Econ 679 – Final Exam

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Contents

1. Problem 1 – Efficient Semiparametric Estimation

Let $Z_i \in \mathbb{R}^d$, $i = 1, 2, \cdots, n$, be i.i.d. continuous random vectors with (unknown) density $f(z)$. Consider the (parametric) estimand $\theta = \mathbb{E}[f(Z)] \in \mathbb{R}$. Using the classical kernel density estimator $\hat{f}_n(z) = n^{-1} h_n^{-d} \sum_{j=1}^{n} K((Z_j - z)/h_n)$, where $z \mapsto K(z)$ integrates to 1 and symmetric, and $h_n \to 0$ as $n \to \infty$ (other assumptions will be required below), consider the following (parametric) plug-in estimator:

$$\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^{n} \hat{f}_n(Z_i) = \frac{1}{n^2 h_n^d} \sum_{i=1}^{n} \sum_{j=1}^{n} K\left(\frac{Z_j - Z_i}{h_n}\right) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} U_n(Z_i, Z_j),$$

where $U_n(Z_i, Z_j) = h_n^{-d} K((Z_j - Z_i)/h_n)$.

1. (**Hoeffding Decomposition**) Verify that

$$\hat{\theta}_n = \frac{K(0)}{n h_n^d} + \frac{n-1}{n} \tilde{\theta}_n, \qquad \tilde{\theta}_n = \binom{n}{2}^{-1} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} U_n(Z_i, Z_j) = \theta_n + \bar{L}_n + \bar{W}_n,$$

where $\theta_n = \mathbb{E}[\tilde{\theta}_n]$,

$$\bar{L}_n = \frac{1}{n} \sum_{i=1}^{n} L_n(Z_i), \qquad L_n(Z_i) = n\left(\mathbb{E}[\tilde{\theta}_n | Z_i] - \theta_n\right),$$

$$\bar{W}_n = \binom{n}{2}^{-1} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} W_n(Z_i, Z_j),$$

$$W_n(Z_i, Z_j) = \binom{n}{2}\left(\mathbb{E}[\tilde{\theta}_n | Z_i, Z_j] - \mathbb{E}[\tilde{\theta}_n | Z_i] - \mathbb{E}[\tilde{\theta}_n | Z_j] + \theta_n\right).$$

In addition, show that $\mathbb{E}[\bar{L}_n] = 0$, $\mathbb{E}[\bar{W}_n] = 0$, and $\mathbb{E}\left[\bar{L}_n \bar{W}_n\right] = 0$.

2. (**Bias**) Provide sufficient conditions so that $\theta_n - \theta = O(h_n^P)$, where P is the Kernel order.

3. (**Linear Term**) Verify that $L_n(Z_i) = 2(\mathbb{E}[U_n(Z_i, Z_j)\,|Z_i] - \theta_n)$ and provide sufficient conditions so that

$$\mathbb{E}\left[|L_n(Z_i) - L(Z_i)|^2\right] = O\left(h_n^{2P}\right), \qquad L(Z_i) = 2\left(f(Z_i) - \theta\right).$$

4. (**Quadratic Term**) Verify that

$$W_n(Z_i, Z_j) = U_n(Z_i, Z_j) - \mathbb{E}[U_n(Z_i, Z_j)\,|Z_i] - \mathbb{E}[U_n(Z_i, Z_j)\,|Z_j] + \theta_n$$

and provide sufficient conditions so that $\mathbb{E}\left[|W_n(Z_i, Z_j)|^2\right] = O\left(h_n^{-d}\right)$.

5. (**Asymptotic Linear Representation**) By considering the decomposition verified in part 1, provide sufficient conditions so that

$$\sqrt{n}(\hat{\theta}_n - \theta) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} L(Z_i) + o_p(1).$$

6. (**Semiparametric Efficiency**) Show that $\hat{\theta}_n$ is a semiparametric efficient estimator.

7. (**Standard Errors**) Provide sufficient conditions so that

$$\hat{V}_n = \frac{4}{n}\sum_{i=1}^{n}\left(\hat{f}_n(Z_i) - \hat{\theta}_n\right)^2 \xrightarrow{p} \mathbb{V}[L(Z)].$$

Conclude that $\sqrt{n}\hat{V}_n^{-1/2}(\hat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}(0,1)$.

2. PROBLEM 2 – REPORT ON HAN AND PHILLIPS (2006)

Write (at most) a four-page report on C. Han and P. Phillips (2006): "GMM with Many Moment Conditions," *Econometrica* 74(1), 147-192.

1. Discuss the motivation and main results of the paper.
2. In addition to address the link between this paper and its related literature, find (at least) one closely related theoretical paper published prior to 2006 but not cited in Han and Phillips (2006). Discuss the link between these papers.