \)). Moreover, \( U \) is \( \tilde{K} \)-split, hence the map \( \tilde{B} \to \tilde{T} \) is surjective (see [Bor91, Corollary 15.7]). This implies \( \tilde{B} = \tilde{T}\tilde{U} \) and we deduce from this a refinement of the Bruhat decomposition: \( \tilde{G} = \bigsqcup_{\dot{w} \in \tilde{N}} \tilde{U}\dot{w}\tilde{U} \). In particular, we may identify the set of \( \tilde{U} \)-double cosets with \( \tilde{N} \). There are the invariant position maps

\[
\text{inv}_{\tilde{U}}: \tilde{G}/\tilde{U} \times \tilde{G}/\tilde{U} \to \tilde{N}
\]

\[
\text{inv}_B: \tilde{G}/\tilde{B} \times \tilde{G}/\tilde{B} \to W_{\text{fin}}.
\]

sending a pair of cosets represented by \( x, y \) to the double coset containing \( x^{-1}y \). These maps commute with the projection \( \tilde{G}/\tilde{U} \to \tilde{G}/\tilde{B} \).

The Iwahori-decomposition states that \( \tilde{G} = \bigsqcup_{\dot{w} \in \tilde{W}} \tilde{I}\dot{w}\tilde{I} \) (where \( \dot{w} \) is some lift of \( w \) to \( \tilde{G} \)). Let \( f \) be as in Section 2.1.5. We write \( D_{G,f} \) for the set of double cosets \( I^f \setminus \tilde{G}/I^f \). There is
the invariant position map
\[ \text{inv}_f : \tilde{G}/I_f \times \tilde{G}/I_f \to D_{G,f} \]
sending a pair of cosets represented by \( x,y \) to the double coset containing \( x^{-1}y \). If \( f_1 \leq f \leq q \), then there are natural projections \( D_{G,g} \to D_{G,f} \to D_{G,f_1} \cong \tilde{W} \). The maps \( \text{inv}_f \) are compatible with the maps \( \mathcal{G}^g \to \mathcal{G}^f \) and these projections.

### 2.2. Deligne–Lusztig sets/varieties.

We have the following direct analog of the classical Deligne–Lusztig varieties introduced in [DL76].

**Definition 2.1.** Let \( w \in W \) and \( b \in \tilde{G} \). We define the Deligne–Lusztig set \( X_w^{(B)}(b) \) as
\[ X_w^{(B)}(b) := \{ x \in \tilde{G}/\tilde{B} : \text{inv}_B(x,b\sigma(x)) = w \} \]

Let moreover \( \tilde{w} \in \tilde{W} \). We define
\[ X_w^{(U)}(b) := \{ x \in \tilde{G}/\tilde{U} : \text{inv}_U(x,b\sigma(x)) = \tilde{w} \} \]

**Remark 2.2.**

(i) Note that \( X_w^{(B)}(b) \) is non-empty if and only if the \( \sigma \)-conjugacy class \([b]\) of \( b \) in \( \tilde{G} \) intersects the double coset \( \tilde{B}w\tilde{B} \). In particular, if \( G = \text{GL}_n \) \((n \geq 2)\) and \( b \) is superbasic, then \( X_1^{(B)}(b) = \emptyset \) (this was observed by E. Viehmann).

(ii) In the context of the classical Deligne–Lusztig theory [DL76], the group \( G \) over \( \mathbb{F}_q \) one starts with, is automatically quasi-split. Similarly, in the context of affine Deligne–Lusztig varieties (see Section 2.3), the group \( G/K \) one starts with, is always residually quasi-split, i.e., \( \tilde{G} \) contains a \( \sigma \)-stable Iwahori subgroup. Observe the discrepancy of these two setups with Definition 2.1, where \( G \) need not to be quasi-split over \( K \).

(iii) However, as we are interested in the representation theory of the groups \( G(K) \) (resp. \( J_b(K) \)), we can assume that \( G \) is quasi-split: indeed, for any \( G/K \), there is always a quasi-split connected reductive group \( G_0/K \), such that \( G \) is an inner form of \( G_0 \). Choosing some \( b \in \tilde{G}_0 \), such that the inner form \( J_b \) of \( G_0 \) satisfies \( J_b \cong G \), we can study representations of \( G(K) \) by considering the Deligne-Lusztig constructions attached to \( G_0 \) and \( b \).

\[ \diamond \]

**Lemma 2.3.** Let \( b \in \tilde{G} \) and let \( w \in W_{\text{fin}} \) with lift \( \tilde{w} \in \tilde{W} \).

(i) Let \( g \in \tilde{G} \). The map \( \tilde{x} \tilde{B} \mapsto gx\tilde{B} \) defines a bijection \( X_w^{(B)}(b) \cong X_{\tilde{w}^{-1}b\sigma(g)}(b) \).

(ii) Let \( g \in \tilde{G} \) and \( t \in \tilde{T} \). The map \( \tilde{x} \tilde{U} \mapsto gx\tilde{t}U \) defines a bijection \( X_{\tilde{w}}^{(U)}(g^{-1}b\sigma(g)) \cong X_{(\tilde{w}^{-1}b\sigma(g))}(g^{-1}b\sigma(g)) \).

(iii) There are actions of \( J_b(K) \) on \( X_w^{(B)}(b) \) given by \( (g, x\tilde{B}) \mapsto gx\tilde{B} \) and of \( J_b(K) \times T_w(K) \) on \( X_{\tilde{w}}^{(U)}(b) \) given by \( (g,t,x\tilde{U}) \mapsto gx\tilde{t}U \). They are compatible with the projection \( X_{\tilde{w}}^{(U)}(b) \to X_w^{(B)}(b) \). Moreover, \( X_{\tilde{w}}^{(U)}(b) \) is a right \( T_w(K) \)-torsor over \( X_w^{(B)}(b) \).

**Proof.** (i) and (ii) follow from the definitions by immediate computations. (iii) follows from (i) and (ii).

Visualizing Lemma 2.3, we have the following actions:

\[ \begin{align*}
J_b(K) & \subset \subset X_{\tilde{w}}^{(U)}(b) \\
& \quad \quad \downarrow \\
J_b(K) & \subset \subset X_w^{(B)}(b)
\end{align*} \tag{2.1} \]
Remark 2.4. The sets $X_w(B)(b)$ and their covers from Definition 2.1 are generalizations of the original construction of the semi–infinite Deligne–Lusztig variety $X = \lim_{h} X_h$ (equipped with an ad hoc scheme structure over $k$ in certain special cases) of Lusztig [Lus79], which was studied by Boyarchenko [Boy12] and the first named author [Cha16b, Cha15, Cha16a]. Indeed, define a map

$$F_b: \tilde{G} \rightarrow \tilde{G}, \quad g \mapsto b\sigma(g)b^{-1}.$$  

Assuming that $(w, b)$ satisfies $w\tilde{B} = b\sigma(\tilde{B})$, so that $w\tilde{B}b^{-1} = F_b(\tilde{B})$,  

$$X_w(B)(b) = \{x\tilde{B} \in \tilde{G}/\tilde{B} : x^{-1}b\sigma(x) \in \tilde{B}w\tilde{B}\}$$

$$= \{x\tilde{B} \in \tilde{G}/\tilde{B} : x^{-1}F_b(x) \in \tilde{B}w\tilde{B}b^{-1}\}$$

$$= \{x\tilde{B} \in \tilde{G}/\tilde{B} : x^{-1}F_b(x) \in \tilde{B}f(\tilde{B})\}$$

$$= \{x \in \tilde{G} : x^{-1}F_b(x) \in F(\tilde{U})/(T^F(\tilde{U} \cap F_b(\tilde{U}))\},$$

where the last equality holds since $\tilde{B} \cap F(\tilde{B}) = \tilde{T}(\tilde{U} \cap F_b(\tilde{U}))$. Similarly, assuming that $(w, b)$ satisfies $w\tilde{U} = b\sigma(\tilde{U})$, so that $w\tilde{U}b^{-1} = F_b(\tilde{U})$,

$$X_w(U)(b) = \{x\tilde{U} \in \tilde{G}/\tilde{U} : x^{-1}b\sigma(x) \in \tilde{U}w\tilde{U}\}$$

$$= \{x\tilde{U} \in \tilde{G}/\tilde{U} : x^{-1}F_b(x) \in \tilde{U}w\tilde{U}b^{-1}\}$$

$$= \{x\tilde{U} \in \tilde{G}/\tilde{U} : x^{-1}F_b(x) \in \tilde{U}F_b(\tilde{U})\}$$

$$= \{x \in \tilde{G} : x^{-1}F_b(x) \in F(\tilde{U})/(\tilde{U} \cap F_b(\tilde{U}))\},$$

which is precisely the definition of the semi–infinite Deligne–Lusztig variety in [Lus79]. \hfill \Box

2.3. Affine Deligne–Lusztig varieties and covers. The affine Deligne-Lusztig set attached to $x \in \tilde{W}$ and $b \in \tilde{G}$ is the subset

$$X_w(b) = \{gI \in \mathcal{F} : \text{inv}_f(gI, b\sigma(g)I) = w\}$$

of $\mathcal{F}$. It was introduced by Rapoport in [Rap95]. In the equal characteristic case, it was shown to be locally closed in $\mathcal{F}$ by [HV11 Corollary 6.5]. In [Iva15a] covers of $X_w(b)$ were studied:

Definition 2.5. Let $b \in \tilde{G}$, let $f$ be a concave function on $\Phi \cup \{0\}$, such that $I^f \subseteq I$ is a $\sigma$-stable subgroup, and $x$ a $I^f$-double coset in $\tilde{G}$. Then we define

$$X^f_x(b) : = \{gI \in \mathcal{F}^f : \text{inv}_f(gI, b\sigma(g)I^f) = x\}.$$

Whenever $X^f_x(b)$ is a locally closed subset of $\mathcal{F}^f$ (see Proposition 2.11 below), we provide it with the reduced induced sub-Ind-scheme resp. Ind perfect algebraic sub-space structure, thus making it to reduced schemes resp. perfect schemes locally of finite type over $k$.

There is a natural $J_b(K)$-action by left multiplication on $X^f_x(b)$ for all $f$ and $x$. If $f \geq g$ and $x' \in D_{G,g}$ lies over $x \in D_{G,f}$, then the natural projection $\tilde{G}/I^g \rightarrow \tilde{G}/I^f$ restricts to a projection $X^f_x(b) \rightarrow X^f_{x'}(b)$. Concerning the right action, we have the following lemma.

Lemma 2.6. Let $f \geq g$ be two concave functions on $\Phi \cup \{0\}$, such that $I^g \subseteq I^f \subseteq I$, $I^g$ is normal in $I^f$, and $I^f, I^g$ are $\sigma$-stable. Let $x' \in D_{G,g}$ lie over $x \in D_{G,f}$ and let $b \in \tilde{G}$.

(i) Any $i \in I^f$ defines an $X^f_x(b)$-isomorphism $X^g_{x'}(b) \rightarrow X^g_{x'-1x'\sigma(i)}(b)$.

(ii) $X^g_{x'}(b)$ is an $(I^f/I^g)x'$-torsor over $X^f_x(b)$, where $(I^f/I^g)x' : = \{i \in I^f : i^{-1}x'\sigma(i) = x'\}/I^g$. 
Proof. (i) is a trivial computation. (ii) follows from (i).

2.4. Scheme structure. The goal of this paragraph is to endow the covers of affine Deligne-Lusztig varieties with a natural (Ind-)scheme resp. (Ind-)perfect scheme structure in some cases. It is enough to show that they are locally closed subsets of the Ind-schemes resp. Ind-perfect schemes $\mathcal{F}$. First we introduce a notation, which will be used in this section only. We write $\Phi := \Phi \cup \{0\}$. Further, let $\Phi_{\aff}$ be the (disjoint) union of $\Phi_{\aff}$ and the set of all pairs $(0, r)$ with $r \in \hat{\mathbb{R}}_{<\infty}$, for which the filtration step $\hat{U}_r/\hat{U}_{r+}$ is non-trivial. We extend the action of $\hat{W}$ on $\Phi, \Phi_{\aff}$ to $\hat{\Phi}, \hat{\Phi}_{\aff}$ by letting it act trivially on $0$ and all $(0, r)$.

For $\alpha \in \hat{\Phi}$ and any $r \in \hat{\mathbb{R}}_{<\infty}$, consider the $O$-group scheme $\hat{U}_{\alpha, r}$ from Section 2.1.3. If $r < s$ in $\hat{\mathbb{R}}_{<\infty}$, there is a unique morphism of group schemes $\hat{U}_{\alpha, s} \to \hat{U}_{\alpha, r}$ which induces the natural inclusion $\hat{U}_{\alpha, s} \hookrightarrow \hat{U}_{\alpha, r}$ on $O$-points. Let $L_{[r, s]}\hat{U}_{\alpha}$ be the fpqc-quotient sheaf

$$L_{[r, s]}\hat{U}_{\alpha} = L^+\hat{U}_{\alpha, s}/L^+\hat{U}_{\alpha, r},$$

It is represented by some finite-dimensional group scheme over $\bar{k}$.

**Lemma 2.7.** Let $f : \hat{\Phi} = \Phi \cup \{0\} \to \hat{\mathbb{R}}_{<\infty}$ be a concave function such that $I^f \subseteq I$ is a normal subgroup. Then there is a well-defined bijective map:

$$\prod_{\alpha \in \Phi \cup \{0\}} L_{[f_1(\alpha), f(\alpha)]}\hat{U}_{\alpha}(\bar{k}) \to I/I^f, \quad (a_\alpha)_{\alpha \in \Phi \cup \{0\}} \to \prod_\alpha \hat{a}_\alpha,$$

where $\hat{a}_\alpha$ is any lift of $a_\alpha$ to $\hat{U}_{\alpha, f_1(\alpha)}$ and the product can be taken in any order.

**Proof.** First, observe that the conclusion of [BT72, 6.4.48] also holds for the Iwahori subgroup, i.e., for the function $f_1$ (this follows from the Iwahori decomposition). Thus there is a bijection

$$\prod_{\alpha \in \Phi \cup \{0\}} L^+\hat{U}_{\alpha, f_1(\alpha)}(\bar{k}) \to I, \quad (a_\alpha)_{\alpha \in \Phi \cup \{0\}} \to \prod_\alpha a_\alpha,$$

given by multiplication in any order, and a similar statement for $I, f_1$ replaced by $I^f, f$. The lemma follows from these bijections by normality of $I^f$ in $I$.

Let $x \in \hat{W}$. We give an explicit parametrization of the set of double cosets $I^f \backslash IxI/I^f$ in certain cases. For reasons of simplicity (and as it does not lead to any confusion), we abuse the notation in the following few lemmas and write $x$ again for any lift of $x$ to $\hat{G}$. We say also that $(\alpha, m) \in \hat{\Phi}_{\aff}$ occurs in a subgroup $J$ of $\hat{G}$, if $\hat{U}_{\alpha, m}$ is contained in $J$. Then $(\alpha, m)$ occurs in $\hat{G}_J$ if and only if $m \geq f(\alpha)$. Let $\hat{\Phi}_{\aff}(J) \subseteq \hat{\Phi}_{\aff}$ denote the set of all pairs $(\alpha, m)$ occurring in $J$. If $J' \subseteq J$ is a normal subgroup, let $\hat{\Phi}_{\aff}(J/J') := \hat{\Phi}_{\aff}(J) \setminus \hat{\Phi}_{\aff}(J')$.

Let $f : \Phi \cup \{0\} \to \hat{\mathbb{R}}_{<\infty}$ be a concave function such that $I^f \subseteq I$ is a normal subgroup. Let $x \in \hat{W}$. We can divide the set of all affine roots $\hat{\Phi}_{\aff}(I/I^f)$ into three disjoint parts:

$$\hat{\Phi}_{\aff}(I/I^f) = A_x \cup B_x \cup C_x,$$

where

$$A_x = \{(\alpha, m) \in \hat{\Phi}_{\aff}(I/I^f) : x.(\alpha, m) \not\in \hat{\Phi}_{\aff}(I)\}$$

$$B_x = \{(\alpha, m) \in \hat{\Phi}_{\aff}(I/I^f) : x.(\alpha, m) \in \hat{\Phi}_{\aff}(I/I^f)\}$$

$$C_x = \{(\alpha, m) \in \hat{\Phi}_{\aff}(I/I^f) : x.(\alpha, m) \in \hat{\Phi}_{\aff}(I')\}.$$

Let $p : \hat{\Phi}_{\aff} \to \hat{\Phi}$ denote the natural projection mapping an affine root onto its vector part.

**Lemma 2.8.** Let $f : \Phi \cup \{0\} \to \hat{\mathbb{R}}_{<\infty}$ be a concave function such that $I^f \subseteq I$ is a normal subgroup. Let $x \in \hat{W}$. Assume that $p^{-1}(p(A_x)) = A_x$, and the same is true for
Then there is a well-defined bijective map:
\[
\prod_{\alpha \in \mathcal{P}(A_{-1})} L_{[f_1(a), f(a)]} U_{\alpha}(\tilde{k}) \times \prod_{\alpha \in \mathcal{P}(B_x)} L_{[f_1(a), f(a)]} U_{\alpha}(\tilde{k}) \times \prod_{\alpha \in \mathcal{P}(A_x)} L_{[f_1(a), f(a)]} U_{\alpha}(\tilde{k}) \to I^f / IxI / I^f
\]
given by \((a_\alpha)_{\alpha \in \mathcal{P}(A_{-1})}, (b_\alpha)_{\alpha \in \mathcal{P}(B_x)}, (a_\alpha)_{\alpha \in \mathcal{P}(A_x)}\) \mapsto \prod_{\alpha \in \mathcal{P}(A_{-1})} \tilde{a}_\alpha \cdot x \cdot \prod_{\alpha \in \mathcal{P}(B_x)} \tilde{b}_\alpha \cdot \prod_{\alpha \in \mathcal{P}(A_x)} \tilde{a}_\alpha,
where \(\tilde{a}_\alpha\) is any lift of \(a_\alpha\) to an element of \(U_{\alpha, f_1(a)}\).

Proof. That the claimed map is well-defined follows from Lemma 2.7. We have an obvious surjective map \(I^f / IxI / I^f \to I^f \setminus IxI / I^f\), given by \((iI^f, jI^f) \mapsto I^f ixjI^f\). Using Lemma 2.7 and the assumption in the proposition, we may write any element of the left \(I^f / IxI / I^f\) as the product \(a_{x-1}b_{x-1}c_{x-1}\), where \(a_{x-1} = \prod_{\alpha \in \mathcal{P}(A_{-1})} a_\alpha\), etc. Thus any element of \(I^f \setminus IxI / I^f\) may be written in the form
\[
I^f \tilde{a}_{x-1} \cdot b_{x-1} \cdot c_{x-1} \cdot x \cdot jI^f,
\]
for some \(j \in I\), where \(\tilde{\cdot}\) denote an (arbitrary) lift of an element to the root subgroup. Bringing \(b_{x-1}c_{x-1}\) to the right side of \(x\) changes it to \(x^{-1}b_{x-1}c_{x-1}x\), which is a product of elements of certain filtration steps of root subgroups, all of which lie in \(I\) by definition of \(B_{x-1}, C_{x-1}\). Thus we may eliminate \(b_{x-1}c_{x-1}\) from (2.3). Now, by Lemma 2.7, we may write any element of the right \(I^f / IxI / I^f\) as the product \(c_xb_xa_x\), with \(c_x = \prod_{\alpha \in \mathcal{P}(C_x)} c_\alpha\), etc. That is, any element of \(I^f \setminus IxI / I^f\) may be written as
\[
I^f \tilde{a}_{x-1} \cdot x \cdot \tilde{c}_x \tilde{b}_x \tilde{a}_x I^f,
\]
for some lifts \(\tilde{c}_x, \tilde{b}_x, \tilde{a}_x\) of \(c_x, b_x, a_x\). Bringing \(\tilde{c}_x\) to the left side of \(x\) in (2.4), makes it to \(x^{-1}\tilde{c}_x\), which is a product of elements of certain filtration steps of root subgroups, all of which lie in \(I^f\) by definition of \(C_x\). By normality of \(I^f\), we may eliminate \(\tilde{c}_x\) from the (2.4). It finally follows that we may write any element of \(I^f \setminus IxI / I^f\) as a product
\[
I^f \tilde{a}_{x-1} \cdot x \cdot \tilde{b}_x \tilde{a}_x I^f,
\]
with \(\tilde{a}_{x-1}, \tilde{b}_x, \tilde{a}_x\) as above. This shows the surjectivity of the map in the lemma. It remains to show injectivity. Suppose there are tuples \((a_{x-1}, b_x, a_x)\) and \((a'_{x-1}, b'_x, a'_x)\) giving the same double coset, i.e.,
\[
\tilde{a}_{x-1}x \tilde{b}_x \tilde{a}_x = i\tilde{a}_{x-1} \tilde{b}_x \tilde{a}_x \tilde{a}^j,
\]
for some \(i, j \in I^f\). This equation is equivalent to
\[
x^{-1}(\tilde{a}_{x-1})^{-1}i\tilde{a}_{x-1}x = \tilde{b}_x \tilde{a}_x j \tilde{a}_{x-1}^{-1} \tilde{b}_x^{-1} \tag{2.4.5}
\]
Here, the right hand side lies in \(I\), hence it follows that \((\tilde{a}_{x-1})^{-1}i\tilde{a}_{x-1} \in I \cap xI^{-1} \cdot x^{-1}\). Let us use Lemma 2.7 in the following form: any element of \(I^f / IxI / I^f\) may be written in a unique way as a product \(s_{x-1}r_x\) with \(s_{x-1} = \prod_{\alpha \in \mathcal{P}(A_{-1})} s_\alpha\) and \(r_x = \prod_{\alpha \in \mathcal{P}(B_{x-1} \cup C_{x-1})} r_\alpha\) with \(s_\alpha, r_\alpha \in L_{[f_1(a), f(a)]} U_{\alpha}(\tilde{k})\). As (by Definition of \(A_{x-1}, B_{x-1}, C_{x-1}\)) the affine roots in \(A_{x-1}\) are precisely those affine roots in \(\Phi_{\text{aff}}(I^f)\), which do not occur in \(I \cap xI^{-1}\), we see that the image of the composed map \(I \cap xI^{-1} \to I \to I^f / IxI / I^f\) is equal to the set of all elements of \(I^f / IxI / I^f\), which in the above decomposition have \(s_{x-1} = 1\). Now we have inside \(I^f / IxI / I^f\) (so in particular, the element \(i \in I^f\) can be ignored)
\[
a_{x-1} = a_x \cdot 1 = a'_{x-1} \cdot (a'_{x-1})^{-1}ia_{x-1},
\]
which gives two decompositions of the element \(a_{x-1} \in I^f / IxI / I^f\). By uniqueness of such a decomposition, we must have \(a'_{x-1} = a_{x-1}\). Now analogous computations (first done for \(a'_{x}, a_x\) and then for \(b'_{x}, b_x\) show that we also must have \(a'_{x} = a_{x}\) and \(b'_{x} = b_{x}\). This finishes the proof of injectivity. \(\square\)
Using the bijection in Lemma 2.8, we can endow $I^f \backslash I x I^f / I^f$ with the structure of a $k$-scheme. The $I^f$-torsor $\mathcal{F}^f \to \mathcal{F}$ can be trivialized over the Schubert cell $I x I^f / I^f (\cong A^f(x))$, hence a choice of any section $I x I^f / I^f$ together with the action of $I^f$ on the fibers of $I x I^f / I^f \to I x I^f$ gives the following parametrization of $I x I^f / I^f$ (the bijectivity on $k$-points is seen in the same straightforward way as in Lemma 2.8).

