Econ 679 – Midterm Exam
Time Series Econometrics

Matias D. Cattaneo
Department of Economics
University of Michigan

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Contents

1. Problem 1 – Regression with Autoregressive Covariate

Let $\{(y_t, x_t)' : 1 \leq t \leq T\}$ be an observed time series satisfying the stochastic equations

$$\begin{aligned} y_t &= \beta x_t + u_t \\ x_t &= \gamma x_{t-1} + v_t \end{aligned}$$

where $\beta \in \mathbb{R}$ and $(u_t, v_t)' \overset{iid}{\sim} (0, \Sigma)$ with

$$\Sigma = \begin{pmatrix} \sigma_{uu} & \sigma_{uv} \\ \sigma_{uv} & \sigma_{vv} \end{pmatrix}.$$

1. Assume that $|\gamma| < 1$. Show that y_t is stationary, and find its univariate representation in terms of u_t and v_t.

2. Assume that $\gamma = 1$. Show that $\{\Delta y_t : 2 \leq t \leq T\}$ has an MA representation. Provide conditions so that this MA presentation is invertible.

3. Assume that $\sigma_{uv} = 0$. Find the (quasi-) log-likelihood function when $(u_t, v_t)' \overset{iid}{\sim} \mathcal{N}(0, \Sigma)$, denoted $\ell_T(\beta, \gamma, \sigma_{uu}, \sigma_{vv})$. Find the (quasi-) ML estimators of β, γ, denoted by $\hat{\beta}, \hat{\gamma}$.

4. Assume that $\sigma_{uv} = 0$ and $|\gamma| < 1$. After appropriate centering and rescaling, and providing the appropriate sufficient conditions, characterize the limiting distribution of $\hat{\gamma}$ and $\hat{\beta}$.

5. Assume that $\sigma_{uv} = 0$ and $\gamma = 1$. After appropriate centering and rescaling, and providing the appropriate sufficient conditions, characterize the limiting distribution of $\hat{\gamma}$ and $\hat{\beta}$.

6. Assume that $\sigma_{uv} \neq 0$ and $|\gamma| < 1$. Find a consistent estimator of β, denoted by $\tilde{\beta}$. After appropriate centering and rescaling, and providing the appropriate sufficient conditions, characterize the limiting distribution of $\tilde{\beta}$.

7. Assume that $\sigma_{uv} \neq 0$ and $\gamma = 1$. Let $\delta = \sigma_{vv}^{-1}\sigma_{uv}$, and show that

$$y_t = \beta x_t + \delta \Delta x_t + \varepsilon_t, \qquad \varepsilon_t \overset{iid}{\sim} (0, \sigma_\varepsilon).$$

Let $(\check{\beta}, \check{\delta})'$ be the OLS estimators using this model. Show that $\check{\beta}$ is a consistent estimator of β. After appropriate centering and rescaling, and providing the appropriate sufficient conditions, characterize the limiting distribution of $\check{\beta}$.

2. PROBLEM 2 – SPURIOUS REGRESSION

Let $\{(y_t, x_t)' : 1 \leq t \leq T\}$ be an observed time series satisfying the stochastic equations

$$\begin{aligned} y_t &= \alpha y_{t-1} + u_t \\ x_t &= \beta x_{t-1} + v_t \end{aligned}$$

where $\alpha, \beta \in [0, 1]$,

$$(u_t, v_t)' \overset{iid}{\sim} \mathcal{N}\left(0, \begin{pmatrix} \sigma_{uu} & 0 \\ 0 & \sigma_{vv} \end{pmatrix}\right),$$

and with initial conditions $x_0 = y_0 = 0$. Let θ represent the regression coefficient for the (misspecified) model $\hat{y}_t = \hat{\theta} x_t + \hat{\varepsilon}_t$, with least squares estimator given by

$$\hat{\theta} = \frac{\sum_{t=1}^T x_t y_t}{\sum_{t=1}^T x_t^2},$$

and R^2 coefficient given by

$$R^2 = \frac{\sum_{t=1}^T y_t^2 - \sum_{t=1}^T \hat{\varepsilon}_t^2}{\sum_{t=1}^T y_t^2} = \frac{\hat{\theta}^2 \sum_{t=1}^T x_t^2}{\sum_{t=1}^T y_t^2} = \frac{\left(\sum_{t=1}^T x_t y_t\right)^2}{\sum_{t=1}^T x_t^2 \sum_{t=1}^T y_t^2}.$$

1. Assume that $|\alpha| < 1$ and $|\beta| < 1$. After appropriate centering and rescaling, and providing the appropriate sufficient conditions, characterize the limiting distribution of $\hat{\theta}$ and R^2.
2. Assume that $\alpha = 1$ and $|\beta| < 1$. After appropriate centering and rescaling, and providing the appropriate sufficient conditions, characterize the limiting distribution of $\hat{\theta}$ and R^2.
3. Assume that $|\alpha| < 1$ and $\beta = 1$. After appropriate centering and rescaling, and providing the appropriate sufficient conditions, characterize the limiting distribution of $\hat{\theta}$ and R^2.
4. Assume that $\alpha = 1$ and $\beta = 1$. After appropriate centering and rescaling, and providing the appropriate sufficient conditions, characterize the limiting distribution of $\hat{\theta}$ and R^2.
5. Compare and discuss the results in parts 1 through 4. Are these results intuitive? Explain.