Econ 679 – Midterm Exam

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1. Problem 1 – Asymmetric Least Squares

Consider the following (robust) least squares based alternative to quantile regression:

$$\hat{\beta}_\pi = \arg\min_{\beta} M_n(\beta), \qquad M_n(\beta) = \frac{1}{n}\sum_{i=1}^{n} \left(Y_i - X_i'\beta\right)^2 \left|\pi - \mathbf{1}\left(Y_i < X_i'\beta\right)\right|,$$

where $\pi \in (0,1)$ and $(Y_i, X_i')'$, $i = 1, 2, \cdots, n$, are i.i.d. satisfying the linear regression relationship $Y_i = X_i'\beta_\pi^* + \varepsilon_i$, where ε_i are independent of X_i for all $i = 1, 2, \cdots, n$.

1. Assuming that X_i contains an intercept and the normalization

$$0 = \arg\min_{m\in\mathbb{R}} \mathbb{E}\left[(\varepsilon_i - m)^2 \left|\pi - \mathbf{1}(\varepsilon_i < m)\right|\right]$$

   holds, provide sufficient conditions for identifiability of β_π^* (as the solution to the limiting M-estimation problem).

2. Provide sufficient conditions so that $\hat{\beta}_\pi = \beta_\pi^* + o_p(1)$.

3. Provide sufficient conditions so that

$$\sqrt{n}(\hat{\beta}_\pi - \beta_\pi^*) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \psi(Y_i, X_i; \beta_\pi^*) + o_p(1),$$

   and conclude that $\sqrt{n}(\hat{\beta}_\pi - \beta_\pi^*) \xrightarrow{d} \mathcal{N}(0, V_\pi)$. Identify the exact form of the influence function $\psi(Y_i, X_i; \beta_\pi^*)$ and limiting covariance matrix V_π.

4. Note that $\pi = 1/2$ corresponds to the classical least squares (LS) regression estimator. Thus, verify that the limiting distribution derived in part 3 agrees with the limiting distribution of LS if $\pi = 1/2$.

2. Problem 2 – Efficient Two-step Estimation

Suppose that the (scalar) parameter of interest β_0 solves the moment condition

$$\mathbb{E}\left[m_1\left(X,\beta_0,\gamma_0\right)\right]=0,$$

where γ_0 is some other unknown (scalar) parameter, which solves the moment condition

$$\mathbb{E}\left[m_2\left(X,\gamma_0\right)\right]=0.$$

Assume that $m\left(X,\beta,\gamma\right)$ is continuously differentiable as needed, and a random sample of observations $X_1,\cdots,X_n$ is available.

1. Suppose γ_0 is known and, using this information, consider the estimator $\tilde{\beta}_n$ given by

$$\frac{1}{n}\sum_{i=1}^{n} m_1(X_i,\tilde{\beta}_n,\gamma_0)=0.$$

   Provide sufficient conditions for consistency and derive the limiting distribution of $\tilde{\beta}_n$.

2. Suppose γ_0 is unknown and consider the two-step estimator $\hat{\beta}_n$ given by

$$\frac{1}{n}\sum_{i=1}^{n} m_1(X_i,\hat{\beta}_n,\hat{\gamma}_n)=0,$$

   where $\hat{\gamma}_n$ is given by

$$\frac{1}{n}\sum_{i=1}^{n} m_2(X_i,\hat{\gamma}_n)=0.$$

   Provide sufficient conditions for consistency and derive the (joint) limiting distribution of $(\hat{\beta}_n,\hat{\gamma}_n)'$.

3. Assume that $\mathbb{E}\left[m_1\left(X,\beta_0,\gamma_0\right)m_2\left(X,\gamma_0\right)\right]=0$. Using the previous two results, verify that $\mathbb{V}[\hat{\beta}_n]\geq\mathbb{V}[\tilde{\beta}_n]$ and derive (alternative) conditions so that $\mathbb{V}[\hat{\beta}_n]=\mathbb{V}[\tilde{\beta}_n]$ (i.e., the feasible estimator is as efficient as the unfeasible one).

4. Assume that $\mathbb{E}\left[m_1\left(X,\beta_0,\gamma_0\right)m_2\left(X,\gamma_0\right)\right]\neq 0$. Is the unfeasible estimator (i.e., the one that uses the information about γ_0) always as efficient as $\hat{\beta}_n$? Provide intuition for your answer.

5. In conclusion, assuming γ_0 is known, how would you estimate β_0 efficiently? Is this result intuitive?