**Lemma 2.9.** Let $f: \Phi \cup \{0\} \to \tilde{\mathbb{R}}_{\leq 0}$ be a concave function such that $I^f \subseteq I$ is a normal subgroup. Let $x \in W$. Assume that $p^{-1}(p(A_{x-1})) = A_{x-1}$. Then there is an isomorphism of $k$-varieties,

$$\prod_{\alpha \in p(A_{x-1})} L_{(f, \alpha)}(A_{x-1}) \cup I^f \to I x I^f$$

given by $(a_{\alpha})_{\alpha \in p(A_{x-1})} \mapsto \prod_{\alpha \in p(A_{x-1})} \tilde{a}_{\alpha} \cdot x \cdot i I^f$, where $\tilde{a}_{\alpha}$ is any lift of $a_{\alpha}$ to an element of $U_{(f, \alpha)}$.

**Lemma 2.10.** Under the assumptions of Lemma 2.8, the natural projection $p: I x I^f / I^f \to I^f \backslash I x I^f / I^f$ is a geometric quotient in the sense of Mumford for the left multiplication action of $I^f$ on $I x I^f / I^f$. Here $I^f \backslash I x I^f / I^f$ is endowed with a structure of a $k$-scheme using the parametrization from Lemma 2.8.

**Proof.** The action of $I^f$ on $I x I^f / I^f$ factors through a finite-dimensional quotient (any subgroup $J \subseteq I^f \cap x I^f x^{-1}$, which is normal in $I^f$ acts trivially on $I x I^f / I^f$). Further, $p$ is surjective orbit map, $I^f \backslash I x I^f / I^f$ is normal and the connected components of $I x I^f / I^f$ are open. Thus by [Bor91, Proposition 6.6], it remains to show that $p$ is a separable morphism of varieties. But in terms of the parametrizations of Lemmas 2.8 and 2.9 it is given by $(a_{x-1}, i = c_x b_x a_x) \mapsto (a_{x-1}, b_x, a_x)$. \hfill \Box

**Proposition 2.11.** Let $f: \Phi \cup \{0\} \to \tilde{\mathbb{R}}_{\leq 0}$ be a concave function such that $I^f \subseteq I$ is a normal subgroup. Let $x$ be an $I^f$-double coset in $\tilde{G}$ with image $x$ in $\tilde{W}$. Assume that $p^{-1}(p(A_{x})) = A_{x}$, and the same is true for $B_x, C_x, A_{x-1}, B_{x-1}, C_{x-1}$, where $A, B, C$ are as in (2.2). Let $b \in \tilde{G}$. Then $X^f_x(b)$ is locally closed in $\mathcal{F}^f$.

**Proof.** By Lemma 2.10, the proposition is a special case of [Val15b, Proposition 2.4]. For convenience, we recall the proof. Let $\mathcal{X} \subseteq \tilde{G}$ be the maximal compact subgroup containing $I$, stabilizing the special vertex $x_0$ of the Bruhat-Tits building of $G$ over $\bar{K}$. By [HV11, Corollary 6.5] in the equal characteristic case, resp. by [Zhu14, Section 3.1] in the mixed characteristic case, the affine Deligne-Lusztig sets $X^f_\mu(b) := \{ g \in N \sigma(g) \in \mathcal{X} \otimes, \mathcal{X} \} \subseteq \tilde{G} / \mathcal{X}$ attached to cocharacters $\mu \in X_s(T)$ are locally closed in the affine Grassmannian $G / \mathcal{X}$. Now, any double coset $\mathcal{X} \otimes I^f$ is a disjoint union of finitely many $I$-double cosets, which implies that under the natural projection $\mathcal{F} \to \tilde{G} / K$, the preimage of $\mathcal{X} \otimes I^f$ in $\mathcal{F}$ decomposes as a disjoint union of finitely many $X^f_\mu(b)$’s. The condition for a point in the preimage of $\mathcal{X} \otimes I^f$ to lie in one of the $X^f_\mu(b)$ is locally closed, hence the Iwahori-level affine Deligne-Lusztig varieties $X^f_\mu(b)$ are locally closed.

Let $\tilde{X}$ be the preimage of $X^f_\mu(b)$ under $\mathcal{F}^f \to \mathcal{F}$. By [PR08, Theorem 1.4], the projection $\beta: LG \to \mathcal{F}^f$ admits sections locally for the étale topology. Let $U \to \tilde{X}$ be étale, such that there is a section $s: U \to \beta^{-1}(U)$ of $\beta$. Consider the composition

$$\psi: U \to \beta^{-1}(U) \times U \to \mathcal{F}^f,$$

where the first map is $g \mapsto (s(g^{-1}), b \sigma(g))$ and the second map is the restriction of the left multiplication action of $G$ on $\mathcal{F}^f$. As $U$ lies over $\tilde{X}$, this composed morphism factors through the inclusion $I x I^f / I^f \subseteq \mathcal{F}^f$. Let $p: I x I^f / I^f \to I^f \backslash I x I^f / I^f$ denote the quotient map, which is
3. Case $G = \text{GL}_n$, $b$ basic, $w$ Coxeter

Now we relate the Deligne–Lusztig sets $X_w^{(b)}(b)$ and their covers $X_w^{(U)}(b)$ to inverse limits of the covers of affine Deligne–Lusztig varieties $X_w(b)$ in the important special case, when $G = \text{GL}_n$, $b \in \tilde{G}$ is basic and $w$ is a special Coxeter element, and moreover, we give a complete and very simple description of these spaces using the isocrystal $(\mathcal{K}^n, b\sigma)$.

3.1. Notation. Fix an integer $n \geq 2$ and let $V_K = \mathcal{K}^n$ be the standard $n$-dimensional vector space over $K$. Set $V := V_K \otimes_K \tilde{K}$. We let now $G$ be the $K$-group $GL(V_K)$. Whenever we make calculations in $V$ or $G$, we work relative to the canonical basis of $V$, thus we may identify $\tilde{G} = \text{GL}_n(\tilde{K})$. Also, in the calculations we always assume that $T$ is the diagonal torus and $B$ is the Borel of upper triangular matrices and we identify $W_{\text{fin}}$ with the symmetric group on $n$ elements.

3.2. Basic $\sigma$-conjugacy classes. Isocrystals. Recall that an element $b \in \tilde{G}$ is called basic, if the corresponding Newton homomorphism $\nu_b$ factors through the center of $G$. Let $B(G)_b$ denote the set of basic $\sigma$-conjugacy classes in $G$. By [RR96], the Kottwitz map

$$\kappa_G = \text{ord} \circ \det : G \to \mathbb{Z}$$

induces a bijection $\kappa_G : B(G)_b \xrightarrow{\sim} \mathbb{Z}$. We fix now once for all a class $[b] \in B(G)_b$, satisfying $\kappa_G(b) \geq 0$. We may do so without loss of generality, as (affine) Deligne–Lusztig constructions do not change essentially, when $b$ is multiplied by a central element. Put $n' := \gcd(n, \kappa_G([b]))$. We may write $n = n'n_0$, $\kappa_G([b]) = n'k_0$ for some coprime integers $k_0, n_0$. For any representative $b \in [b]$, we have

$$J_b(K) \cong GL_{n'}(D_{k_0/n_0})$$

where $D_{k_0/n_0}$ denotes the central simple division algebra over $K$ with invariant $k_0/n_0 \in \mathbb{Q}/\mathbb{Z}$.

Recall that an isocrystal over $\bar{k}$ is a pair consisting of a finite-dimensional $\bar{k}$-vector space $W$ and a Frobenius-semilinear bijective endomorphism of $W$. For a representative $b$ of $[b]$, let $F$ denote the $\sigma$-linear operator $b\sigma$ on $V$. We denote by $V_b$ the $\bar{k}$-isocrystal $(V, F)$. Note that the isomorphy type of $V_b$ only depends on $[b]$, not on $b$. Further, $V_b$ is isomorphic to the direct sum of $n'$ copies of the simple isocrystal with slope $k_0/n_0$. We note that the group $J_b(K)$ from $[3.1]$ is the group of automorphisms of $V_b$. It can be seen as the group of units of the semi-simple algebra of endomorphisms of $V_b$.

3.3. The admissible subset of $V_b$ and $F$-cyclic lattices. We attach to $V_b$ the following subset of the underlying $\tilde{K}$-vector space $V$:

$$V_b^{\text{adm}} := \{\text{the } n \text{ vectors } \{F^i(x)\}_{i=0}^{n-1} \text{ span } V\}$$

Up to isomorphism, $V_b^{\text{adm}}$ only depends on the $\sigma$-conjugacy class $[b]$ of $b$. Elements $x \in V_b^{\text{adm}}$ will be called $b$-admissible. Note that $x \in V$ lies in $V_b^{\text{adm}}$ if and only if the $O$-submodule of $V$ generated by $x, F(x), \ldots, F^{n-1}(x)$ is an $O$-lattice. We denote this lattice by $\mathcal{L}_b(x)$ and call it an $F$-cyclic lattice with generator $x$.

Note that if $b' = g^{-1}b\sigma(g)$, then $g\mathcal{L}_b(x) = \mathcal{L}_b(gx)$. In particular, if $g \in J_b(K)$, then $\mathcal{L}_b(gx) = g\mathcal{L}_b(x)$. With other words, if $x \in V_b^{\text{adm}}$, then $gx \in V_b^{\text{adm}}$. Hence there is a natural action of $J_b(K)$ on $V_b^{\text{adm}}$. 

a geometric quotient by Lemma 2.10. The composition $p \circ \psi$ is independent of the choice of the section $s$. It sends a $\bar{k}$-point $gI^f$ to the double coset $I^f g^{-1}b\sigma(g)I^f$. Thus étale locally $X'_{\tilde{f}}(b)$ is just the preimage of the point $x$ point under $p \circ \psi$. This finishes the proof. 

□
We have the following useful lemma, which essentially follows from the properties of the Newton-polygon. Its simple proof was explained to the authors by Viehmann.

**Lemma 3.1.** An $F$-cyclic lattice in $V$ is $F$-stable, i.e. for any $x \in V^{adm}_b$, there exist unique elements $\lambda_i \in O$ such that $F^n(v) = \sum_{i=0}^{n-1} \lambda_i F^{i}(v)$. Moreover, ord($\lambda_0$) = $\kappa_G(b)$.

**Proof.** We first prove (a). The Newton polygon of $V_b$ is the straight line segment connecting the points $(0,0)$ and $(n, \kappa_G(b))$ in the plane. Now, let $K[F]$ be the non-commutative ring, defined by the relation $aF = F\sigma(a)$. A vector $v \in V$ is called cyclic, if $V$ is generated by $v$ as a $K[F]$-module. Then the Newton polygon of the characteristic polynomial of $v$ (which is an element of $K[F]$) is equal to the Newton polygon of $V_b$ (see e.g. [Bea09]). Observe that any $v \in V^{adm}_b$ is cyclic. Then the point $(i, \text{ord}(a_i))$ in the plane, where $a_i$ is the coefficient of $F^{n-i}$ in the characteristic polynomial lies over that Newton polygon. This simply means $\text{ord}(a_i) \geq \frac{i \kappa_G(b)}{n} \geq 0$, as $\kappa_G(b) \geq 0$. Hence $F^n(v) = \sum_{i=1}^{n} a_i F^{n-i}(v)$ lies in the $O$-lattice generated by $v, Fv, \ldots, F^{n-1}v$. This proves the first assertion. The second statement follows as $(n, \text{ord}(a_n))$ has necessarily to be the rightmost vertex of the Newton polygon, which is $(n, \kappa_G(b))$.

We have two explicit examples of $V^{adm}_b$. For $b = 1$, the set $V^{adm}_1$ is just the Drinfeld upper halfspace. As another example, if $\kappa_G(b)$ is coprime to $n$, then $V^{adm}_b = V \setminus \{0\}$ (as $V_b$ has no proper non-trivial sub-isocrystals).

## 3.4. Comparison and uniformization of DLV and ADLV.

To state our main result, we introduce the following notation.

- $I^n$ (with $m \geq 0$) denotes the preimage under the projection $G(O) \to G(O/\varpi^{m+1}O)$, of all upper triangular matrices in $G(O/\varpi^{m+1}O)$ whose entries over the main diagonal lie in $\varpi^{m}O/\varpi^{m+1}O$.

- $I^n$ (with $m \geq 0$) denotes the subgroup of $I^n$ consisting of all elements whose diagonal entries are congruent 1 modulo $\varpi^{m+1}$

- $X^{m}_x(b)$ (resp. $\hat{X}^{m}_x(b)$) denote an affine Deligne–Lusztig variety of level $I^n$ (resp. $\hat{I}^n$)

- For $r \geq 0$ and $x \in V^{adm}_b$, let $g_{b,r}(x) \in G$ denote the matrix whose $i$-th column is $\varpi^{r(i-1)}F^{r-1}(x)$. Set $g_b(x) := g_{b,0}(x)$.

- For $r, m \geq 0$, define the equivalence relations $\sim_{b,m,r}$ and $\sim_{b,m,r}$ on $V^{adm}_b$ by

\[
x \sim_{b,m,r} y \in V^{adm}_b \iff y \in g_{b,r}(x) \cdot (O^{x}p^{m+1} \ldots p^{m+1})^T,
\]

\[
x \sim_{b,m,r} y \in V^{adm}_b \iff y \in g_{b,r}(x) \cdot (1 + p^{m+1} \ldots p^{m+1})^T.
\]

- For $r \geq 0$, set

\[
\hat{\omega}_r = \left(\begin{array}{cccc}
\varpi^{-r} & \varpi^{-r} & \cdots & \varpi^{-r} \\
\varpi^{-r} & \varpi^{-r} & \cdots & \varpi^{-r} \\
\vdots & \vdots & \ddots & \vdots \\
\varpi^{-r} & \varpi^{-r} & \cdots & \varpi^{-r}
\end{array}\right) \in \hat{G},
\]

and denote again by $\hat{\omega}_r$ the image of $\hat{\omega}_r$ in all the sets $I^n\hat{G}/I^n$ and $\hat{I}^n\hat{G}/\hat{I}^n$ for $m \geq 0$.

- Let $w$ denote the image of $\hat{\omega}_r$ in $W_{\text{fin}} \cong S_n$. It is the cycle $(1,2,\ldots,n)$ and does not depend on $r$.
Remark 3.2. The image of \( \tilde{w}_r \) in \( \widetilde{W} \) satisfies the assumptions of Proposition 2.11, hence \( X^m_{\tilde{w}_r}(b) \) resp. \( \tilde{X}^m_{\tilde{w}_r}(b) \) are locally closed inside \( \tilde{G}/I^m \) resp. \( \tilde{G}/I^m \), and hence we may (and do) endow them with the natural sub-Ind-scheme resp. sub-Ind perfect scheme structure. Moreover, as the affine Deligne-Lusztig varieties of Iwahori-level are locally of finite type and as the fibers of \( \mathcal{F}^m \to \mathcal{F} \) are of finite type, it follows that the varieties \( \tilde{X}^m_{\tilde{w}_r}(b) \) are locally of finite type.

The following theorem (and especially its proof) is very much inspired by [DL76, Section 2.2].

**Theorem 3.3.** Let \( b, w, \tilde{w}_r \) be as above.

(i) There is a commutative diagram (of sets)

\[
\begin{array}{c}
\{ x \in V_b^{\text{adm}} : \det(g_b(x)) \in K^\times \} \sim \to X^U_{w_0}(b) \\
V_b^{\text{adm}}/\tilde{K}^\times \sim \to X^U_w(b)
\end{array}
\]

in which horizontal arrows are isomorphisms. All maps are \( J_b(K) \)-equivariant.

(ii) Assume \( r \geq m \geq 0 \). There is a commutative diagram (of sets)

\[
\begin{array}{c}
\{ x \in V_b^{\text{adm}} : \det(g_{b,r}(x)) \in K^\times \} \sim_b,m,r \to X^m_{w_0}(b)(\bar{k}) \\
V_b^{\text{adm}}/\sim_b,m,r \sim \to X^m_w(b)(\bar{k})
\end{array}
\]

in which horizontal arrows are isomorphisms. All maps are \( J_b(K) \)-equivariant.

Before proving the theorem, we need some preparations. Observe that by Lemmas 2.3 and 2.6 in the proof of Theorem 3.3 we may replace \( b \) by an \( \sigma \)-conjugate element of \( \tilde{G} \).

**Lemma 3.4.** Let \( r > 0 \). Let \( x, y \in V_b^{\text{adm}} \). Then

\[
\begin{align*}
x \sim_{b,m,r} y & \iff g_{b,r}(x)I^m = g_{b,r}(y)I^m, \quad (3.2) \\
x \sim_{b,m,r} y & \iff g_{b,r}(x)I^m = g_{b,r}(y)I^m. \quad (3.3)
\end{align*}
\]

**Proof.** Indeed, \( g_{b,r}(y) \in g_{b,r}(x)I^m \) is equivalent to

\[
y \in xO^\times + \omega^{m+1+r}F(x)O + \cdots + \omega^{m+1+r(n-1)}F^{n-1}(x)O
\]

\[
\omega^rF(y) \in \omega^m xO + \omega^rF(x)O^\times + \omega^{m+1+2r}F^2(x)O + \cdots + \omega^{m+r(n-1)}F^{n-1}(x)O
\]

\[
\vdots
\]

\[
\omega^{r(n-1)}F^{n-1}(y) \in \omega^m xO + \cdots + \omega^{m+r(n-2)}F^{n-2}(x)O + \omega^{r(n-1)}F^{n-1}(x)O^\times
\]

By definition, the first equation is equivalent to \( x \sim_{b,m,r} y \). But once the first equation holds, then the \((i+1)\)-th equation must also hold by applying \( \omega^rF^i \) to the first equation. (Recall from Lemma 3.1 that \( F^n(x) \) can be written as a linear combination of \( x, F(x), \ldots, F^{n-1}(x) \) with coefficients in \( O \).) Hence the Equation (3.2) follows, and a similar proof gives Equation (3.3). \( \square \)

**Lemma 3.5.** Let \( r \geq 0 \) and \( x \in V_b^{\text{adm}} \). Then

\[
F(g_{b,r}(x)) = g_{b,r}(x)\tilde{w}_rA,
\]
where $A \in \tilde{G}$ is a matrix, which can differ from the identity matrix only in the last column. Moreover, the lower right entry of $A$ lies in $\mathcal{O}^\times$, and if $r \geq m \geq 0$, then $A \in F^m$.

**Proof.** By definition, we have

\[
F(g_{b,r}(x)) = \left( F(x) \; \varpi^r F^2(x) \; \ldots \; \varpi^r(n-2) F^{n-1}(x) \; \varpi^r(n-1) F^n(x) \right),
\]

\[
g_{b,r}(x) \tilde{w}_r = \left( F(x) \; \varpi^r F^2(x) \; \ldots \; \varpi^r(n-2) F^{n-1}(x) \; \varpi^r(n-1)+\kappa_G(b) x \right),
\]

As the first $n-1$ columns of these matrices coincide, it follows that $A$ can at most differ from the identity matrix in the last column. By Lemma [3.1] we may write

\[
F^n(x) = \sum_{i=0}^{n-1} \alpha_i \cdot F^i(x)
\]

\[
= \frac{\alpha_0}{\varpi^{r(n-1)+\kappa_G(b)} x} \cdot \varpi^{r(n-1)+\kappa_G(b)} x + \sum_{i=1}^{n-1} \frac{\alpha_i}{\varpi^{r(i-1)}} \cdot \varpi^{r(i-1)} F^i(x),
\]

where $\alpha_0, \ldots, \alpha_{n-1} \in \mathcal{O}$ and $\text{ord}(\alpha_0) = \kappa_G(b)$. By construction, the last column of $A$ is

\[
\left( \varpi^{r(n-1)} \alpha_1, \varpi^{r(n-2)} \alpha_2, \varpi^{r(n-3)} \alpha_3, \ldots, \varpi^{r} \alpha_{n-1}, \frac{\alpha_0}{\varpi^{\kappa_G(b)}} \right)^\top.
\]

We then see that the lower right entry of $A$ is $\frac{\alpha_0}{\varpi^{\kappa_G(b)}} \in \mathcal{O}^\times$ and that if $r \geq m+1$, then all the entries above $\frac{\alpha_0}{\varpi^{\kappa_G(b)}}$ lie in $\mathcal{O}^{m+1}$ and $A \in F^m$. \hfill \Box

**Proof of Theorem [3.3] (i):** Lemma [3.5] for $r = 0$ implies the existence of the map

\[
V_b^{\text{adm}} \to X_w^{(B)}(b), \quad x \mapsto g_b(x)\tilde{B}.
\]

We claim this map is surjective. Let $g\tilde{B} \in X_w^{(B)}(b)$, i.e., $g^{-1} F(g) \in \tilde{B} \tilde{w}_0 \tilde{B}$. Replacing $g$ by another representative in $g\tilde{B}$ if necessary, we may assume that $F(g) \in g\tilde{w}_0 \tilde{B}$. Moreover, this assumption does not change, whenever we replace $g$ by another representative $g' = gc$ with $c \in \hat{B} \cap {}^b \hat{B}$ (here $^b \hat{B} = b\hat{B}b^{-1}$). A direct computation shows that replacing $g$ by $gc$ for an appropriate $c \in \hat{B} \cap {}^\circ \tilde{w}_0 \tilde{B}$, we find a representative $g$ of $g\tilde{B}$ with columns $g_1, g_2, \ldots, g_n$ satisfying $g_{i+1} = F(g_i)$ for $i = 1, \ldots, n-1$. This means precisely $g = g_0(x)$. All this shows the surjectivity claim. For $x, y \in V_b^{\text{adm}}$, one has $g_b(x)\tilde{B} = g_b(y)\tilde{B}$ if and only if $x, y$ differ by a constant in $\tilde{K}^\times$. This shows the lower horizontal isomorphism in part (i) of the theorem.

We construct now the upper isomorphism. We may write an element of $\tilde{G}/\tilde{U}$ lying over $g_b(x)\tilde{B} \in X_w^{(B)}(b)$ as $\tilde{g}\tilde{U} = g_b(x)t\tilde{U}$ for some $t \in \tilde{T}$. The $\sigma$-linearity of $F$ implies $F(g_b(x)t) = F(g_b(x))\sigma(t)$. Using Lemma [3.5] (and the notation from there) we see that

\[
\tilde{g}^{-1} F(\tilde{g}) = t^{-1} g_0(x)^{-1} F(g_b(x))\sigma(t) = t^{-1} \tilde{w}_0 A \sigma(t) = \tilde{w}_0 A (\tilde{w}_0^{-1} t \tilde{w}_0) \sigma(t),
\]

the last equation being true as $A \in \tilde{U}$. Hence a necessary and sufficient condition for $g_b(x)t\tilde{U}$ to lie in $X_w^{(U)}(b)$ is $(\tilde{w}_0^{-1} t \tilde{w}_0) \sigma(t) = 1$. Writing $t_0, t_1, \ldots, t_{n-1} \in \tilde{K}^\times$ for the diagonal entries of $t$, we deduce the necessary condition $t_{i+1} = \sigma(t_i)$ for $0 \leq i \leq n-2$. We may assume this condition. In particular, it implies that $g_b(x)t = g_b(x)t_0$. With other words, replacing $x$ by $xt_0$, we may assume that $\tilde{g} = g_b(x)$. It remains to determine all $x \in V_b^{\text{adm}}$, for which $g_b(x)\tilde{U} \in X_w^{(U)}(b)$, i.e., $g_b(x)^{-1} F(g_b(x)) \in \tilde{U} \tilde{w}_0 \tilde{U}$. Comparing the determinants on both sides we deduce $\det(g_b(x)) \in K^\times$ as a necessary condition. Assume this holds. With notations as in Lemma [3.3] we deduce $\det(A) = 1$. Moreover, Lemma [3.5] also shows that $\det(A) = 1$ is equivalent to $A \in \tilde{U}$. All this shows the upper isomorphism in part (i). The commutativity of the diagram and $J_b(K)$-equivariance of the involved maps are clear from the construction.
(ii): Lemma 3.5 for \( r > m \geq 0 \) implies the existence of the map
\[
V_{b}^{\text{adm}} \to X_{w}^{m}(b), \quad x \mapsto g_{b,r}(x)I^{m}.
\]
We claim it is surjective. Let \( g^{m} \in X_{w}^{m}(b) \), i.e., \( g^{-1}F(g) \in I^{m}w_{r}I^{m} \). Replacing \( g \) by another representative of \( gI^{m} \) if necessary, we may assume that \( F(g) \in gw_{r}I^{m} \). Moreover, this assumption does not change, whenever we replace \( g \) by another representative \( g' = gj \) with \( j \in I^{m} \cap gw_{r}I^{m} \). In the rest of the proof, we call such transformations allowed. We compute
\[
I^{m} \cap w_{r}I^{m} = \begin{pmatrix}
\mathcal{O}^{\times} & p^{m} & \cdots & p^{m} \\
p^{m} & \mathcal{O}^{\times} & p^{m+1} & \cdots & p^{m+1} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
p^{m} & \cdots & p^{m} & \mathcal{O}^{\times} & p^{m+1} \\
p^{m} & \cdots & \cdots & \cdots & \mathcal{O}^{\times}
\end{pmatrix}
\]
(on the main diagonal entries can lie in \( \mathcal{O}^{\times} \), under the main diagonal in \( p^{m} \), in the first row, beginning from the second entry, in \( p^{m} \), and above the main diagonal, except for the first row, in \( p^{m+1} \)). Let \( g_{1}, \ldots, g_{n} \) denote the columns of \( g \), seen as elements of \( V \). Then \( gw_{r} \in F(g)I^{m} \) is equivalent to the following \( n \) equations:
\[
g_{2} \in \sigma^{r}F(g_{1})\mathcal{O}^{\times} + \sigma^{r+m}F(g_{2})\mathcal{O} + \cdots + \sigma^{r+m}F(g_{n})\mathcal{O} \\
g_{3} \in \sigma^{r+m+1}F(g_{1})\mathcal{O} + \sigma^{r}F(g_{2})\mathcal{O}^{\times} + \sigma^{r+m}F(g_{3})\mathcal{O} + \cdots + \sigma^{r+m}F(g_{n})\mathcal{O} \\
\vdots \\
g_{n} \in \sigma^{r+m}F(g_{1})\mathcal{O} + \cdots + \sigma^{r+m}F(g_{n-2})\mathcal{O} + \sigma^{r}F(g_{n-1})\mathcal{O}^{\times} + \sigma^{r+m}F(g_{n})\mathcal{O} \\
\sigma^{r^{n}m+1}g_{1} \in \sigma^{r+2m+1}F(g_{1})\mathcal{O} + \cdots + \sigma^{r+2m+1}F(g_{n-1})\mathcal{O} + \sigma^{r+m}F(g_{n})\mathcal{O}^{\times}.
\]
A linear algebra exercise shows that after some allowed transformations these equations can be rewritten as
\[
g_{2} \in \sigma^{r}F(g_{1})\mathcal{O}^{\times} \\
g_{3} \in \sigma^{r}F(g_{2})\mathcal{O}^{\times} \\
\vdots \\
g_{n} \in \sigma^{r}F(g_{n-1})\mathcal{O}^{\times} \\
I^{r(n-1)}g_{1} \in \sigma^{m+1}F(g_{1})\mathcal{O} + \cdots + \sigma^{m+1}F(g_{n-1})\mathcal{O} + F(g_{n})\mathcal{O}^{\times}.
\]
This shows that \( g = g_{b,r}(g_{1}) \), and hence the claimed surjectivity. Lemma 3.4 shows that the lower map in part (ii) is an isomorphism. Exactly as in the proof of (i), one shows that the claim of (ii) is true if one replaces the upper left entry by \( \left\{ x \in V_{b}^{\text{adm}} \mid \text{det}(g_{b,r}(x)) \mod \sigma^{m+1} \right\} \). As \( x \sim_{b,m,r} xu \) for any \( u \in 1 + p^{m+1} \), the original claim of (ii) follows from this modified claim along with the surjectivity of the map \( 1 + p^{m+1} \to 1 + p^{m+1}, \ u \mapsto \prod_{i=0}^{r-1} \sigma^{i}(u) \). □

**Corollary 3.6.** Let \( r > m \geq 0 \). Then we have maps
\[
X_{w_{r+1}}^{m}(b) \to X_{w_{r}}^{m}(b), \quad g_{b,r+1}(x)I^{m} \mapsto g_{b,r}(x)I^{m} \\
X_{w_{r+1}}^{m+1}(b) \to X_{w_{r}}^{m}(b), \quad g_{b,r+1}(x)I^{m+1} \mapsto g_{b,r}(x)I^{m} \\
\]
so that we have a commutative diagram

\[ \begin{array}{c}
\cdots \\
X^2_{w_3}(b) \leftarrow \\
X^1_{w_2}(b) \leftarrow X^1_{w_3}(b) \leftarrow \\
X^0_{w_1}(b) \leftarrow X^0_{w_2}(b) \leftarrow X^0_{w_3}(b) \leftarrow \\
\cdots
\end{array} \]

and analogously for \( \hat{X}^m_{w_r}(b) \).

**Definition 3.7.** Define the following inverse limits of schemes resp. perfect schemes over \( k \):

\[
X^\infty_w(b) := \lim_{r \to m} X^m_{w_r}(b)
\]

\[
\hat{X}^\infty_w(b) := \lim_{r \to m} \hat{X}^m_{w_r}(b)
\]

**Corollary 3.8.**

(i) There are natural \( J_b(K) \)-equivariant bijections

\[
X^\infty_w(b)(\bar{k}) = V^\text{adm}_b/O^\times
\]

\[
\hat{X}^\infty_w(b)(\bar{k}) = \{ x \in V^\text{adm}_b : \det(g_b(x)) \in K^\times \}
\]

In particular, this endows the sets on the right with the structure of pro-\( k \)-schemes resp. pro-perfect \( k \)-schemes.

(ii) There is a commutative diagram with \( J_b(K) \)-equivariant maps:

\[
X^{(U)}_w(b) \xleftarrow{\sim} \{ x \in V^\text{adm}_b : \det(g_b(x)) \in K^\times \} \xrightarrow{\sim} \hat{X}^\infty_w(b)
\]

\[
X^{(B)}_w(b) \xleftarrow{\sim} V^\text{adm}_b/K^\times \xleftarrow{\sim} V^\text{adm}_b/O^\times \xrightarrow{\sim} X^\infty_w(b)
\]

In particular, this endows \( X^{(U)}_w(b), X^{(B)}_w(b) \) with natural pro-\( k \)-scheme resp. pro-perfect \( k \)-scheme.

**Proof.** This follows immediately from the theorem. \( \square \)

3.5. **Connected components.** This section completely relies on the work of Viehmann \cite{Vie08}, where connected components of moduli spaces of \( p \)-divisible groups, or equivalently, affine Deligne-Lusztig varieties of maximal compact level were determined. For \( r > m \geq 0 \), \( X^m_{w_r}(b) \) is a scheme resp. perfect scheme locally of finite type, but not of finite type. We investigate its decomposition into a disjoint union of connected varieties, all of which are of finite type and describe the connected components slightly more explicitly.

By Lemma \[2.3\] and its obvious counterpart for affine Deligne-Lusztig varieties, we may choose \( b \) to be a special representative of the class \([b]\) defined in the following way (see also \cite[Section 4]{Vie08}). Let

\[
V_b = \bigoplus_{i=1}^{n'} V_{b,i}
\]
be the decomposition of $V_b$ into simple isocrystals. Recall that $n' = \gcd(n, \kappa_G([b]))$, $n = n'n_0$, $\kappa_G([b]) = n'k_0$ for some coprime integers $k_0, n_0$. Choose $c, d \in \mathbb{Z}$ with
\[ cn_0 + dk_0 = 1. \]
For $i = 1, \ldots, n'$, let $e_{i,0} \in V_{b,i} \setminus \{0\}$ be such that
\[ F^{n_0} e_{i,0} = \varpi^{k_0} e_{i,0}. \tag{3.4} \]
For $l \in \mathbb{Z}_{\geq 0}$, let $e_{i,l}$ be such that
\[ e_{i,l} = (\varpi^c F^d)^l (e_{i,0}). \tag{3.5} \]
In particular,
\[ e_{i,l+n_0} = (\varpi^c F^d)^{l+n_0}(e_{i,0}) = (\varpi^c F^d)^l \varpi^{c_0 n_0} F^{d n_0}(e_{i,0}) \]
\[ = (\varpi^c F^d)^l \varpi^{1-dk_0} F^{d n_0}(e_{i,0}) = \varpi(\varpi^c F^d)^l (\varpi^{-k_0} F^{n_0})^d(e_{i,0}) = \varpi e_{i,l}. \]
Then the set $\{e_{i,l}: i = 1, 2, \ldots, n'; l = 0, 1, \ldots, n_0 - 1\}$ is a basis of $V$. We order the basis elements $e_{i,l}$ lexicographically (first with respect to $i$, then with respect to $l$). Recalling that $F = b\sigma \in \text{End}(V)$, we see that Equations (3.4) and (3.5) defines a unique representative $b$ (up to a multiple of $\pi$ viewed as a central element of $\text{GL}_{n'}(\bar{K})$ of the $\sigma$-conjugacy class $[b]$. In particular, it is easy to check that $b$ is independent of the choice of $c, d$.

Remark 3.9. When $\kappa_G([b])$ is coprime to $n$, the representative $b$ is a length-0 element of the extended affine Weyl group $W$ and therefore is a standard representative in the sense of [GHKR10 Section 7.2]. In general, $b$ is block-diagonal with blocks consisting of the standard representative of size $n' \times n'$ and determinant $k_0$.

Example 3.10. Here are the standard representatives $b$ associated to the inner forms of $\text{GL}_6(K)$, where we take the convention that $b$ acts on $V$ via multiplication on column vectors.

\[
\begin{align*}
\kappa &= 0, & J_0(K) &\cong \text{GL}_{n}(K), & b &= \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}; \\
\kappa &= 1, & J_0(K) &\cong \text{GL}_1(D_{1/6}), & b &= \begin{pmatrix} 1 & \varpi & \varpi & \varpi & \varpi & \varpi \\
\varpi & 1 & \varpi & \varpi & \varpi & \varpi \\
\varpi & \varpi & 1 & \varpi & \varpi & \varpi \\
\varpi & \varpi & \varpi & 1 & \varpi & \varpi \\
\varpi & \varpi & \varpi & \varpi & 1 & \varpi \\
\varpi & \varpi & \varpi & \varpi & \varpi & 1 \end{pmatrix}; \\
\kappa &= 2, & J_0(K) &\cong \text{GL}_2(D_{1/3}), & b &= \begin{pmatrix} 1 & \varpi & \varpi & \varpi & \varpi & \varpi \\
\varpi & 1 & \varpi & \varpi & \varpi & \varpi \\
\varpi & \varpi & 1 & \varpi & \varpi & \varpi \\
\varpi & \varpi & \varpi & 1 & \varpi & \varpi \\
\varpi & \varpi & \varpi & \varpi & 1 & \varpi \\
\varpi & \varpi & \varpi & \varpi & \varpi & 1 \end{pmatrix}; \\
\kappa &= 3, & J_0(K) &\cong \text{GL}_3(D_{1/2}), & b &= \begin{pmatrix} 1 & \varpi & \varpi & \varpi & \varpi & \varpi \\
\varpi & 1 & \varpi & \varpi & \varpi & \varpi \\
\varpi & \varpi & 1 & \varpi & \varpi & \varpi \\
\varpi & \varpi & \varpi & 1 & \varpi & \varpi \\
\varpi & \varpi & \varpi & \varpi & 1 & \varpi \\
\varpi & \varpi & \varpi & \varpi & \varpi & 1 \end{pmatrix}; \\
\kappa &= 4, & J_0(K) &\cong \text{GL}_2(D_{2/3}), & b &= \begin{pmatrix} 1 & \varpi & \varpi & \varpi & \varpi & \varpi \\
\varpi & 1 & \varpi & \varpi & \varpi & \varpi \\
\varpi & \varpi & 1 & \varpi & \varpi & \varpi \\
\varpi & \varpi & \varpi & 1 & \varpi & \varpi \\
\varpi & \varpi & \varpi & \varpi & 1 & \varpi \\
\varpi & \varpi & \varpi & \varpi & \varpi & 1 \end{pmatrix}; \\
\kappa &= 5, & J_0(K) &\cong \text{GL}_1(D_{5/6}), & b &= \begin{pmatrix} 1 & \varpi & \varpi & \varpi & \varpi & \varpi \\
\varpi & 1 & \varpi & \varpi & \varpi & \varpi \\
\varpi & \varpi & 1 & \varpi & \varpi & \varpi \\
\varpi & \varpi & \varpi & 1 & \varpi & \varpi \\
\varpi & \varpi & \varpi & \varpi & 1 & \varpi \\
\varpi & \varpi & \varpi & \varpi & \varpi & 1 \end{pmatrix}.
\end{align*}
\]

Let $\mathcal{O}_0$ denote the $\mathcal{O}$-lattice in $V$, generated by $e_{i,l}$ with $l \geq 0$. By [Vie08 Remark 4.3(ii)], $J_0(K) \cap \text{Stab}(\mathcal{O}_0)$ is a maximal compact subgroup of $J_0(K)$ (the intersection taken inside $\tilde{G}$). For $a \in \bar{k}$, let $[a]$ denote $a$ if $F$ has equal characteristics and the Teichmüller lift of $a$ otherwise.
Proposition 3.11 (cf. [Vie08 Section 4]). Let \( r > m \geq 0 \) and assume \( \kappa_G(b) \geq 0 \). Let \( b \) be a special representative of \([b]\) and \( \mathcal{L}_0 \) as above. Then, scheme-theoretically,

\[
X_{\tilde{w}_r}(b) = \prod_{g \in J_0(K)/(J_0(K) \cap \text{Stab}(\mathcal{L}_0))} g \cdot \mathcal{L}_{adm}^{0,b} / \sim_{b,r,m},
\]

where \( \mathcal{L}_{adm}^{0,b} / \sim_{b,r,m} \) is connected and

\[
\mathcal{L}_{adm}^{0,b} := \left\{ v = \sum_{i=1}^{n'} \sum_{l \geq 0} [a_{i,l}] e_{i,l} \in \mathcal{L}_0 : \begin{array}{l}
a_{i,0} \text{ for } 1 \leq i \leq n' \text{ are linearly independent over } \mathbb{F}_{q^0}
\end{array} \right\}.
\]

Proof. As \( \varpi \in \tilde{G} \) is central, we have \( X_{\tilde{w}_r}(b) = X_{\tilde{w}_r \varpi}(b \varpi^r) \) (as locally closed subsets of \( \tilde{G}/I^m \)), so we may and do replace \( b, \tilde{w}_r \) in this proof by \( b \varpi^r, \tilde{w}_r \varpi^r \). We do so to have a better overlap with the setup in [Vie08 Section 4]. Note that the new \( b \) is still a standard representative of its basic \( \sigma \)-conjugacy class, whose image under the Kottwitz map is \( \kappa_G(b) + rn \). Let \( \mu \) be the image of \( \tilde{w}_r \) under \( \tilde{W} = I/\tilde{G}/I \to \text{Stab}(\mathcal{L}_0) \). \( \tilde{G}/\text{Stab}(\mathcal{L}_0) \) = \( X_r(T) \). Then \( \mu = (0,0,\ldots,0,\kappa_G(b) + nr) \), which differs only by a scalar multiple from a minuscule cocharacter. We consider the natural projection onto the \( \text{Stab}(\mathcal{L}_0) \)-level affine Deligne-Lusztig variety:

\[
V_{b}^{adm} / \sim_{b,m,r} \cong X_{\tilde{w}_r}(b) \to X_{\tilde{w}_r}^{\text{Stab}(\mathcal{L}_0)}(b) \subseteq \tilde{G}/\text{Stab}(\mathcal{L}_0).
\]

Points of \( X_{\tilde{w}_r}^{\text{Stab}(\mathcal{L}_0)}(b) \) can be interpreted as \( F \)-cyclic lattices in \( V \). The lattice corresponding to \( v \in V_{b}^{adm} \) is the \( F \)-stable lattice \( \mathcal{L}(v) \subseteq V \), generated by \( v, F(v), \ldots, F^{n-1}(v) \) (here we already profit from our rescaling: \( \mathcal{L}(v) \) is only \( F \)-stable with respect to the new \( F = b \sigma \), not the old one. Working with the old \( b \), we would be forced to work with \( \varpi F \) instead of \( F \), which seems unnatural).

Let \( \pi_1, \sigma_1 : V \to V \) be the additive maps defined by \( \pi_1 := \varpi^c F^d \) and \( \sigma_1 := \varpi^{-k_0} F^{m_0} \) (just as the \( \pi_j \)'s and \( \sigma_j \)'s in [Vie08 Section 4.1]). To an \( F \)-stable lattice \( \mathcal{L} \) we attach the lattice

\[
P(\mathcal{L}(v)) := \mathcal{L} + \sigma_1(\mathcal{L}) + \sigma_1^2(\mathcal{L}) + \ldots
\]

This lattice is the smallest one containing \( \mathcal{L} \) and stable under \( F, \pi_1, \sigma_1 \), which coincide with the notion of the lattice \( P(\cdot) \) introduced in [Vie08] before Lemma 4.10. Let \( v \in V_{b}^{adm} \). By [Vie08 Lemmas 4.8, 4.10], there exists some \( g \in J_0(K) \) such that \( P(\mathcal{L}(v)) = g \mathcal{L}_0 \) and \( \text{ord} (\det(g)) \) is maximal among those \( g \in J_0(K) \) with \( \mathcal{L} \subseteq g \mathcal{L}_0 \). The class of \( g \) in \( J_0(K)/(J_0(K) \cap \text{Stab}(\mathcal{L}_0)) \) depends only on \( \mathcal{L}(v) \), not on \( v \). Moreover, [Vie08 Lemma 4.8] shows also that an element \( v \in V_{b}^{adm} \) satisfies \( P(\mathcal{L}(v)) = g \mathcal{L} \) if and only if

\[
v = \sum_{i=1}^{n'} \sum_{l \geq 0} a_{i,l} g(e_{i,l})
\]

with \( a_{i,l} \in \bar{k} \) and \( a_{0,l} \) linearly independent over \( \mathbb{F}_{q^m} \). This shows the claim of the proposition holds at least set-theoretically.

We show that each \( g \cdot \mathcal{L}_{adm}^{0,b} / \sim_{b,r,m} \) is a connected component. For a lattice \( \mathcal{L} \) in \( V \), let

\[
\text{vol}(\mathcal{L}) := \text{lg}(\mathcal{L}/\mathcal{L}_0 \cap \mathcal{L}) - \text{lg}(\mathcal{L}/\mathcal{L}_0 \cap \mathcal{L}).
\]

If \( \mathcal{L} = g \mathcal{L}_0 \), we have \( \text{vol}(\mathcal{L}) = -\text{ord}(\det(g)) \).

Lemma 3.12. On connected components of \( X_{\tilde{w}_r}(b) \), the volume of the lattice \( P(\mathcal{L}(v)) \) is constant.
We have a natural isomorphism $D$ where we have $\pi_0(\tilde{G}/I^m) = \mathbb{Z}$, the connected component of the point $gI^m$ given by $\ord(\det(g)) = -\vol(gL_0)$, i.e., on connected components $\tilde{G}/I^m$, the volume of the lattice attached to a point $(gL_0)$ is attached to the point $gI^m$ is constant. In particular, the same is also true on connected components of $X^m_{\bar{w}_r}(b)$. By the above-mentioned result of Viehmann, we deduce the lemma. 

Proof. By [Vie08, Theorem 4.11(ii)], for $v \in V^\text{adm}_b$ the difference $\vol(P(L(v))) - \vol(L(v))$ is constant, depending only on $b$, not on $L(v)$. We have $\pi_0(\tilde{G}/I^m) = \mathbb{Z}$, the connected component of the point $gI^m$ given by $\ord(\det(g)) = -\vol(gL_0)$, i.e., on connected components $\tilde{G}/I^m$, the volume of the lattice attached to a point $(gL_0)$ is attached to the point $gI^m$ is constant. In particular, the same is also true on connected components of $X^m_{\bar{w}_r}(b)$. By the above-mentioned result of Viehmann, we deduce the lemma. 

Let $g \in J_b(K)$. Write $S(g)$ for the subset $gL_0^\text{adm}/\sim_{b,m,r}$ of $X^m_{\bar{w}_r}(b)$. Taking into account the definition of the equivalence relation $\sim_{b,m,r}$ in Section 3.4, $S(1)$ is the preimage under the “reduction modulo $\varpi$”-map of the complement of finitely many hyperplanes in an affine space. Hence $S(1)$ is connected, and hence the same is true for any $g$. We show now that $S(g)$ is open and closed. Let $\mathcal{M} := gL_0$. The condition of being contained in a lattice is closed, hence the closure of $S(g)$ inside $\tilde{G}/I^m$ is contained in the subset

$$\{xI^m \in \tilde{G}/I^m : \sum_{i=0}^{n} \sigma_1(xL_0) \leq \mathcal{M}\}.$$ 

Now let $g' \in J_b(K)$. Assume $h'I^m \in S(g')$ is contained in the closure of $S(g)$. By the above argument, we have $g'L_0 \subseteq \mathcal{M}$. By Lemma 3.12 we have $\vol(gL_0) = \vol(\mathcal{M})$. It follows that $gL_0 = \mathcal{M}$. This shows that $S(g)$ is closed. As $X^m_{\bar{w}_r}(b)$ locally of finite type (Remark 3.2), the disjoint union of the $S(g)$ is locally finite. Hence $S(g)$ is also open.

Definition 3.13. Define $F^i_{\text{red}}(v) := \frac{1}{\varpi^{[k_0/i/n_0]}} \cdot F^i(v)$ and for a column vector $v \in V$, define $g^\text{red}_b(v) := \left( v \mid F^1_{\text{red}}(v) \mid F^2_{\text{red}}(v) \mid \ldots \mid F^{n-1}_{\text{red}}(v) \right)$ to be the $n \times n$ matrix whose $i$th column is $F^{i-1}_{\text{red}}(v)$. Observe that $g_b(v) = g^\text{red}_b(v) \cdot D_{k,n}$, where $D_{k,n}$ is the diagonal matrix whose $(i, i)$th entry is $\varpi^{[k_0/i/n_0]}$.

Lemma 3.14. Let $b' = h^{-1}b\sigma(h)$ for some $h \in \text{Stab}(L_0)$. Then

$$\left\{ v \in L_0 : g^\text{red}_b(v) \in \mathcal{O}_K^x \right\} / \sim_{b',m,r}$$

is a connected component of $X^m_{\bar{w}_r}(b')$, and

$$\left\{ v \in L_0 : g^\text{red}_b(v) \in \mathcal{O}_K^x \right\} / \sim_{b',m,r}$$

is a connected component of $X^m_{\bar{w}_r}(b')$.

Proof. By Proposition 3.11, with respect to the basis $\{e_{i,l} : 1 \leq i \leq n', 0 \leq l \leq n_0 - 1\}$ of $V$, we have

$$L_0^\text{adm} = \left\{ v \in V^\text{adm}_b : g^\text{red}_b(v) \in \text{GL}_n(\mathcal{O}) \right\} = \left\{ v \in V^\text{adm}_b : g_b(v) \in \text{GL}_n(\mathcal{O}) \cdot D_{k,n} \right\},$$

where $D_{k,n}$ is the $n \times n$ diagonal matrix whose $(i, i)$th entry is $\varpi^{[k_0/i/n_0]}$. as in Definition 3.13.

We have a natural isomorphism

$$V^\text{adm}_b/\sim_{b,m,r} \cong X^m_{\bar{w}_r}(b) \xrightarrow{\sim} X^m_{\bar{w}_r}(b') \cong V^\text{adm}_b/\sim_{b',m,r} \xrightarrow{gI^m \mapsto h^{-1}gI^m.}$$
where the first and third isomorphisms hold by Theorem 3.3 (ii). Hence the image of $\mathcal{L}^{\text{adm}}_{b,0} / \sim_{b,m,r}$ in $V^{\text{adm}}_b / \sim_{b', m, r}$ is a connected component of $X^m_{w_r}(b')$. Recalling that

$$V^{\text{adm}}_b / \sim_{b,m,r} \rightarrow X^m_{w_r}(b), \quad v \mapsto g_{b,r}(v) I^m,$$

we see that for the diagonal matrix $D^{(r)}_{k,n}$ whose $(i, i)$th entry is $\varpi^{r \lfloor k/n \rfloor}$,

$$\left\{ h^{-1} g_{b,r}(v) I^m : v \in \mathcal{L}^{\text{adm}}_{0,b} \right\} = \left\{ h^{-1} g_{b}^{\text{red}}(v) D^{(r)}_{k,n} I^m : v \in \mathcal{L}^{\text{adm}}_{0,b} \right\}$$

$$= \left\{ g D^{(r)}_{k,n} I^m \in X^m_{w_r}(b') : g \in \text{GL}_n(O) \right\}$$

$$= \left\{ g_{b'}^{\text{red}}(v) D^{(r)}_{k,n} I^m : v \in \mathcal{L}_0 \text{ and } \det(g_{b'}^{\text{red}}(v)) \in O^\times \right\}$$

$$\approx \left\{ v \in \mathcal{L}_0 : \det(g_{b'}^{\text{red}}(v)) \in O^\times \right\} / \sim_{b', m, r}$$

is a connected component of $X^m_{w_r}(b')$. The proof of the corresponding claim for $\hat{X}^m_{w_r}(b')$ is completely analogous. \qed
Part 2. Cohomology

The aim of this part of the paper is to provide a complete description of the cohomology of the affine Deligne–Lusztig variety at infinite level $X_b^\infty(b)$ in the case that $b$ is a Coxeter element. In Section 7, we use Proposition 3.11 to relate the homology of the infinite-dimensional variety $\hat{X}_b^\infty(b)$ to the cohomology of the finite-type variety $X^m_{nr}(b)_{\mathcal{L}_{0,b}}$. In Section 6, we use a natural open/closed decomposition of the finite-type variety $X^m_{nr}(b)_{\mathcal{L}_{0,b}}$ to relate its cohomology to classical Deligne–Lusztig theory for finite reductive groups. It will turn out that there is a top chunk of degrees where the cohomology of $\hat{X}_b^\infty(b)$ to the cohomology of the finite-type variety $X^m_{nr}(b)_{\mathcal{L}_{0,b}}$ is completely determined by the cohomology of the classical Deligne–Lusztig variety associated to maximal nonsplit torus inside the finite reductive quotient of $J_b(K)$. The remaining nonzero cohomology groups (the bottom chunk) are completely determined by the cohomology of what can be thought of as the “unipotent part” of $X^m_{nr}(b)_{\mathcal{L}_{0,b}}$. Using techniques established in [Cha16a], we give a complete description of the cohomology of the unipotent part in Section 5.

4. Definitions

For any integer $l$, define $[l]$ to be the unique integer with $1 \leq [l] \leq n$ such that $l \equiv [l]$ modulo $n$. (Note the conflict of notation between this and the $\sigma$-conjugacy class $[b]$. It should be clear from the context which we mean.) In this part of the paper, we assume that $\kappa_G([b])$ is the unique integer with $0 \leq \kappa_G([b]) \leq n - 1$ with $\kappa_G([b]) \equiv \text{ord}([\det(b)])$ modulo $n$ (we may multiply $b$ by a central element without changing the affine Deligne–Lusztig varieties and the group $J_b$). Recall that for a $\sigma$-conjugacy class $[b]$, we set $\kappa = \kappa_G([b])$, $n' = \gcd(n, \kappa_G([b]))$, $n_0 = n/n'$, and $k_0 = \kappa_G([b])/n'$. By construction, $k_0$ and $n_0$ are coprime, and

$$J_b(K) \cong \text{GL}_{n'}(D_{k_0/n_0}),$$

where $D_{k_0/n_0}$ is the division algebra of dimension $k_0^2$ over $K$ of Hasse invariant $k_0/n_0$. Set

$$b_0 := \begin{pmatrix} 0 & 1_{n-1} \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad t_{\kappa,n} := \begin{cases} \text{diag}(\overbrace{\pi, \ldots, \pi}^{k}, 1, \ldots, 1) & \text{if } (k,n) = 1, \\ \text{diag}(t_{k_0,n_0}, \ldots, t_{k_0,n_0}) & \text{otherwise}. \end{cases}$$

Fix an integer $\epsilon_{\kappa,n}$ such that $(\epsilon_{\kappa,n}, n) = 1$ and $\epsilon_{\kappa,n} \equiv k_0 \mod n_0$. (It is clear that $\epsilon_{\kappa,n}$ exists.) Now define

$$b_{\kappa,n} := b_0^{\epsilon_{\kappa,n}} \cdot t_{\kappa,n}.$$  \hspace{1cm} (4.1)

From now on, we fix $\kappa, n$ and write $b := b_{\kappa,n}$.

**Example 4.1.** We explicitly write out the representatives $b = b_{\kappa,n}$ in the case $n = 6$. In this case, there is a unique choice of $\epsilon_{\kappa,n}$ for each $\kappa$.

- $\kappa = 0$ \quad $J_b(K) \cong \text{GL}_6(K) \quad b = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ \quad $b_0$;

- $\kappa = 1$ \quad $J_b(K) \cong \text{GL}_4(D_{1/6}) \quad b = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \omega & \omega & \omega & \omega \end{pmatrix} = b_0 \cdot \text{diag}(\pi, 1, 1, 1, 1, 1);$

- $\kappa = 2$ \quad $J_b(K) \cong \text{GL}_2(D_{1/3}) \quad b = \begin{pmatrix} 1 & 1 \\ \omega & 1 \end{pmatrix} = b_0 \cdot \text{diag}(\pi, 1, 1, \pi, 1, 1);$
\[ \kappa = 3 \quad J_b(K) \cong \text{GL}_3(D_{1/2}) \quad b = \begin{pmatrix} 1 & 1 & \infty \\ \infty & \infty & \infty \\ \infty & \infty & \infty \end{pmatrix} = b_0 \cdot \text{diag}(\pi, 1, \pi, 1, \pi, 1); \]

\[ \kappa = 4 \quad J_b(K) \cong \text{GL}_2(D_{2/3}) \quad b = \begin{pmatrix} 1 & \infty \\ \infty & 1 \end{pmatrix} = b_0^5 \cdot \text{diag}(\pi, 1, \pi, 1, \pi, 1); \]

\[ \kappa = 5 \quad J_b(K) \cong \text{GL}_1(D_{1/6}) \quad b = \begin{pmatrix} 1 & \infty \\ \infty & \infty \\ \infty & \infty \end{pmatrix} = b_0^5 \cdot \text{diag}(\pi, 1, \pi, 1, \pi, 1). \]

Consider the twisted polynomial ring \( \mathcal{L}(\Pi) \) determined by the commutation relation

\[ \Pi \cdot a = \sigma^l(a) \cdot \Pi, \quad \text{where } l \text{ is an integer satisfying } \varepsilon_{n,M} l \equiv 1 \pmod{n}. \]

**Lemma 4.3.** The natural homomorphism

\[ \Phi : \mathcal{L}(\Pi)/(\Pi^n - \pi^l) \to M_n(\tilde{K}) \]

given by \( \Pi \mapsto b_{n,n}^e \) and \( a \mapsto D(a) := \text{diag}(a, \sigma^l(a), \sigma^2(a), \ldots, \sigma^{(n-1)l}(a)) \) for \( a \in L \), induces an isomorphism

\[ \mathcal{L}(\Pi)/(\Pi^n - \pi^l) \cong M_n(\tilde{K})^F. \]

**Proof.** We first prove that the map is well-defined. Recall that \( F(g) = b_{n,n}^e \sigma(g) b_{n,n}^e \) where \( e = \varepsilon_{n,M} \). We have

\[ b_{n,n}^e \sigma(\text{diag}(a, \sigma^l(a), \ldots, \sigma^{(n-1)l}(a))) b_{n,n}^e = \text{diag}(\sigma(n-l)l+1(a), \sigma(n-l+1)(a), \ldots, \sigma^{(n-1)l+1}(a)) = \text{diag}(a, \sigma^l(a), \ldots, \sigma^{(n-1)l}(a)). \]

It follows that \( \Phi \) is a homomorphism of algebras. Furthermore, this calculation shows that \( L \subset M_n(\tilde{K})^F \). It is clear that \( \Phi(\Pi) \subset M_n(\tilde{K})^F \), and so we see that the image of \( \Phi \) is a \( n^2 \)-dimensional \( K \)-vector space. Since \( M_n(\tilde{K})^F \) is also a \( n^2 \)-dimensional \( K \)-vector space, \( \Phi \) must define an isomorphism onto \( M_n(\tilde{K})^F \). \( \square \)

**Lemma 4.3.**

(a) \( J_b(K) \cap \text{GL}_{m}(O) \) is a maximal compact subgroup of \( J_b(K) \).

(b) \( J_b(K) \cap T(\tilde{K}) \) is isomorphic to \( L^\times \).

**Proof.** Consider the order

\[ \Lambda := \bigoplus_{i=0}^{n-1} \frac{1}{\varepsilon_{i/M}} \mathcal{O}_L \cdot \Pi^i \subset L(\Pi)/(\Pi^n - \pi^l). \]

By the definition of \( b \), it is clear that \( \Phi(\Lambda) \subset M_n(\tilde{O}) \) and \( \Phi(\Lambda^\times) = J_b(K) \cap \text{GL}_n(O) \). I claim that \( \Lambda \) is a maximal order. Indeed, if \( \Lambda' \supseteq \Lambda \), then \( \Lambda' \) must contain an element of the form

\[ \frac{1}{\varepsilon_{m/M}} \cdot \frac{1}{\varepsilon_{i/M}} \cdot \Pi^i \]

for some \( 0 \leq i \leq n - 1 \) and \( m > 0 \). This implies

\[ \left( \frac{1}{\varepsilon_{m/M}} \cdot \frac{1}{\varepsilon_{i/M}} \cdot \Pi^i \right)^n = \left( \frac{\varepsilon_{n^l/M}}{\varepsilon_{m^l/M} + [i/n_0]} \right) \notin \mathcal{O}_L, \]

where the last assertion holds since \( m + [i/n_0] \geq 1 + [i/n_0] > i/n_0 \) and therefore \( mn + [i/n_0] > in' \). But then this implies that \( \Lambda' \) would have to contain \( L \), which is not possible. Thus \( \Lambda \) must be a maximal order and (a) follows.

We now prove (b). Let \( g = \text{diag}(g_1, \ldots, g_n) \). Then \( b^{-1}\sigma(g)b = \text{diag}(\sigma(g_{w(1)}), \ldots, g_{w(n)}) \) for a length-\( n \)-cycle \( w \in S_n \). Thus the defining condition \( g = b^{-1}\sigma(g)b \) implies that

\[ g_1 = \sigma(g_{w(1)}) = \sigma^2(g_{w^2(1)}) = \cdots = \sigma^{n-1}(g_{w^{n-1}(1)}) = \sigma^n(g_1), \]
and hence we see that $g$ is determined by $g_1$ and $g_1 \in (\tilde{K}^\times)^{\sigma^n} = L^\times$. □

**Example 4.4.** We give a comparison between $b$ and the special representative $b_{sp}$ defined in Section 3.5. Let $n = 6$.

\[
\begin{align*}
\kappa &= 0 & b_{sp} &= \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & & 1 & \\ & & & & & 1 \\ & & & & & \end{pmatrix} & b &= \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & & 1 & \\ & & & & & 1 \\ & & & & & \end{pmatrix} = b_0; \\
\kappa &= 1 & b_{sp} &= \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \omega & & & \\ & & & \omega & & \\ & & & & \omega & \\ & & & & & 1 \end{pmatrix} & b &= \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \omega & & & \\ & & & \omega & & \\ & & & & \omega & \\ & & & & & 1 \end{pmatrix} = b_0 \cdot \text{diag}(\pi, 1, 1, 1, 1); \\
\kappa &= 2 & b_{sp} &= \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \omega & & & \\ & & & \omega & & \\ & & & & \omega & \\ & & & & & 1 \end{pmatrix} & b &= \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \omega & & & \\ & & & \omega & & \\ & & & & \omega & \\ & & & & & 1 \end{pmatrix} = b_0 \cdot \text{diag}(\pi, 1, 1, \pi, 1); \\
\kappa &= 3 & b_{sp} &= \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \omega & & & \\ & & & \omega & & \\ & & & & \omega & \\ & & & & & 1 \end{pmatrix} & b &= \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \omega & & & \\ & & & \omega & & \\ & & & & \omega & \\ & & & & & 1 \end{pmatrix} = b_0^5 \cdot \text{diag}(\pi, 1, \pi, 1, \pi, 1); \\
\kappa &= 4 & b_{sp} &= \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \omega & & & \\ & & & \omega & & \\ & & & & \omega & \\ & & & & & 1 \end{pmatrix} & b &= \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \omega & & & \\ & & & \omega & & \\ & & & & \omega & \\ & & & & & 1 \end{pmatrix} = b_0^5 \cdot \text{diag}(\pi, \pi, 1, \pi, \pi, 1); \\
\kappa &= 5 & b_{sp} &= \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \omega & & & \\ & & & \omega & & \\ & & & & \omega & \\ & & & & & 1 \end{pmatrix} & b &= \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \omega & & & \\ & & & \omega & & \\ & & & & \omega & \\ & & & & & 1 \end{pmatrix} = b_0^5 \cdot \text{diag}(\pi, \pi, \pi, \pi, \pi, 1).
\end{align*}
\]

**Lemma 4.5.** Let $b_{sp}$ be the special representative of $[b]$ defined in Section 3.5. There exists $g \in \text{GL}_n(\tilde{O})$ such that $g^{-1}b_{sp}\sigma(g) = b$.

**Proof.** First assume that $\kappa$ is coprime to $n$. It is easy to see that $b_{sp}$ is conjugate to $b$ via a permutation matrix. Now assume that $(\kappa, n) = n' > 1$. By construction, $b_{sp} = \text{diag}(b_{sp,0}, \ldots, b_{sp,n'})$ where $b_{sp,0}$ is a matrix of size $n_0 \times n_0$. Observe that by definition, $b_{sp,0}$ is $\sigma$-invariant. Write $b_{sp} \cdot w$ for the action of $w \in S_{n'}$ permuting the blocks of $b_{sp}$.

**Claim 1.** If $w$ has order $n'$, then $b_{sp}$ is $\sigma$-conjugate to $b_{sp} \cdot w$ via an element of $\text{GL}_n(\tilde{O})$.

We first explain why Claim 1 implies the lemma. (Note that Claim 1 is true for any $w \in S_{n'}$ and the general argument requires only slightly more reasoning, but we will only need the claim as stated.) Since $b_{sp,0}$ has order $n_0$ by definition, the element $b_{sp} \cdot w$ is the product of an order-$n$ permutation matrix with a diagonal matrix with $\kappa$’s and $(n - \kappa)$’s. It is now easy to see that one can reorder the basis vectors to obtain $b$; equivalently, $b_{sp} \cdot w$ is conjugate to $b$ via a permutation matrix.

It now remains to prove Claim 1. Suppose that

\[
g := (g_1 | \cdots | g_{n'}) \in \text{GL}_n(\tilde{K})
\]

where each $g_i$ is a matrix of size $n \times n_0$. If $g$ has the property that $b_{sp} \cdot \sigma(g) = g \cdot b_{sp}$, then we must have

\[
(b_{sp,0} \ast \sigma(g_1) | \cdots | b_{sp,0} \ast \sigma(g_{n'})) = (g_{w(1)} \ast b_{sp,0} | \cdots | g_{w(n')} \ast b_{sp,0}),
\]

where we view each $g_i$ as a block-matrix of $n_0 \times n_0$ and multiply each of the $n'$ blocks by $b_{sp,0}$. Since $w$ has order $n'$, the above equation shows that each $g_i$ can be written in terms of $g_1$ and $b_{sp,0}$, and that $g_1 = (c_1 | \cdots | c_{n'})^T$ must satisfy $c_i = b_{sp,0}' \cdot \sigma^{n'}(c_i) \cdot b_{sp,0}^{-n'}$ for each $i$. To finish the lemma, we need to argue that one can find such a $g_1$ with $\tilde{O}$-coefficients such that $\det(g) \in \mathcal{O}^\times$. We may take $c_i = \text{diag}(a_{i,1}, \ldots, a_{i,n_0})$ where we first
pick \((a_{1,1}, \ldots, a_{n',1})\) in \(\tilde{O}^{\oplus n'}\) to be fixed by \(F_{\text{bsp}, 0}\) mod \(\varpi\) but not fixed by any smaller power of \(F_{\text{bsp}, 0} := b_{\text{sp}, 0} \cdot \sigma\) mod \(\varpi\). Then the condition \(c_i = b_{\text{sp}, 0} - \sigma^{-n'}(a_{i,j}) \cdot b_{\text{sp}, 0}\) may determine some of the remaining \(a_{i,j}\)'s. Repeat this process for any remaining undetermined \(a_{i,j}\). It is easy to check now these choices give a \(g\) with \(\det g \neq 0\) modulo \(\varpi\), which is equivalent to producing an appropriate \(g\) in \(\text{GL}_n(\tilde{O})\). This completes the proof of the claim and therefore the lemma. 

**Lemma 4.6.** Let \(b_{\text{sp}}\) be the special representative of \([b]\) defined in Section 3.5. The assignment

\[
\lambda(v_1, \ldots, v_n) := \sum_{i=1}^{n} \frac{1}{\varpi^{(i-1)/n_0}} \text{diag}(v_i, \sigma^i(v_i), \ldots, \sigma^{(n-1)i}(v_i)) \cdot b^{i-1}
\]

defines an embedding

\[\lambda : \mathcal{L}_{b_{\text{sp}}^\text{adm}} \rightarrow \Lambda^\times.\]

**Proof.** Recall from Lemma 4.5 that \(b_{\text{sp}}\) is \(\sigma\)-conjugate to \(b\) via \(\text{GL}_n(\tilde{O})\). This means we can apply Lemma 3.14 and obtain an isomorphism

\[
\left\{ x \in \mathcal{L}_0 : \det g_{b_{\text{sp}}}^{\text{red}}(x) \in \mathcal{O}_K^\times \right\} \cong \left\{ x \in \mathcal{L}_0 : \det g_b^{\text{red}}(x) \in \mathcal{O}_K^\times \right\}.
\]

To complete the proof of the lemma, it remains to show that \(\det g_b^{\text{red}}(x) \in \mathcal{O}_K^\times\) if and only if \(\det(\lambda(x)) \in \mathcal{O}_K^\times\). To see this, observe that the integer \(l\) is such that \(b^l\) corresponds to the permutation \((1 2 \cdots n) \in S_n\) and so one can reorder the columns of \(g_b(x)\) to obtain \(\lambda(x)\).

We may consider the \(\mathcal{O}_K\)-group scheme \(\mathcal{G}\) defined by setting

\[\mathcal{G}((\mathcal{O}_K)) := J_0(K) \cap \text{GL}_n(\mathcal{O}).\]

For each integer \(h \geq 1\), define the \(\mathbb{F}_q\)-group scheme \(\mathcal{G}_h\) to be

\[\mathcal{G}_h(A) := \mathcal{G}(\mathcal{W}_{h^{-1}}(A)).\]

Define

\[\mathcal{T}_h(A) := \mathcal{G}_h(A) \cap \{\text{diagonal matrices}\}.\]

We set \(\mathcal{G}_h^{(1)}\) to be the \(k\)-subgroup scheme of \(\mathcal{G}_h\) determined by the condition that \(\mathcal{G}_h^{(1)}(\bar{k})\) is the pro-unipotent subgroup of \(\mathcal{G}_h(\bar{k})\) (that is, \(\mathcal{G}_h^{(1)}(\bar{k})\) is the minimal normal subgroup of \(\mathcal{G}_h(\bar{k})\) such that the quotient \(\mathcal{G}_h(\bar{k})/\mathcal{G}_h^{(1)}(\bar{k})\) is a finite reductive group over \(k\)). Define \(\mathcal{T}_h^{(1)}\) to be the subgroup scheme of \(\mathcal{G}_h^{(1)}\) consisting of diagonal matrices.

**Remark 4.7.** \(\mathcal{G}(\mathcal{O}) = G_{x,0}\) for a point \(x\) on the building \(\mathcal{B}(\text{GL}_n)\). Let \(G_{x,r}, G_{x,r^+}\) denote the subgroups in the Moy–Prasad filtration of \(G_{x,0}\). Then we have \(\mathcal{G}_h(\bar{k}) = G_{x,0}/G_{x,h}\) and \(\mathcal{G}_h^{(1)}(\bar{k}) = G_{x,0^+}/G_{x,h}\).

**Definition 4.8.** Let \(k_n\) be the degree-\(n\) extension of \(k\) and for each integer \(h \geq 1\), let \(\varphi_h : \mathcal{G}(\bar{k}) \rightarrow \mathcal{G}_h(\bar{k})\) denote the natural quotient map.

(a) Define

\[X_h(\bar{k}) := \varphi_h(\mathcal{L}_{b_{\text{sp}}^\text{adm}}(\mathcal{O})).\]

Note that by descent, \(X_h\) is defined over \(k_n\) (but not \(k\)) because \(X_h(\bar{k})\) is stable under \(F^n\).

(b) Define the closed subscheme \(Z_h\) of \(X_h\) by

\[Z_h := X_h \cap \varphi_h(\varphi_1^{-1}(\mathcal{G}_1(k))).\]
Define the closed subscheme $Z^{(1)}_h$ of $Z_h$ and the open subscheme $Y_h$ of $X_h$ to be

$$Z^{(1)}_h := X_h \cap G^{(1)}_h,$$

$$Y_h := X_h \setminus Z_h.$$

$Z^{(1)}_h$, $Z_h$, and $Y_h$ are all defined over $k_n$.

The $k$-scheme $X_h$ has a natural action by $G_h \times T_h$ given by left-multiplication by $G_h$ and right-multiplication by $T_h$:

$$(g, t) \cdot x := g^{-1} xt, \quad \text{for } g \in G_h, t \in T_h, x \in X_h.$$ 

Both $Z_h$ and $Y_h$ are stabilized by $G_h \times T_h$, but the stabilizer of $Z^{(1)}_h$ in $G_h \times T_h$ is

$$\Gamma_h := \left( G^{(1)}_h \times T^{(1)}_h \right) \cdot \{(t, t^{-1}) : t \in T_h \}.$$ 

**Proposition 4.9.**

(a) $X_h$ is a smooth affine scheme of dimension $(n-1)(h-1) + (n'-1)$.

(b) $Z_h$ and $Z^{(1)}_h$ are smooth affine schemes of dimension $(n-1)(h-1)$.

**Proof.** The proof is very similar to that of [Cha16a Proposition 3.10]. \qed
5. Deligne–Lusztig varieties for unipotent groups

In this section we study the cohomology groups of $Z_h^{(1)}$ as representations of $T_h^{(1)} \times G_h^{(1)}$.

5.1. Combinatorial set-up. Fix $n \geq 1$ and a divisor $m$ of $n$. Pick any $k \geq 1$ coprime to $n$ with $km < n$. Define $b := b_{m,n}$. In addition, we fix an integer $h \geq 1$. For any integer $l \geq 1$, define $[b^l] := \pi^{-[lm/n]} \cdot b^l \in M_n(\mathbb{W}_h(\mathbb{F}_q))$. Define $[b^l]$ to be the operator given by replacing all $\pi$’s in $[b^l]$ with the Verschiebung morphism $V$.

Definition 5.1. Define $U(\mathbb{F}_q)$ to be the set of matrices

$$i(x_1, \ldots, x_n) := x_1 + x_2[b] + \cdots + x_n[b^{n-1}] \in GL_n(\mathbb{W}_h(\mathbb{F}_q))$$

where

$$x_j = \text{diag}(x_{1,j}, x_{2,j+1}, \ldots, x_{n,j+n-1})$$

$$x_{i,j+i-1} = \begin{cases} [1, x_{i,i+1}, x_{i,i+2}, \ldots, x_{i,i+h-1}] \in \mathbb{W}_h^{(1)}(\mathbb{F}_q) & \text{if } j = 1, \\
[1, x_{i,i+1}, x_{i,i+2}, \ldots, x_{i,i+h-1}] \in \mathbb{W}_{h-1}(\mathbb{F}_q) & \text{if } j > 1 \text{ and } \frac{n}{m} \text{ divides (at least) one of } \{i, i + 1, \ldots, i + j - 2\}, \\
& \text{otherwise.} \end{cases}$$

We define $\mathcal{A}^+$ to be the collection of all $\lambda = (i, j, l)$ such that $x_{\lambda}$ appears in $i(x_1, \ldots, x_n)$.

The Frobenius morphism

$$F^* : U(\mathbb{F}_q) \to U(\mathbb{F}_q^e), \quad g \mapsto b^{-1} \varphi(g)b$$

endows $U$ with an $\mathbb{F}_q$-rational structure.

Definition 5.2. We define three bijections $\sigma, \sigma', \tau$ associated to $b$ and $\varpi := \left(\begin{smallmatrix} 0 & 1^{n-1} \\ \pi^m & 0 \end{smallmatrix}\right)$. First $\gamma_0$ be the permutation matrix satisfying the following two conditions:

(i) $\gamma_0 \cdot e_1 = e_1$ where $e_1$ is the first elementary column vector.

(ii) There exists a $\lambda \in \mathbb{Z}^{\mathbb{P}n}$ such that $\gamma := \gamma_0 \cdot t^\lambda$ satisfies $b = \gamma^{-1} \cdot \varpi \cdot \gamma$.

Now define:

1. Let $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be the permutation such that $F_2(\text{diag}(y_1, \ldots, y_n)) = \text{diag}(\varphi(y_{\sigma(1)}), \ldots, \varphi(y_{\sigma(n)})$).

2. Let $\sigma' : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be the permutation such that $F_1(\text{diag}(y_1, \ldots, y_n)) = \text{diag}(\varphi(y_{\sigma'(1)}), \ldots, \varphi(y_{\sigma'(n)}))$.

3. Let $\tau : \{1, \ldots, n\} \to \{0, \ldots, n - 1\}$ be the bijection such that $\gamma_0^{-1} \cdot \text{diag}(y_1, \ldots, y_n) \cdot \gamma_0 = \text{diag}(y_{\tau(1)+1}, \ldots, y_{\tau(n)+1}$).

By the assumption that $e_1 \cdot \gamma_0 = e_1$, we have $\tau(1) = 0$.

Definition 5.3. Consider the $\mathbb{F}_q$-subscheme $X \subset U$ defined by

$$X(\mathbb{F}_q) := \left\{ x = i(x_1, \ldots, x_n) \in U(\mathbb{F}_q) : i_{x,\sigma'}(i) = \varphi^{(i)}(x_{1,\sigma'(1)}), \varphi(\text{det}(x)) = \text{det}(x) \in \mathbb{W}_h^{(1)}(\mathbb{F}_q) \right\}.$$

Remark 5.4. It is a straightforward computation to see that $X(\mathbb{F}_q^e) = U(\mathbb{F}_q)$.

---

Note that the notation in this subsection is inconsistent with the notation in the rest of the paper.
Let \( \mathcal{I}_{n,h} \) denote the set of characters \( \chi: W_q^{(1)}(F_q^n) \to \overline{\mathbb{Q}}_l^\times \). Recall that we have natural surjections \( \text{pr}: W_q^{(1)} \to W_{q_{n-1}}^{(1)} \) and injections \( \mathbb{G}_a \to W_q^{(1)} \) given by \( x \mapsto [1, 0, \ldots, 0, x] \). Furthermore, for any subfield \( F \subset L \), the norm map \( L^\times \to F^\times \) induces a map \( \text{Nm}: W_q^{(1)}(k_L) \to W_q^{(1)}(k_F) \). These maps induce
\[
\text{pr}^*: \mathcal{I}_{n,h'} \to \mathcal{I}_{n,h}, \quad \text{for } h' < h, \\
\text{Nm}^*: \mathcal{I}_{m,n} \to \mathcal{I}_{n,h}, \quad \text{for } m \mid n.
\]
By pulling back along \( \mathbb{G}_a \to W_q^{(1)} \), we may restrict characters of \( W_q^{(1)}(F_q^n) \) to characters of \( F_q^n \). We say that \( \chi \in \mathcal{I}_{n,h} \) has conductor \( m \) if the stabilizer of \( \chi|_{F_q^n} \) in \( \text{Gal}(F_q^n/F_q) \) is \( \text{Gal}(F_q^n/F_{q^n}) \). If \( \chi \in \mathcal{I}_{n,h} \) has conductor \( n \), we say that \( \chi \) is primitive. We write \( \mathcal{I}_{n,h}^0 \subset \mathcal{I}_{n,h} \) denote the subset of primitive characters.

We can decompose \( \chi \in \mathcal{I}_{n,h} \) into primitive components in the sense of Howe [How77 Corollary after Lemma 11].

**Definition 5.5.** A Howe factorization of a character \( \chi \in \mathcal{I}_{n,h} \) is a decomposition
\[
\chi = \prod_{i=1}^r \chi_i, \quad \text{where } \chi_i = \text{pr}^* \text{Nm}^* \chi_i^0 \text{ and } \chi_i^0 \in \mathcal{I}_{m_i,h_i}^0,
\]
such that \( m_i < m_{i+1}, m_i \mid m_{i+1} \), and \( h_i > h_{i+1} \). It is automatic that \( m_i \leq n \) and \( h_i \geq h_i \).

For any integer \( 1 \leq t \leq r \), define
\[
\chi_{\geq t} := \prod_{i=t}^r \chi_i \in \mathcal{I}_{n,h_i}.
\]

Observe that the choice of \( \chi_i \) in a Howe factorization \( \chi = \prod_{i=1}^r \chi_i \) is not unique, but we may always take \( h_1 = h \), and once we do this, the \( m_i \) and \( h_i \) only depend on \( \chi \). Hence the Howe factorization attaches to each character \( \chi \in \mathcal{I}_{n,h} \) a pair of well-defined sequences
\[
1 =: m_0 \leq m_1 < m_2 < \cdots < m_r \leq m_{r+1} := n \\
h =: h_0 = h_1 > h_2 > \cdots > h_r \geq h_{r+1} := 1
\]
satisfying the divisibility \( m_i \mid m_{i+1} \) for \( 0 \leq i \leq r \).

**Definition 5.6.** We define some subsets of the indexing set \( \mathcal{A}^+ \) defined in Definition 5.1
\[
\mathcal{A} := \{(i, \sigma^j(i), l) \in \mathcal{A}^+: j < n\}, \\
\mathcal{A}^- := \{(i, \sigma^j(i), l) \in \mathcal{A}: i = 1\}.
\]

Define \( h := h_1 \). Given two sequences of integers
\[
1 =: m_0 \leq m_1 < m_2 < \cdots < m_r \leq m_{r+1} := n, \quad m_i \mid m_{i+1}, \\
h =: h_0 = h_1 > h_2 > \cdots > h_r \geq h_{r+1} := 1,
\]
we can define the following subsets of \( \mathcal{A} \) for \( 0 \leq s, t \leq r \):
\[
\mathcal{A}_{s,t}^+ := \{(i, \sigma^j(i), l) \in \mathcal{A}: j \equiv 0 \pmod{m_s}, \ j \neq 0 \pmod{m_{s+1}}, \ 1 \leq l \leq h_t - 1\}, \\
\mathcal{A}_{s,t}^- := \{(i, \sigma^j(i), l) \in \mathcal{A}_{s,t}: i = 1\}.
\]

**Definition 5.7.** We will need to understand which \( \lambda \in \mathcal{A}^+ \) are such that \( x_\lambda \) contributes to \( \det(i(x_1, \ldots, x_n)) \). We denote the set of all such \( \lambda \) by \( \mathcal{A}^{+, \min} \). It can be explicitly described as follows:
\[
\mathcal{A}^{+, \min} := \bigcup_{j=1}^n \{(i, \sigma^j(i), l) : 1 \leq i \leq n, \ 1 \leq l \leq h(j) - 1\},
\]
Proof. Note that this defines a total ordering on $|\lambda|$ and so we see that if $\lambda \vdash j$, then $|\lambda| \geq j$. We define

$$x_i,\sigma'(i) \in \begin{cases} \mathbb{W}_{h}^{(1)} & \text{if } j = n, \\ \mathbb{W}_{h-2} & \text{if } \frac{n}{m} \mid j \text{ and } j < n, \\ \mathbb{W}_{h-1} & \text{if } \frac{n}{m} \nmid j. \end{cases}$$

For every set $S$ defined in Definition 5.6, we define $S^{\min} := S \cap A^+$. min.

**Definition 5.8.** Define a norm on $A^+$. min as follows:

$$|(i, j, l)| := |j - i| + n(l - 1).$$

This defines a total ordering on $A^+$. min. For $\lambda = (i, j, l) \in A^+$. min, define

$$\lambda' := \begin{cases} (j, i, h - l) & \text{if } [i] \neq [j], \\ (j, i, h - 1 - l) & \text{if } [i] = [j]. \end{cases}$$

To apply the arguments of [Cha16a], we will need to consider the following decomposition of $A_{s,t}^{+\min}$:

$$\mathcal{T}_{s,t}^{\min} := \{(1, j, l) \in A_{s,t}^{+\min} : |(1, j, l)| > n(h_t - 1)/2\},$$

$$\mathcal{J}_{s,t}^{\min} := \{(1, j, l) \in A_{s,t}^{+\min} : |(1, j, l)| \leq n(h_t - 1)/2\}.$$

**Lemma 5.9.** There is an order-reversing injection $\mathcal{T}_{s,t}^{\min} \hookrightarrow \mathcal{J}_{s,t}^{\min}$ that is a bijection if and only if $\#A_{s,t}^{+\min}$ is even. Explicitly, it is given by

$$\mathcal{T}_{s,t}^{\min} \hookrightarrow \mathcal{J}_{s,t}^{\min}, \quad (1, \sigma^j(1), l) \mapsto \begin{cases} (1, \sigma^{n-j}(1), h_t - 1 - l) & \text{if } \frac{n}{m} \mid j, \\ (1, \sigma^{n-j}(1), h_t - l) & \text{if } \frac{n}{m} \nmid j, \end{cases}$$

and $\#A_{s,t}^{+\min}$ is even unless

(a) $m$ is even, $h_t$ is odd, and $\frac{n}{m} \equiv 0$ modulo $m_s$ and $\frac{n}{m} \neq 0$ modulo $m_{s+1}$, or

(b) $m$ is odd and $n, h_t$ are both even, and $\frac{n}{m} \equiv 0$ modulo $m_s$ and $\frac{n}{m} \neq 0$ modulo $m_{s+1}$.

Note that $\sigma^j(1) + \sigma^{n-j}(1) = n + 2$ and that if $\lambda \mapsto \lambda'$, then $|\lambda| + |\lambda'| = n(h_t - 1) - 1$.

**Proof.** We first note that $\sigma^j(1) = [kj + 1]$ so that $\sigma^j(1) + \sigma^{n-j}(1) = n + 2$. Thus

$$\begin{cases} |(1, \sigma^j(1), l)| + |(1, \sigma^{n-j}(1), h_t - 1 - l)| = n + n(l - \frac{1}{2} + h_t - 1 - l - \frac{1}{2}) = n(h_t - 1) & \text{if } \frac{n}{m} \mid j, \\ |(1, \sigma^j(1), l)| + |(1, \sigma^{n-j}(1), h_t - l)| = n + n(l - 1 + h_t - l - 1) = n(h_t - 1) & \text{if } \frac{n}{m} \nmid j, \end{cases}$$

and so we see that if $|(1, \sigma^j(1), l)| > n(h_t - 1)/2$, then

$$|(1, \sigma^{n-j}(1), l)| = n(h_t - 1) - |(1, \sigma^j(1), l)| < n(h_t - 1)/2.$$ The assertion regarding when $\mathcal{T}_{s,t}^{\min} \hookrightarrow \mathcal{J}_{s,t}^{\min}$ is clear.

**Lemma 5.10.** Write $M = i(M_1, \ldots, M_n) \in X(\mathbb{F}_q)$. Assume that for $\lambda_1, \lambda_2 \in A^+$. min with $|\lambda_1| \leq |\lambda_2|$, the variables $M_{\lambda_1}$ and $M_{\lambda_2}$ appear in the same monomial in $\det(M) \in \mathbb{W}_{h}^{(1)}$ for some $h \leq h$.

(a) Then $|\lambda_1| + |\lambda_2| \leq \frac{n}{m}(h' - 1)$.

(b) Assume in addition that $|\lambda_1| + |\lambda_2| = \frac{n}{m}(h' - 1)$ and the monomial only consists of variables $M_\mu$ with $|\lambda_1| \leq |\mu| \leq |\lambda_2|$. Then $\lambda_1 = \lambda_2^\vee$. 

\[\square\]
Proof. The $m=1$ case of this lemma was proved in [Cha16a]. The general proof is nearly identical, and we reproduce it in the case that $K$ has characteristic $p$ here. (See [Cha16a] for the necessary minor modifications needed in the case $K$ has characteristic 0.)

Let $A_{i,j} = \sum_{l \geq 0} V^l r(A_{i,j,l})$ denote the $(i,j)$th entry of $M$. Then

$$A_{i,j,l} = \begin{cases} M_{(i,j,l)} & \text{if } [i] \geq [j], \\ M_{(i,j,l+1)} & \text{if } [i] < [j]. \end{cases}$$

By definition,

$$\det(M) = \sum_{\gamma \in S_n} \prod_{i=1}^{n} A_{i,\gamma(i)} \in \mathbb{W}_h^{(1)}(\mathbb{F}_q).$$

Let $l \leq h' - 1$. Then the contributions to the $\pi^l$-coefficient coming from $\gamma \in S_n$ are of the form

$$\prod_{i=1}^{n} A_{i,\gamma(i),l_i} = \prod_{i=1}^{n} M_{(i,\gamma(i),l_i^*)},$$

where $(l_1, \ldots, l_n)$ is a partition of $l$. Then

$$|(i, \gamma(i), l_i^*)| = |\gamma(i) - i| + \frac{n}{m} (l_i^* - 1) = |\gamma(i) - i| + \frac{n}{m} \cdot l_i,$$

and therefore

$$\sum_{i=1}^{n} |(i, \gamma(i), l_i^*)| = \sum_{i=1}^{n} |\gamma(i) - i| + \frac{n}{m} \cdot l_i = \sum_{i=1}^{n} \frac{n}{m} \cdot l_i \leq \frac{n}{m}(h' - 1).$$

It is now clear that (a) holds. The assumption in (b) implies that the monomial corresponds to a transposition $\gamma$, so that if $\lambda_1 = (i, j, l)$, then $\lambda_2 = (j, i, l')$. By assumption,

$$\frac{n}{m} (h' - 1) = |\lambda_1| + |\lambda_2| = |j - i| + \frac{n}{m} (l - 1) + |i - j| + \frac{n}{m} (l' - 1) = \begin{cases} \frac{n}{m} (l + l' - 1) & \text{if } [i] \neq [j], \\ \frac{n}{m} (l + l' - 2) & \text{if } [i] = [j], \end{cases}$$

and solving for $l'$ shows that $\lambda_2 = \lambda_1^\gamma$. \hfill \Box

Example 5.11. We explicitly calculate the indexing sets $\mathcal{A}^+, \mathcal{A}^+, \min$ in the cases $m = 1$ and $m = n$.

(i) The case $m = 1$ corresponds to the division algebra case. If $(i, [j + i - 1])$ satisfies the condition that $n$ divides one of $\{i, i+1, \ldots, i+j-2\}$, then necessarily $n \leq i+j-2$ and so $n < i+j-1$. This implies $j-1 > n-i$ and so $i > [i+j-1]$, which means that $(i, [j+i-1])$ lies below the diagonal. Hence we have that $(i,j,l) \in \mathcal{A}^+$ if and only if

$$\begin{cases} 1 \leq l \leq h-1 & \text{if } i = j; \text{ i.e. } (i,j) \text{ lies on the diagonal}, \\ 1 \leq l \leq h-1 & \text{if } i > j; \text{ i.e. } (i,j) \text{ lies below the diagonal}, \\ 1 \leq l \leq h & \text{if } i < j; \text{ i.e. } (i,j) \text{ lies above the diagonal}. \end{cases}$$

Every entry of $x$ below the diagonal lies in $V \mathcal{W}_{h-1}$. Note that the contribution of a $\pi^{h-1}$ term above the diagonal to $\det(x) \in \mathbb{W}_h^{(1)}$ must be paired with a term below the diagonal. Since all terms below the diagonal lie in $V \mathcal{W}_{h-1}$, this implies that the $\pi^{h-1}$ can only contribute to $V^h \mathcal{W}$, which is trivial in $\mathbb{W}_h^{(1)}$. Similarly elementary considerations imply that the only $\lambda$ in $\mathcal{A}^+$ wherein $x_\lambda$ contributes nontrivially to
\[ \det(x) \text{ are exactly those } \lambda \text{ in } \]
\[ A^{+, \min} = \bigcup_{j=1}^{n} \{(i, j, l) : 1 \leq i \leq n, 1 \leq l \leq h - 1\}. \]

(ii) The case \( m = n \) corresponds to the split form \( \text{GL}_m(F) \). We have \((i, j, l) \in A^+\) if and only if
\[ \begin{cases} 1 \leq l \leq h - 1 & \text{if } i = j; \text{i.e. } (i, j) \text{ lies on the diagonal}, \\ 1 \leq l \leq h - 1 & \text{if } i \neq j; \text{i.e. } (i, j) \text{ does not lie on the diagonal}. \end{cases} \]
Every entry of \( x \) off the diagonal lies in \( V^\mathcal{W}_{h-1} \) and hence if \( \lambda = (i, j, l) \) where \( i \neq j \), then \( x_\lambda \) can contribute nontrivially to \( \det(x) \in \mathcal{W}_h^1 \) if and only if \((l-1)+2 \leq h-1\). Hence we have
\[ A^{+, \min} = \{(i, i, l) : 1 \leq i \leq n, 1 \leq l \leq h - 1\} \cup \]
\[ \bigcup_{j=1}^{n-1} \{(i, \sigma^j(i), l) : 1 \leq i \leq n, 1 \leq l \leq h - 2\}. \]

Remark 5.12. A foreshadowing remark about cohomological degrees. Specialize to \( m = 1 \). The indexing set \( A^{+, \min} \) defined in Definition 5.6 corresponds to the indexing set \( A^+ \) of [Cha16a]. By construction (see Definition 5.3), we see that the conditions cutting out \( X \) can be written in terms of polynomials in \( x_\lambda \) for \( \lambda \in A^- \setminus A^{-, \min} \). Comparing this to the definition of \( X_h \) in [Cha16a, Definition 2.7, 2.9], we see that \( X = X_h \times \mathbb{A}^{n-1} \), where we note that \( \#(A^- \setminus A^{-, \min}) = n - 1 \). The dimension of \( X \) is \((n-1)h\). We therefore have
\[ H_c^{l+2(n-1)}(X, \overline{\mathbb{Q}}_\ell) \cong H_c^j(X_h, \overline{\mathbb{Q}}_\ell)(n-1), \]
where the \((n-1)\) denotes the \( n \)th Tate twist; that is, that the \( \text{Fr}_{q^n} \)-action is multiplied by \( q^{-n(n-1)} \). In particular, if \( \chi \) is a primitive character of level \( n \), the \( \chi \)-eigenspace of the cohomology of \( X \) is concentrated in degree \((n-1)(h+1)\) (non-middle degree), whereas the \( \chi \)-eigenspace of the cohomology of \( X_h \) is concentrated in degree \((n-1)(h-1)\) (middle degree). This might be disconcerting, but is ok for the following reason: moving to \( \acute{\text{e}} \)tale homology, we have
\[ H_i(X, \overline{\mathbb{Q}}_\ell) \cong H_c^{2(n-1)h-i}(X, \overline{\mathbb{Q}}_\ell)((n-1)h) \]
\[ \cong H_c^{2(n-1)h-i-2(n-1)}(X_h, \overline{\mathbb{Q}}_\ell)((n-1)h - (n-1)) \]
\[ \cong H_c^{2(n-1)(h-1)-i}(X_h, \overline{\mathbb{Q}}_\ell)((n-1)(h-1)) \cong H_i(X_h, \overline{\mathbb{Q}}_\ell). \]
This is a concrete example of why it is better to consider the homology, rather than the cohomology, of pro-schemes.

5.2. Cohomology of \( Z_h^{(1)} \). The following theorems now hold by the nearly the same proofs as in [Cha16a] (see [Cha16a, Remark 5.6]). We record how the various results in this section
correspond to the various results in [Cha16a] in the following table:

<table>
<thead>
<tr>
<th>Current paper</th>
<th>Cha16a</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 5.13</td>
<td>Theorem 4.1</td>
</tr>
<tr>
<td>Corollary 5.15</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>Theorem 5.17</td>
<td>Theorem 5.2</td>
</tr>
<tr>
<td>Theorem 5.18</td>
<td>Theorem 5.3</td>
</tr>
<tr>
<td>Theorem 5.19</td>
<td>Theorem 5.4</td>
</tr>
<tr>
<td>Theorem 5.20</td>
<td>Theorem 5.5</td>
</tr>
<tr>
<td>Theorem 5.21</td>
<td>Theorem 5.6</td>
</tr>
</tbody>
</table>

By far the most nontrivial result is Theorem 5.13, which we prove in Section 5.2.1. Once this has been established, then one can obtain Theorems 5.17 and 5.18 by combining Theorem 5.13 with Theorem 5.16, a theorem of Lusztig [Lus04] (char \( K > 0 \)) and Stasinski [Sta09] (char \( K = 0 \)). Theorems 5.19 and 5.20 are proved using the Deligne–Lusztig fixed-point formula together with Theorems 5.17 and 5.19, though the latter is significantly more nontrivial than the former. Theorem 5.21 is an easy corollary of Theorems 5.13, 5.17, 5.18, and 5.20.

**Theorem 5.13.** For any character \( \chi : \mathbb{T}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times \),

\[
\text{Hom}_{\mathcal{U}(\mathbb{F}_q)} \left( \text{Ind}_{\mathcal{T}(\mathbb{F}_q)}^{\mathcal{U}(\mathbb{F}_q)}(\chi), H^r_c(X, \overline{\mathbb{Q}}_\ell) \right) = \begin{cases} 
\overline{\mathbb{Q}}_\ell \otimes ((-q^{n/2})^{r_\chi})^{\deg} & \text{if } i = r_\chi, \\
0 & \text{otherwise},
\end{cases}
\]

where

\[
d_\chi = \sum_{i=1}^r \left( \frac{n}{m_{i-1}} - \frac{n}{m_i} \right) (h_i - 1),
\]

\[
r_\chi = 2(n' - 1) + \sum_{i=1}^r \left( \frac{n}{m_{i-1}} - \frac{n}{m_i} \right) (h_i - 1) + 2 \left( \frac{n}{m_i} - 1 \right) (h_i - h_{i+1})
\]

Moreover, \( \text{Fr}_{q^n} \) acts on \( H^r_c(X, \overline{\mathbb{Q}}_\ell) \) by multiplication by the scalar \((-1)^i q^{ni/2}\).

**Remark 5.14.** Following Katz [Kat01], we write \( \alpha^{\deg} \) to mean the rank-1 \( \overline{\mathbb{Q}}_\ell \)-sheaf with the action of \( \text{Fr}_{q^n} \) given by multiplication by \( \alpha \).  

**Corollary 5.15.** Let \( \pi \) be an irreducible constituent of \( H^r_c(X_h, \overline{\mathbb{Q}}_\ell) \) for some \( r \). Then \( \text{Hom}_{\mathcal{U}_h(\mathbb{F}_q)}(\pi, H^r_c(X_h, \overline{\mathbb{Q}}_\ell)) = 0 \), for all \( i \neq r \).

**Theorem 5.16** (Lusztig [Lus04], Stasinski [Sta09]). Let \( R_\chi := \sum_i (-1)^i H^i(X, \overline{\mathbb{Q}}_\ell)[\chi] \). For each \( \chi : \mathbb{T}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times \), the \( \mathcal{U}(\mathbb{F}_q) \)-representation \( \pm R_\chi \) is irreducible. If \( \chi \neq \chi' \), then \( \pm R_\chi, \pm R_{\chi'} \) are nonisomorphic.

**Theorem 5.17.** \( X \) is a maximal variety in the sense of Boyarchenko–Weinstein [BW16]. That is, for each \( i \geq 0 \), we have \( H^i_c(X, \overline{\mathbb{Q}}_\ell) = 0 \) unless \( i \) or \( n \) is even, and the Frobenius morphism \( \text{Fr}_{q^n} \) acts on \( H^i_c(X, \overline{\mathbb{Q}}_\ell) \) by the scalar \((-1)^i q^{ni/2}\).

**Theorem 5.18.** For any \( \chi : \mathbb{T}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times \), the cohomology groups \( H^i_c(X, \overline{\mathbb{Q}}_\ell)[\chi] \) are nonzero in a single degree \( i = s_\chi \), and \( H^c(X, \overline{\mathbb{Q}}_\ell)[\chi] \) is an irreducible representation of \( \mathcal{U}(\mathbb{F}_q) \). Moreover, \( H^c(X, \overline{\mathbb{Q}}_\ell)[\chi] \cong H^{c_x}(X, \overline{\mathbb{Q}}_\ell)[\chi'] \) if and only if \( \chi = \chi' \).

**Theorem 5.19.** For any \( \zeta \) of \( \mathbb{F}_{q^n}^\times \) with trivial stabilizer in \( \text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \),

\[
\text{Tr}((\zeta, 1, g)^*: H^{c_x}(X, \overline{\mathbb{Q}}_\ell)[\chi]) = (-1)^{s_x} \chi(g).
\]
Theorem 5.20. For any $\chi: T(F_q) \to \mathcal{O}_x^+$, 
\[ \text{Hom}_U(T(F_q)) \left( \text{Ind}_{W(T(F_q))}^U(\chi), H^*_c(X, \mathcal{O}_\ell)[\chi] \right) \neq 0. \]
In particular, $s_\chi = r_\chi$.

Theorem 5.21. The zeta function of $X$ is
\[ Z(t) = \prod_{i=0}^{2 \dim X} \left( 1 - (-q^{n/2})^i \cdot t \right)^{(-1)^{i+1} \dim H^i_c(X, \mathcal{O}_\ell)}, \]
where
\[ \dim H^i_c(X, \mathcal{O}_\ell) = \sum_{\chi: T(F_q) \to \mathcal{O}_\ell^+} q^{n_{\chi}/2}. \]
Moreover, if $n$ is odd, then $Z(t)^{-1}$ is a polynomial.

5.2.1. Proof of Theorem 5.13. Recall that over $F_q$, the quotient $U/T$ can be identified with the affine space $A[A]$. There is a natural section of the quotient map $U \to U/T$ given by
\[ (x_\lambda)_{\lambda \in A} \mapsto 1_n + x_2[b] + \cdots + x_n[b^{n-1}], \]
where we write
\[ x = \text{diag}(x_{1,1}, x_{2,2} + 1, \ldots, x_{n,n} + 1), \quad x_{i,j} = [x_{(i,j,1)}, \ldots, x_{(i,j,h_{ij})}] \]
Here, $h_{ij}$ is the maximal $l$ such that $(i, j, l) \in A$. Let $Y := L_q(X)$ and note that since $X$ is an affine variety and $L_q$ is an étale morphism, $Y$ is affine. Define
\[ \beta: (U/T) \times T \to U, \quad (x, g) \mapsto s(F(x)) \cdot g \cdot s(x)^{-1}. \]
Now the $F_q$-scheme $\beta^{-1}(Y) \subset (U/T) \times T$ comes with two maps:
\[ \text{pr}_1: \beta^{-1}(Y) \to U/T = A[A], \quad \text{pr}_2: \beta^{-1}(Y) \to T. \]
Then by [Cha16a] Proposition 4.4., for $i \geq 0$, we have $F_q$-compatible isomorphisms
\[ \text{Hom}_U(T(F_q)) \left( \text{Ind}_{W(T(F_q))}^U(\chi), H^i_c(X, \mathcal{O}_\ell) \right) \cong H^i_c(\text{pr}_1^{-1}(A[A]), P^*\mathcal{L}_\chi), \tag{5.1} \]
where $P: \text{pr}_1^{-1}(A[A]) \to W_h^{(1)}$ is the morphism $(x, g) \mapsto L_q(\det(s(x)))^{-1}$. By construction, $\det(s(x))^{-1}$ only depends on the coordinates coming from $A_{\text{min}}$, and so
\[ \text{pr}_1^{-1}(A[A]) = \text{pr}_1^{-1}(A[A_{\text{min}}]) \times \{ (x_\lambda)_{\lambda \in A_{\text{min}} : x_{i,\sigma(i)} = \varphi^{(i)}(x_{1,\sigma(i)})} \}
= \text{pr}_1^{-1}(A[A_{\text{min}}]) \times A[A^- \setminus A_{\text{min}}]. \]
Moreover, we have the diagram
\[ \begin{array}{ccc}
\text{pr}_1^{-1}(A[A]) & \xrightarrow{P} & W_h^{(1)} \\
& \searrow & \swarrow P \\
& \text{pr}_1^{-1}(A[A_{\text{min}}]) & \\
\end{array} \]
and therefore
\[ H^i_c(\text{pr}_1^{-1}(A[A]), P^*\mathcal{L}_\chi) \cong H^i_c(\text{pr}_1^{-1}(A[A_{\text{min}}]), P^*\mathcal{L}_\chi)[2r_{\text{min}}] \otimes ((q^n)^{r_{\text{min}}})^{\text{deg}}, \tag{5.2} \]
where $r_{\text{min}} = \#(A^- \setminus A_{\text{min}})$.

Let $A_{\text{d,\ell}}$ be the indexing sets associated to $\chi: W_h^{(1)}(F_q) \cong T(F_q) \to \mathcal{O}_\ell^+$. Recall that $A_{\text{min}} = A^0_{\geq 0,1}$. Recall the following two propositions from [Cha16a].
Proposition 5.22. We have $\text{Fr}_{q^n}$-compatible isomorphisms

$$H_c^i(\text{pr}_1^{-1}(\mathbb{A}[A_{\geq t}^{\min}])), P^*L_{X_{\geq t}}) \cong H_c^i(\text{pr}_1^{-1}(\mathbb{A}[A_{\geq t,t+1}]^{\min}), P^*L_{X_{\geq t+1}}, [2e_t] \otimes (-(q^{n/2})^{2e_t})^{\deg},$$

where $e_t = \#(A_{\geq t,t} \setminus A_{\geq t,t+1})$.

Proposition 5.23. For $1 \leq t \leq r$, we have $\text{Fr}_{q^n}$-compatible isomorphisms

$$H_c^i(\text{pr}_1^{-1}(\mathbb{A}[A_{\geq t-1,t}^{\min}])), P^*L_{X_{\geq t}}) \cong H_c^i(\text{pr}_1^{-1}(\mathbb{A}[A_{\geq t-1,t}^{\min}])), P^*L_{X_{\geq t}}, [d_t] \otimes (-(q^{n/2})^{d_t})^{\deg},$$

where $d_t = \#A_{t-1,t}^{\min}$.

Define

$$d_\chi := d_1 + \cdots + d_r,$$

$$e_\chi := e_0 + e_1 + \cdots + e_r,$$

where $e_0 = (n - 1)(h - h_1)$ and for $1 \leq t \leq r$,

$$d_t = \#A_{t-1,t}^{\min}$$

$$= \{(1, j, l) : m_{t-1} \mid j - 1, m_t \mid j - 1, 1 \leq l \leq h_t - 1\}$$

$$= \left(\frac{n}{m_{t-1}} - \frac{n}{m_t}\right)(h_t - 1),$$

$$e_t = \#(A_{t-1,t}^{\min} \setminus A_{t-1,t+1}^{\min})$$

$$= \#\{(1, j, l) : m_t \mid j - 1, j \neq 1, \frac{n}{m}(h_{t+1} - 1) < l \leq \frac{n}{m}(h_t - 1)\}$$

$$= \left(\frac{n}{m_t} - 1\right)(h_t - h_{t+1}).$$

We therefore have

$$H_c^i(\text{pr}_1^{-1}(\mathbb{A}[A_{\geq 0,0}^{\min}], P^*L_{X}))$$

$$\cong H_c^i(\text{pr}_1^{-1}(\mathbb{A}[A_{\geq 0,1}^{\min}], P^*L_{X})[2e_0] \left(-q^{n/2})^{2e_0}\right)^{\deg}$$

(Prop 5.22)

$$\cong H_c^i(\text{pr}_1^{-1}(\mathbb{A}[A_{\geq 1,1}^{\min}], P^*L_{X}) \otimes q^{d_1/2} \left(-q^{n/2})^{2e_0+d_1}\right)^{\deg}$$

(Prop 5.23),

$$\cong H_c^i(\text{pr}_1^{-1}(\mathbb{A}[A_{\geq 1,2}^{\min}], P^*L_{X}) \otimes q^{d_1} \left(-q^{n/2})^{2e_0+d_1+2e_1}\right)^{\deg}$$

(Prop 5.22)

$$\cong H_c^i(\text{pr}_1^{-1}(\mathbb{A}[A_{\geq r+1}^{\min}], P^*L_{X}) \otimes q^{d_\chi} \left(-q^{n/2})^{d_\chi+2e_\chi}\right)^{\deg}$$

(Prop 5.23, 5.22)

$$= \begin{cases} Q_{\ell}^{2q^{d_\chi/2}} \otimes (-(q^{n/2})^{d_\chi+2e_\chi})^{\deg} & \text{if } i = d_\chi + 2e_\chi, \\ 0 & \text{otherwise.} \end{cases}$$

Combining this with Equations (5.1) and (5.2), we obtain

$$\text{Hom}_{\mathbb{U}(\mathbb{F}_q)}(\text{Ind}_{\mathbb{U}(\mathbb{F}_q^{\chi})}^{\mathbb{U}(\mathbb{F}_q^{\chi})}(X), H_c^i(X, \mathbb{Q}_{\ell}))$$

$$\cong \begin{cases} Q_{\ell}^{2q^{d_\chi/2}} \otimes (-(q^{n/2})^{2r_{\min}+d_\chi+2e_\chi})^{\deg} & \text{if } i = 2r_{\min} + d_\chi + 2e_\chi, \\ 0 & \text{otherwise.} \end{cases}$$
To finish, observe that
\[ d_\chi = \sum_{t=1}^r \left( \frac{n}{m_{t-1}} - \frac{n}{m_t} \right) (h_t - 1), \]
\[ r_\chi := 2r_{\min} + d_\chi + 2e_\chi \]
\[ = 2(n' - 1) + \sum_{t=1}^r \left( \frac{n}{m_{t-1}} - \frac{n}{m_t} \right) (h_t - 1) + 2 \left( \frac{n}{m_t - 1} \right) (h_t - h_{t+1}) \].

6. Deligne–Lusztig varieties for reductive groups over finite rings

In this section, we compute the cohomology of \( X_h \). In Section 6.1, we will see that there is a very close relationship between the cohomologies of \( Z_h \) and \( Z_h^{(1)} \). The relationship between \( X_h \), \( Y_h \), and \( Z_h \) is explained by the following observation, which is exploited in detail in Section 6.2. The open/closed decomposition
\[ Y_h \to X_h \to Z_h. \]

This induces a short exact sequence of sheaves
\[ 0 \to j_* \overline{\mathcal{O}}_\ell \to \overline{\mathcal{O}}_\ell \to i_* \overline{\mathcal{O}}_\ell \to 0 \]
which in turn induces a long exact sequence of cohomology groups
\[ 0 \to H^0_c(Y_h, \overline{\mathcal{O}}_\ell) \to H^0_c(X_h, \overline{\mathcal{O}}_\ell) \to H^0_c(Z_h, \overline{\mathcal{O}}_\ell) \]
\[ \to H^1_c(Y_h, \overline{\mathcal{O}}_\ell) \to H^1_c(X_h, \overline{\mathcal{O}}_\ell) \to H^1_c(Z_h, \overline{\mathcal{O}}_\ell) \to \cdots \]

Hence we see that to understand the cohomology of \( X_h \), we need to understand the cohomology of \( Y_h \), the cohomology of \( Z_h^{(1)} \), and the morphisms between them.

Our aim is to understand the \( G_h \)-representations \( H^i_c(X_h, \overline{\mathcal{O}}_\ell)[\theta] \) for any character \( \theta: T_h \to \overline{\mathcal{O}}_\ell^\times \). The behavior is roughly as follows. If \( \theta \) is very degenerate, then \( H^i_c(X_h, \overline{\mathcal{O}}_\ell)[\theta] \) is controlled by the cohomology of \( X_1 \), a classical Deligne–Lusztig variety associated to \( \GL_n(k) \). Otherwise, \( H^i_c(X_h, \overline{\mathcal{O}}_\ell)[\theta] \) is controlled by the cohomology of \( Z_h \), whose cohomology is intern controlled by the cohomology of \( Z_h^{(1)} \), an analogue of a Deligne–Lusztig variety for unipotent groups. To distinguish between these two vastly different behaviors, we introduce the following notation:

\[ T_h := \{ \theta: T_h \cong \mathcal{O}_L^\times / U \to \overline{\mathcal{O}}_\ell^\times \}, \]
\[ T_h^{\text{gen}} := \{ \theta \in T_h: \theta|_{U_L^1} \text{ does not factor through } \text{Nm} \}, \]
\[ T_h^0 := \{ \theta \in T_h: \theta|_{U_L^1} \text{ factors through } \text{Nm} \} = T_h \setminus T_h^{\text{gen}}, \]
where \( \text{Nm}: U_L^1 \to U_K^1 \) is the norm map. If \( \theta \in T_h^0 \), then we may write \( \theta = \overline{\theta} \otimes \theta_0(\text{Nm}) \) for some \( \overline{\theta}: \mathcal{O}_K^\times \to \overline{\mathcal{O}}_\ell^\times \) and \( \theta_0: U_K^1 \to \overline{\mathcal{O}}_\ell^\times \). With respect to this decomposition, set \( \overline{\theta} := \theta_0(\text{det}) \).

We now record the main theorem of this section, which we prove in Section 6.2.

**Theorem 6.1.** For each character \( \theta: T_h \to \overline{\mathcal{O}}_\ell^\times \),
\[ H^i_c(X_h, \overline{\mathcal{O}}_\ell)[\theta] = \begin{cases} H^i_c(Z_h, \overline{\mathcal{O}}_\ell)[\theta] & \text{if } i \leq 2(n-1)(h-1) \text{ and } \theta \in T_h^{\text{gen}}, \\ H^i_c(\mathcal{O}_L/\mathcal{O}_K)[\overline{\theta}] \otimes \overline{\theta} & \text{if } i > 2(n-1)(h-1) \text{ and } \theta \in T_h^0. \end{cases} \]

We note that \( X_1 \) is a Deligne–Lusztig variety associated to the finite reductive group \( \GL_{\text{gcd}(n,\ell)}(\mathbb{F}_q) \), whose cohomology is known. Therefore, for Theorem 6.1, to give a complete description of the cohomology of \( H^i_c(X_h, \overline{\mathcal{O}}_\ell) \), it remains to understand \( H^i_c(Z_h, \overline{\mathcal{O}}_\ell) \). To this end, we have the following theorem, which we prove in Section 6.1.
Theorem 6.2. Let \( \theta : T_h \to \overline{\mathbb{Q}}_\ell^\times \) be a character and set \( \chi := \theta|_{T_h^{(1)}} \). Then as representations of \( T_h \times G_h \),
\[
H^i_c(Z_h, \overline{\mathbb{Q}}_\ell)[\theta] \cong \theta \otimes \text{Ind}^{G_h}_{T_h \times G_h} (\eta_\theta^i),
\]
where \( \eta_\theta^i \) is the extension of the \( G_h^{(1)} \)-representation \( H^i_c(Z_h^{(1)}, \overline{\mathbb{Q}}_\ell)[\chi] \) where for any \( \zeta \in T_h \) whose image in \( \mathbb{F}_{q^n}^\times \) has trivial \( \text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \)-stabilizer, \( \text{Tr} (\zeta; \eta_\theta) = (-1)^i \theta(\zeta) \).

6.1. Cohomology of \( Z_h \). In this section, we prove some results relating the varieties \( Z_h \) and \( Z_h^{(1)} \). The variety \( Z_h \) is a disjoint union of copies of \( Z_h^{(1)} \) in a precise way, which allows us to describe the \( T_h \times Z_h \)-representation \( H^i_c(Z_h, \overline{\mathbb{Q}}_\ell) \) in terms of the \( G_h \)-representation \( H^i_c(Z_h^{(1)}, \overline{\mathbb{Q}}_\ell) \).

Consider the following subgroups of \( T_h \times G_h \):
\[
\Gamma_h := \{(\zeta, \zeta^{-1}) : \zeta \in T_h \}
\]
\( (T_h^{(1)} \times G_h^{(1)}) \),
\[
\tilde{\Gamma}_h := (T_h \times \{1\}) \cdot \Gamma_h.
\]
Observe that \( \Gamma_h \) is the stabilizer of \( Z_h^{(1)} \subset Z_h \) in \( T_h \times G_h \).

Proposition 6.3. (a) Let \( g \in (G_h \times T_h) \setminus \Gamma_h \). Then
\[
Z_h^{(1)} \cap g \cdot Z_h^{(1)} = \emptyset.
\]
Moreover,
\[
Z_h = \bigcup_{g \in (G_h \times T_h)/\Gamma_h} g \cdot Z_h^{(1)}.
\]

(b) As representations of \( G_h \times T_h \),
\[
H^i_c(Z_h, \overline{\mathbb{Q}}_\ell) \cong \text{Ind}_{\Gamma_h \times T_h}^{G_h \times T_h} \left( H^i_c(Z_h^{(1)}, \overline{\mathbb{Q}}_\ell) \right).
\]

Corollary 6.4. For each character \( \theta : T_h \to \overline{\mathbb{Q}}_\ell^\times \), there exists an integer \( r \) with \( (n-1)(h-1) + (\gcd(\kappa_{G}(b))) = 2(n-1)(h-1) \) such that
\[
H^r_c(Z_h, \overline{\mathbb{Q}}_\ell)[\theta] \neq 0 \quad \iff \quad i = r.
\]
Moreover, \( r = r_\chi \), where \( \chi := \theta|_{T_h^{(1)}} \) and \( r_\chi \) as in Theorem 5.13.

Proof. This follows from Proposition 6.3(b), Theorem 5.18, and Theorem 5.20.

We now prove Theorem 6.2.

Proof of Theorem 6.2. By Proposition 6.3, we have that
\[
H^i_c(Z_h, \overline{\mathbb{Q}}_\ell) \cong \text{Ind}_{\Gamma_h \times T_h}^{G_h \times T_h} \left( H^i_c(Z_h^{(1)}, \overline{\mathbb{Q}}_\ell) \right).
\]
By Corollary 6.4, we need only prove the assertion of the theorem for when \( i = r := r_\chi \), where \( \chi := \theta|_{T_h^{(1)}} \). We define \( \eta_\theta \) to be the \( \Gamma_h \)-representation \( H^i_c(Z_h^{(1)}, \overline{\mathbb{Q}}_\ell)[\chi] \). By Theorem 6.19, we know that for any \( \zeta \in T_h \cong \mathcal{O}_L^\times / U_L^h \) whose image in \( \mathbb{F}_{q^n}^\times \) has trivial \( \text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \)-stabilizer,
\[
\text{Tr} ((\zeta^{-1}, \zeta); \eta_\theta) = (-1)^r.
\]
Then \( \eta_\theta \) extends to a representation \( \tilde{\eta}_\theta \) of \( \tilde{\Gamma}_h \) whose restriction to \( T_h \times \{1\} \) is given by \( \theta \). Since \( (1, \zeta) = (\zeta, 1) \cdot (\zeta^{-1}, \zeta) \) and \( (\zeta, 1) \) acts on \( H^i_c(Z_h, \overline{\mathbb{Q}}_\ell)[\theta] \) by \( \theta(\zeta) \) by assumption, the statement of the theorem follows.

The following proposition is crucial for the work in Section 6.2.

Proposition 6.5. Fix \( \theta \in \mathcal{T}_h^0 \) and let \( r_\chi \) be as in Theorem 5.13 for \( \chi := \theta|_{T_h^{(1)}} \).

(a) Then
\[
H^i_c(Z_h, \overline{\mathbb{Q}}_\ell)[\theta] = 0, \quad \text{for } i < 2(n-1)(h-1).
\]
(b) Moreover, as representations of $T_h \times G_h$,

$$H^0_c(Z_h, \overline{\mathbb{Q}_l})[\theta] \cong H^0_c(Z_1, \overline{\mathbb{Q}_l})[\theta] \otimes (\theta_1 \otimes \overline{\theta}).$$

Proof. Part (a) is clear from Theorem 6.2. Moreover,

$$H^1_c(Z_h(1), \overline{\mathbb{Q}_l})[\theta_1] \neq 0 \iff i = 2(n - 1)(h - 1)$$

and $\text{Fr}_{q^n}$ acts by multiplication by $q^{(n-1)(h-1)}$. Then Lemma 6.12 implies that for any $g \in G_h(1)$,

$$\text{Tr} \left( g; H^2_c(n-1)(h-1)(Z_h(1), \overline{\mathbb{Q}_l})[\theta_1] \right) = \frac{1}{C} \sum_{t \in T_h(1)} \theta_1(t) \cdot \#\{ x \in Z_h(1)(\overline{\mathbb{F}}_q) : F_{q^n}(x) \cdot g = t \cdot x \}. $$

Here, $C = q^{2(n-1)(h-1)} \cdot \#T_h(1)$. It is clear that if $\det(t) \neq \det(g)$, then $A_{t,g} = 0$. Hence

$$\text{Tr} \left( g; H^2_c(n-1)(h-1)(Z_h(1), \overline{\mathbb{Q}_l})[\theta_1] \right) = \frac{1}{C} \sum_{\substack{t \in T_h(1) \\det(t) = \det(g)}} \theta_0(\det(t)) \cdot A_{t,g} = C' \cdot \theta_0(\det(g)).$$

Since $H^2_c(n-1)(h-1)(Z_h(1), \overline{\mathbb{Q}_l})[\theta_1]$ is nonzero and irreducible, then in fact the above equation implies that $C' = 1$. We therefore see that as representations of $T_h(1) \times G_h(1)$, we have

$$H^2_c(n-1)(h-1)(Z_h(1), \overline{\mathbb{Q}_l})[\theta_1] \cong \theta_1 \otimes \overline{\theta}.$$
for some $C \in \mathbb{F}_q$ determined by $z$. Making the change-of-variables $y_h \mapsto x_0^{1/q} \left( y_h^q + \frac{y_0}{y_0^{1/q}} \cdot x_h \right)$ changes the equation to

$$x_0^q x_1 - y_0^q x_0 - x_0^{1/q} \cdot y_0^q y_1 - x_0^{1/q} \frac{y_0}{y_0^{1/q}} \cdot x_1 = C,$$

and hence the coefficient of $x_1$ is $x_0^q - x_0^{1/q} y_0^{1-1/q}$, and this is equal to zero if and only if $y_0 \in \mathbb{F}_q^2 \cdot x_0$, which implies $x_0, y_0 \in \mathbb{F}_q^2$. But this is not the case since $z \in Y_h$ by assumption, and therefore the map

$$\{(x_h, y_h) : x_0 x_h^q + x_0^q x_h - y_0 y_h - y_0^q y_h = C\} \xrightarrow{pr_2} \mathbb{A}^1$$

is an isomorphism.

\[\square\]

**Lemma 6.7.** As representations of $T_h \times G_h$,

$$H^*_c(Y_h, \overline{\mathbb{k}}_\ell) \cong \text{Ind}^{T_h \times G_h}_{\Gamma'_h} \left( H^*_{c-2(n-1)(h-1)}(Y_1, \overline{\mathbb{k}}_\ell) \right)$$

where $\Gamma'_h \subset T_h \times G_h$ is the subgroup

$$\Gamma'_h := (T_1 \times G_1) \cdot \{ (t, g) \in T_h^{(1)} \times G_h^{(1)} : (t, g) \in \ker(\det) \},$$

where $\det : T_h^{(1)} \times G_h^{(1)} \to (T_h^{(1)})^\sigma \times (T_h^{(1)})^\sigma \to (T_h^{(1)})^\sigma$. Moreover, as representations of $T_h \times G_h$,

$$H^*_c(Y_h, \overline{\mathbb{k}}_\ell)[\theta] \cong \begin{cases} 0 & \text{if } \theta \in T_h^0, \\ H^*_{c-2(n-1)(h-1)}(Y_1, \overline{\mathbb{k}}_\ell)[\theta] \otimes (\theta_1 \otimes \theta) & \text{if } \theta \in T_h^{\text{gen}}, \end{cases}$$

where $H^*_c(Y_1, \overline{\mathbb{k}}_\ell)[\theta]$ is the $(T_h \times G_h)$-representation obtained by pulling back along the natural surjection $T_h \times G_h \to T_1 \times G_1$.

**Proof.** We first show that $\Gamma'_h$ is the stabilizer of $i(Y_1 \times \mathbb{A}^{(n-1)(h-1)})$ in $T_h \times G_h$, where

$$i : Y_1 \times \mathbb{A}^{(n-1)(h-1)} \hookrightarrow Y_h.$$ 

By construction $Y_h$ has natural actions of $T_h \times G_h$. It is clear that $T_1 \times G_1 \hookrightarrow T_h \times G_h$ stabilizes $i(Y_1) \times \mathbb{A}^{(n-1)(h-1)}$. By Lemma 6.6, we know that $(t, g) \in T_h^{(1)} \times G_h^{(1)}$ stabilizes $i(Y_1 \times \mathbb{A}^{(n-1)(h-1)})$ if and only if $(t, g)$ fixes the determinant. This implies that

$$\text{Stab} \left( i(Y_1 \times \mathbb{A}^{(n-1)(h-1)}) \right) = \Gamma'_h. \tag{6.4}$$

Let $\Gamma''_h := \Gamma'_h \cap (T_h^{(1)} \times G_h^{(1)})$. Then $\Gamma''_h$ stabilizes $\{ x \} \times \mathbb{A}^{(n-1)(h-1)} \subset Y_1 \times \mathbb{A}^{(n-1)(h-1)}$ and so as representations of $\Gamma''_h$,

$$H^*_c(Y_1 \times \mathbb{A}^{(n-1)(h-1)}, \overline{\mathbb{k}}_\ell) \cong H^*_{c-2(n-1)(h-1)}(Y_1, \overline{\mathbb{k}}_\ell) \otimes H^*_c(Y_1 \times \mathbb{A}^{(n-1)(h-1)}, \overline{\mathbb{k}}_\ell).$$

Since the cohomology of $\mathbb{A}^{(n-1)(h-1)}$ is concentrated in degree $2(n-1)(h-1)$, by [Boy12, Lemma 2.12], we know that the subspace $V$ wherein $\Gamma''_h$ acts by the trivial character has the property that

$$\text{Tr}(1; V) = \frac{(-1)^{2(n-1)(h-1)}}{q^{2(n-1)(h-1)} \cdot \#\Gamma''_h} \sum_{\gamma \in \Gamma''_h} \# \{ x \in \mathbb{A}^{(n-1)(h-1)}(\overline{\mathbb{F}}_q) : F_q(x) = \gamma \ast x \}.$$

To show that $V = H^*_{c-2(n-1)(h-1)}(\mathbb{A}^{(n-1)(h-1)}, \overline{\mathbb{k}}_\ell)$, it is enough to exhibit a $\gamma \in \Gamma''_h$ such that $A_{\gamma} \neq 0$. This is not so hard: for any $t \in T_h^{(1)}$, we have $\gamma_t := (t^{-1}, t) \in \Gamma''_h$ and $\gamma_t$ acts on $0 \in \mathbb{A}^{(n-1)(h-1)}$ trivially. Moreover, $F_q(0) = 0$, so we see that $A_{\gamma_t} \neq 0$. We have thus shown
that as $\Gamma_h'$-representations,

$$H^i_c(Y_1 \times \mathbb{A}^{(n-1)(h-1)}, \overline{\mathcal{O}}_\ell) \cong H^i_c(\mathbb{A}^{(n-1)(h-1)}(Y_1, \overline{\mathcal{O}}_\ell)),$$

where the $\Gamma_h'$-action on $H^i_c(Y_1, \overline{\mathcal{O}}_\ell)$ is the pullback of the natural action along the surjection $\Gamma_h' \to T_1 \times G_1$. Putting this together with Equation (6.4), we have

$$H^i_c(Y_h, \overline{\mathcal{O}}_\ell) \cong \text{Ind}_{\Gamma_h}^{T_h \times G_h} \left( H^i_c(\mathbb{A}^{(n-1)(h-1)}(Y_1, \overline{\mathcal{O}}_\ell)) \right).$$

(6.5)

Note that by Equation (6.3), it is clear that if $\theta \in T^\text{gen}_h$, then $H^i_c(Y_h, \overline{\mathcal{O}}_\ell)[\theta] = 0$ since $\theta$ is not trivial on $\Gamma_h' \cap T_h$. Now pick $\theta \in T^0_h$. Notice that the subgroup of $T_h \times G_h$ generated by $\Gamma_h'$ and $T_h \times \{1\}$ is all of $T_h \times G_h$ and therefore

$$H^i_c(Y_h, \overline{\mathcal{O}}_\ell)[\theta] \cong \eta_\theta,$$

where $\eta_\theta$ is the extension of $H^i_c(\mathbb{A}^{(n-1)(h-1)}(Y_1, \overline{\mathcal{O}}_\ell))$ whose restriction to $T_h \times \{1\}$ is $\theta \otimes 1$. To see that $\eta_\theta$ has the asserted form, it remains to calculate the action of $G_h$. For $g \in G_h$, we may write $g = \overline{g} \cdot g_1$, where $\overline{g} \in G_1$ and $g_1 \in G_h^{(1)}$. Then

$$(1, g) = (1, \overline{g}) \cdot (1, g_1) = (1, \overline{g}) \cdot (t^{-1}, g_1) \cdot (t, 1) \in T_h \times G_h,$$

where $t \in T_h$ has the property that $\det(t) = \det(g_1)$. Since $(1, \overline{g}) \in T_1 \times G_1 \subset \Gamma_h'$ and $(t^{-1}, g_1) \in \Gamma_h^{(1)}$, it now follows that $(1, g)$ acts on $\eta_\theta$ by multiplication by $\theta(t) = \theta_0(\det(t)) = \theta_0(\det(g))$. This completes the proof.

Lemma 6.8. Let $\theta \in T_h$ and set $d := 2(n-1)(h-1)$ for convenience. Then

$$H^i_c(Y_h, \overline{\mathcal{O}}_\ell)[\theta] \cong \begin{cases} 0 & \text{if } i \leq d \text{ or } \theta \in T^\text{gen}_h, \\ H^i_c(Z_h, \overline{\mathcal{O}}_\ell)[\theta] \oplus H^i_c(X_1, \overline{\mathcal{O}}_\ell)[\theta] \otimes (\theta_1 \otimes \overline{\theta}) & \text{if } i = d + 1 \text{ and } \theta \in T^0_h, \\ H^{i-d}_c(X_1, \overline{\mathcal{O}}_\ell)[\theta] \oplus (\theta_1 \otimes \overline{\theta}) & \text{if } i > d + 1 \text{ and } \theta \in T^0_h. \end{cases}$$

Proof. By Lemma 6.7, we have, as $(T_h \times G_h)$-representations,

$$H^i_c(Y_h, \overline{\mathcal{O}}_\ell)[\theta] \cong \begin{cases} 0 & \text{if } \theta \in T^\text{gen}_h, \\ H^{i-2(n-1)(h-1)}_c(Y_1, \overline{\mathcal{O}}_\ell)[\theta] \otimes (\theta_1 \otimes \overline{\theta}) & \text{if } \theta \in T^0_h. \end{cases}$$

(6.6)

This immediately gives the vanishing assertion for $i < 2(n-1)(h-1)$ and $\theta \in T^\text{gen}_h$. For the remainder of the proof, we assume $\theta \in T^0_h$.

We have a long exact sequence

$$0 \to H^0_c(Y_1, \overline{\mathcal{O}}_\ell) \to H^0_c(X_1, \overline{\mathcal{O}}_\ell) \to H^0_c(Z_1, \overline{\mathcal{O}}_\ell) \to H^1_c(Y_1, \overline{\mathcal{O}}_\ell) \to H^1_c(X_1, \overline{\mathcal{O}}_\ell) \to H^1_c(Z_1, \overline{\mathcal{O}}_\ell) \to \cdots$$

Since $X_1$ is a smooth affine variety of dimension $n - 1 > 0$, we have that $H^0_c(X_1, \overline{\mathcal{O}}_\ell) = 0$. This proves the vanishing assertion for $i = 2(n-1)(h-1)$. Since $Z_1$ is a dimension-0 variety, we have that $H^i_c(Z_1, \overline{\mathcal{O}}_\ell) = 0$ for $i > 0$. Hence

$$H^i_c(Y_1, \overline{\mathcal{O}}_\ell) = \begin{cases} 0 & \text{if } i = 0, \\ H^0_c(Z_1, \overline{\mathcal{O}}_\ell) \oplus H^1_c(X_1, \overline{\mathcal{O}}_\ell) & \text{if } i = 1, \\ H^0_c(X_1, \overline{\mathcal{O}}_\ell) & \text{if } i > 1. \end{cases}$$

The assertion of the lemma for $i = d + 1$ now follows from Proposition 6.5(b) and Equation (6.6). Finally, the assertion of the lemma for $i > d+1$ follows directly from Equation (6.6).

Lemma 6.9. For $\theta \in T_h$, consider the restriction of the boundary map

$$\delta_\theta : H^i_c(n-1)(h-1)(X_h, \overline{\mathcal{O}}_\ell)[\theta] \to H^{i+2(n-1)(h-1)+1}_c(Y_h, \overline{\mathcal{O}}_\ell)[\theta].$$
Then
\[
\delta_\theta = \begin{cases} 
\text{injective} & \text{if } \theta \in \mathcal{T}_h^0, \\
0 & \text{if } \theta \in \mathcal{T}_h^{\text{gen}}.
\end{cases}
\]

**Proof.** For notational convenience, write \(d := 2(n-1)(h-1)\). Taking \(\theta\)-eigenspaces is exact and hence we have
\[
0 \to H_c^d(X_h, \overline{Q}_\ell)[\theta] \to H_c^d(Z_h, \overline{Q}_\ell)[\theta] \xrightarrow{\delta_\theta} H_c^{d+1}(Y_h, \overline{Q}_\ell)[\theta] \to H_c^{d+1}(X_h, \overline{Q}_\ell)[\theta] \to 0.
\]
By Lemma 6.8, we see that for \(\theta\)-eigenspaces and using Proposition 6.5(a), we see that for \(\theta\) injective if \(\theta \in \mathcal{T}_h^{\text{gen}}\) and hence
\[
\delta_\theta = 0, \quad \text{if } \theta \in \mathcal{T}_h^{\text{gen}}.
\]
To finish the proof, assume \(\theta \in \mathcal{T}_h^0\). By Proposition 6.5 we have
\[
H_c^d(Z_h, \overline{Q}_\ell)[\theta] \cong H_c^0(Z_1, \overline{Q}_\ell)[\theta] \otimes (\theta_1 \otimes \tilde{\theta}),
\]
and by Lemma 6.7 we have
\[
H_c^{d+1}(Y_h, \overline{Q}_\ell)[\theta] \cong H_c^1(Y_1, \overline{Q}_\ell)[\theta] \otimes (\theta_1 \otimes \tilde{\theta}).
\]

The long exact sequence for \(h = 1\) gives
\[
0 \to H_c^0(Z_1, \overline{Q}_\ell) \to H_c^1(Y_1, \overline{Q}_\ell) \to H_c^1(X_1, \overline{Q}_\ell) \to 0,
\]
and hence the restriction of the boundary map
\[
\delta_\theta : H_c^0(Z_1, \overline{Q}_\ell)[\theta] \to H_c^1(Y_1, \overline{Q}_\ell)[\theta]
\]
is an injection. It now follows that
\[
\delta_\theta : H_c^d(Z_h, \overline{Q}_\ell)[\theta] \to H_c^{d+1}(Y_h, \overline{Q}_\ell)[\theta]
\]
is injective. \(\square\)

### 6.2.1. Proof of Theorem 6.1

We now combine the Lemmas to obtain Theorem 6.1. Recall that we have a long exact sequence
\[
0 \to H_c^0(Y_h, \overline{Q}_\ell) \to H_c^0(X_h, \overline{Q}_\ell) \to H_c^0(Z_h, \overline{Q}_\ell) \\
\to H_c^1(Y_h, \overline{Q}_\ell) \to H_c^1(X_h, \overline{Q}_\ell) \to H_c^1(Z_h, \overline{Q}_\ell) \to \cdots
\]
By Lemma 6.8 we immediately obtain isomorphisms
\[
H_c^i(X_h, \overline{Q}_\ell) \cong H_c^i(Z_h, \overline{Q}_\ell), \quad \text{for } i < 2(n-1)(h-1).
\]
Taking \(\theta\)-eigenspaces and using Proposition 6.5(a), we see that for \(i < 2(n-1)(h-1)\),
\[
H_c^i(X_h, \overline{Q}_\ell)[\theta] \cong \begin{cases} 
H_c^i(Z_h, \overline{Q}_\ell)[\theta] & \text{if } \theta \in \mathcal{T}_h^{\text{gen}}, \\
0 & \text{if } \theta \in \mathcal{T}_h^0.
\end{cases}
\] (6.7)

Since \(\dim Z_h = (n-1)(h-1)\), then we have \(H_c^i(Z_h, \overline{Q}_\ell) = 0\) for \(i > 2(n-1)(h-1)\) and hence we immediately obtain isomorphisms
\[
H_c^i(X_h, \overline{Q}_\ell) \cong H_c^i(Y_h, \overline{Q}_\ell), \quad \text{for } i > 2(n-1)(h-1) + 1.
\]
By Lemma 6.8 we see that for \(i > 2(n-1)(h-1) + 1\),
\[
H_c^i(X_h, \overline{Q}_\ell)[\theta] \cong \begin{cases} 
0 & \text{if } \theta \in \mathcal{T}_h^{\text{gen}}, \\
H_c^{i-2(n-1)(h-1)}(X_1, \overline{Q}_\ell)[\theta] \otimes (\theta_1 \otimes \tilde{\theta}) & \text{if } \theta \in \mathcal{T}_h^0.
\end{cases}
\] (6.8)
It remains to calculate the cohomology degrees \( d := 2(n - 1)(h - 1) \) and \( d + 1 = 2(n - 1)(h - 1) + 1 \) for \( X_h \). Taking \( \theta \)-eigenspaces of the long exact sequence gives

\[
0 \to H^d_c(X_h, \overline{Q}_\ell)[\theta] \to H^d_c(Z_h, \overline{Q}_\ell)[\theta] \xrightarrow{\partial_0} H^{d+1}_c(Y_h, \overline{Q}_\ell)[\theta] \to H^{d+1}_c(X_h, \overline{Q}_\ell)[\theta] \to 0.
\]

By Lemma 6.9, if \( \theta \in \mathcal{T}^\text{gen}_h \), then

\[
H^d_c(X_h, \overline{Q}_\ell)[\theta] \cong H^d_c(Z_h, \overline{Q}_\ell)[\theta], \quad H^{d+1}_c(X_h, \overline{Q}_\ell)[\theta] \cong H^{d+1}_c(Y_h, \overline{Q}_\ell)[\theta] = 0.
\]

If \( \theta \in \mathcal{T}^0_h \), then

\[
H^d_c(X_h, \overline{Q}_\ell)[\theta] = 0, \quad H^{d+1}_c(Y_h, \overline{Q}_\ell)[\theta] \cong H^d_c(Z_h, \overline{Q}_\ell)[\theta] \oplus H^{d+1}_c(X_h, \overline{Q}_\ell)[\theta],
\]

and since \( H^{d+1}_c(Y_h, \overline{Q}_\ell)[\theta] \cong H^d_c(Z_h, \overline{Q}_\ell)[\theta] \oplus \left( H^1_c(X_1, \overline{Q}_\ell)[\theta] \otimes (\theta_1 \otimes \tilde{\theta}) \right) \), we have

\[
H^{d+1}_c(X_h, \overline{Q}_\ell)[\theta] \cong H^1_c(X_1, \overline{Q}_\ell)[\theta] \otimes (\theta_1 \otimes \tilde{\theta}).
\]

Hence

\[
H^{2(n-1)(h-1)}_c(X_h, \overline{Q}_\ell)[\theta] \cong \begin{cases} H^{2(n-1)(h-1)}_c(Z_h, \overline{Q}_\ell)[\theta] & \text{if } \theta \in \mathcal{T}^\text{gen}_h, \\ 0 & \text{if } \theta \in \mathcal{T}^0_h, \end{cases} \tag{6.9}
\]

\[
H^{2(n-1)(h-1)+1}_c(X_h, \overline{Q}_\ell)[\theta] \cong \begin{cases} 0 & \text{if } \theta \in \mathcal{T}^\text{gen}_h, \\ H^1_c(X_1, \overline{Q}_\ell)[\theta] \otimes (\theta_1 \otimes \tilde{\theta}) & \text{if } \theta \in \mathcal{T}^0_h. \end{cases} \tag{6.10}
\]

Combining Equations (6.7), (6.8), (6.9), and (6.10) completes the proof.

7. Affine Deligne–Lusztig varieties at finite level

In this section, we use the results of the Sections 5 and 6 to give an explicit description of the cohomology of the affine Deligne–Lusztig varieties \( \mathcal{X}^m_{w_0}(b) \). We make use of Section 3.5 especially Remark ??.

Lemma 7.1. Let \( r \geq 1 \) and \( m \geq 0 \) be integers and pick column vectors \( x, y \in \mathcal{L}^{\text{adm}}_0 \). Then \( x_{b,m,r}^{-1} y \) if and only if there exists \( u \in \mathcal{O} \) such that \( y = x \cdot (1 + \varpi^m u) \) modulo \( \varpi^{m+r} \mathcal{O} \).

Proof. By definition,

\[
x_{b,m,r}^{-1} y \iff y \in g_{b,r}(x) \cdot (1 + \varpi^m \mathcal{O}, \varpi^m \mathcal{O}, \ldots, \varpi^m \mathcal{O})^\top,
\]

where

\[
g_{b,r}(x) = \left( x \mid \varpi^r F(x) \mid \cdots \mid \varpi^{r(n-1)} F^{n-1}(x) \right) \in \text{GL}_n(\mathcal{O}).
\]

It is clear that if \( x_{b,m,r}^{-1} y \), then there exists \( u \in \mathcal{O} \) such that \( y = x \cdot (1 + \varpi^m u) \) modulo \( \varpi^{m+r} \). It remains to show the converse.

Assume that there exists \( u \in \mathcal{O} \) such that \( y = x \cdot (1 + \varpi^m u) \) modulo \( \varpi^{m+r} \). We must show that \( y \in g_{b,r}(x) \cdot (1 + \varpi^m \mathcal{O}, \varpi^m \mathcal{O}, \ldots, \varpi^m \mathcal{O})^\top \). To this end, first observe that \( g_{b,r}(x)^{-1} \cdot x = (1, 0, \ldots, 0)^\top \) and hence, writing \( y_i = x_i (1 + \varpi^m u) + \varpi^{m+r} u_i \) for some \( u_i \in \mathcal{O} \), we have

\[
g_{b,r}(x)^{-1} \cdot y = g_{b,r}(x)^{-1} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} (1 + \varpi^m u) + \varpi^{m+r} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \\
\in (1 + \varpi^m \mathcal{O}, 0, \ldots, 0)^\top + \varpi^{m+r} (\mathcal{O}, \mathcal{O}, \ldots, \mathcal{O})^\top.
\]

We let

\[
\mathcal{X}_w^m(b, \mathcal{L}^{\text{adm}}_0) := \{ x \in \mathcal{L}^{\text{adm}}_0 : \det(g_b(x)) \in \mathcal{O}_K \} / \sim_{b,m,r}
\]
Proposition 7.2. Let $r \geq 1$ and $m \geq 0$ be integers. The natural morphism
\[
\hat{X}^m_{w_r}(b)_{\mathcal{O}_{0,\hat{b}}} \to \hat{X}^m_{w_1}(b)_{\mathcal{O}_{0,\hat{b}}}
\]
is an affine fibration with fibers isomorphic to the affine space $\mathbb{A}^{(n-1)(r-1)}$.

Proof. The existence of the morphism $\hat{X}^m_{w_r}(b)_{\mathcal{O}_{0,\hat{b}}} \to \hat{X}^m_{w_1}(b)_{\mathcal{O}_{0,\hat{b}}}$ follows from the observation that $x^{\sim}_{b,m,r}$ implies $x^{\sim}_{b,m,1}$. By Theorem 6.1, it is sufficient to prove that we have natural isomorphisms$
\begin{align*}
\text{Lemma 7.4.} & \quad \text{for any } r \geq 1, \text{ we have } \\
\hat{X}^m_{w_r}(b)_{\mathcal{O}_{0,\hat{b}}} &= \left\{x \in \mathcal{O}_{0,\hat{b}} : \det(g_b(x)) \in \mathcal{O}_K^{\times}/\mathcal{O}_{b,m,r}\right\}.
\end{align*}

Since the determinant condition imposes no constraints on $\mathcal{O}_K^{\times}/\mathcal{O}_{b,m,1}$, the desired conclusion follows. □

Proposition 7.3. Let $r > m \geq 0$ be integers. There is a natural $(T_h \times G_h)$-equivariant morphism
\[
\hat{X}^m_{w}(b)_{\mathcal{O}_{0,\hat{b}}} \to X_{m+1}
\]
where the fibers are isomorphic to an affine space.

7.1. Homology of Deligne–Lusztig varieties at infinite level. Let
\[
W_h := \ker(T_h \to T_{h-1}), \\
K_h := \ker(W_h \to W^\sigma_h),
\]
where $\sigma : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q$ is the arithmetic Frobenius $x \mapsto x^q$.

Lemma 7.4. We have a natural isomorphism
\[
H_k^i(X_h, \overline{\mathbb{Q}})_{W_h} \cong H_k^{i-2(n-1)}(X_{h-1}, \overline{\mathbb{Q}}).
\]
In particular, we have a canonical inclusion
\[
H_i(X_{h-1}, \overline{\mathbb{Q}}) \hookrightarrow H_i(X_h, \overline{\mathbb{Q}}).
\]

Proof. By Theorem 6.1, it is sufficient to prove that we have natural isomorphisms
\[
H_k^i(Z_h, \overline{\mathbb{Q}})_{W_h} \cong H_k^{i-2(n-1)}(Z_{h-1}, \overline{\mathbb{Q}}), \\
H_k^i(Y_h, \overline{\mathbb{Q}})_{W_h} \cong H_k^{i-2(n-1)}(Y_{h-1}, \overline{\mathbb{Q}}).
\]
The first isomorphism follows from a slightly more general version of Proposition 6.5(b), and the second isomorphism follows from Lemma 6.7. □

Remark 7.5. There is likely a more straightforward proof of Lemma 7.4. ⊙

For a smooth scheme $S$ of dimension $d$, define $H_i(S, \overline{\mathbb{Q}}_\ell) := H_c^{2d-i}(S, \overline{\mathbb{Q}}_\ell)$ as $\overline{\mathbb{Q}}_\ell$-vector spaces. By Proposition 7.3, for any $r > m \geq 0$, we have canonical isomorphisms
\[
H_i(\hat{X}^m_{w_r}(b)_{\mathcal{O}_{0,\hat{b}}}, \overline{\mathbb{Q}}_\ell) \cong H_i(X_{m+1}, \overline{\mathbb{Q}}_\ell).
\]
Hence by Lemma 7.4, we have canonical inclusions
\[
H_i(\hat{X}^{m_1}_{w_{r_1}}(b)_{\mathcal{O}_{0,\hat{b}}}, \overline{\mathbb{Q}}_\ell) \hookrightarrow H_i(\hat{X}^{m_2}_{w_{r_2}}(b)_{\mathcal{O}_{0,\hat{b}}}, \overline{\mathbb{Q}}_\ell),
\]
for any $r_1 > m_1$ and $r_2 > m_2$ with $m_1 < m_2$. We may therefore define
\[
H_i\left(\hat{X}^m_{w}(b)_{\mathcal{O}_{0,\hat{b}}}, \overline{\mathbb{Q}}_\ell\right) := \lim_{r > m \geq 0} H_i\left(\hat{X}^m_{w_r}(b)_{\mathcal{O}_{0,\hat{b}}}, \overline{\mathbb{Q}}_\ell\right),
\]
and by Proposition 3.11
\[ H_i \left( X_w^{\infty}(b), \mathbb{Q}_\ell \right) = \bigoplus_{J_b(K)/J_b(O_K)} H_i \left( g.X_w^{\infty}(b), \mathbb{Q}_\ell \right). \]

We write
\[ H_\bullet \left( X_w^{\infty}(b), \mathbb{Q}_\ell \right) := \bigoplus_{i \in \mathbb{Z}} H_i \left( X_w^{\infty}(b), \mathbb{Q}_\ell \right). \]

The results of Sections 6 and 3 can now be translated into theorems about the homology of \( X_w^{\infty}(b) \), the affine Deligne–Lusztig variety at infinite level.

**Theorem 7.6.** Let \( \theta \) be a smooth character of \( L^\times \) whose restriction to \( U_L^{m+1} \) is trivial.

(a) If the restriction of \( \theta \) to \( O_L^\times \) factors through \( \text{Nm} : O_L^\times \to O_K^\times \), then
\[ H_i \left( X_w^{\infty}(b), \mathbb{Q}_\ell \right) [\theta] = 0 \quad \text{for } i \geq \gcd(\kappa_G([b]), n). \]

(b) Otherwise, there exists a unique integer \( r_\theta \) with \( \gcd(\kappa_G([b]), n) \leq r_\theta \leq (n-1)(h-1) + \gcd(\kappa_G([b]), n) - 1 \) such that
\[ H_i \left( X_w^{\infty}(b), \mathbb{Q}_\ell \right) [\theta] \neq 0 \quad \iff \quad i = r_\theta. \]

**Theorem 7.7.** If \( \theta : L^\times \to \mathbb{Q}_\ell^\times \) has trivial \( \text{Gal}(L/K) \)-stabilizer, then the \( J_b(K) \)-representation \( H_\bullet \left( X_w^{\infty}(b), \mathbb{Q}_\ell \right) [\theta] \) is irreducible.

**Part 3. Character formulae**

We will use the results of Part 2 to write down certain trace formulae for the representations arising in the cohomology of affine Deligne–Lusztig varieties at infinite level.

**8. On very regular elements**

We say that an element \( x \) of \( L^\times \) is very regular if \( x \in O_L^\times \) and its image in the residue field \( k_n \) does not lie in any proper subfield. In many situations, the character of a supercuspidal representation on very regular elements is rigid enough to specify the representation.

We announce the two following theorems, whose proofs essentially follow from Theorem 6.2 in a similar way to the proof of [Cha15, Theorem 7.12]

**Theorem 8.1** (Announcement). Let \( \theta : L^\times \to \mathbb{Q}_\ell^\times \) be a smooth character with trivial \( \text{Gal}(L/K) \)-stabilizer. Then for any very regular element \( x \in O_L^\times \),
\[ \text{Tr} \left( x; H_\bullet \left( X_w^{\infty}(b), \mathbb{Q}_\ell \right) [\theta] \right) = (-1)^{r_\theta} \sum_{\gamma \in \text{Gal}(L/K)} \theta^\gamma(x). \]

By appealing to methods of Henniart, Theorem 8.1 implies

**Theorem 8.2** (Announcement). Let \( \theta : L^\times \to \mathbb{Q}_\ell^\times \) be a smooth character with trivial \( \text{Gal}(L/K) \)-stabilizer. Then for any basic \( \sigma \)-conjugacy classes \([b]\) and \([b']\), the \( J_b(K) \)-representation \( H_{r_\theta}(X_w^{\infty}(b), \mathbb{Q}_\ell) \) and the \( J_{b'}(K) \)-representation \( H_{r_\theta}(X_w^{\infty}(b'), \mathbb{Q}_\ell) \) correspond under the Jacquet–Langlands correspondence.

**References**


