Econ 671 – Midterm Exam

MATIAS D. CATTANEO
DEPARTMENT OF ECONOMICS
UNIVERSITY OF MICHIGAN

October 28, 2008

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1. PART I – SHORT QUESTIONS (25 POINTS)

1. (**5 points**) Let X be a random variable with Moment Generating Function $M_X(t)$. Using the Markov's Inequality, show that

$$\mathbb{P}(X > x) \leq \frac{M_X(t)}{\exp(tx)}, \qquad \text{for all } x \in \mathbb{R} \text{ and } t > 0.$$

Note: Applying $\inf_{t>0}$ on the right hand side gives the *Chernoff's Inequality*.

2. (**5 points**) Let X and Y be random variables with finite variance. Show that if $\mathbb{V}[X] = \mathbb{V}[Y]$ and $\mathbb{E}[Y|X] = X$, then $\mathbb{P}(Y = X) = 1$.

3. (**15 points**) Let $\hat{\theta}_n$ be an unbiased estimator of θ with finite second moment. A very important result is:

$$\hat{\theta}_n \text{ is UMVU} \Longleftrightarrow \mathbb{C}ov_\theta[\hat{\theta}_n, \tilde{\theta}_n - \hat{\theta}_n] = 0, \text{ for all } \tilde{\theta}_n \in \mathscr{S}_u,$$

where $\mathscr{S}_u$ is the set of all unbiased estimators of θ with finite second moment. Show this result by considering the following three steps:

(a) (**5 points**) For $\lambda \in [0, 1]$, verify that the estimator

$$\bar{\theta}_{\lambda,n} = (1 - \lambda)\hat{\theta}_n + \lambda\tilde{\theta}_n = \hat{\theta}_n + \lambda(\tilde{\theta}_n - \hat{\theta}_n)$$

is unbiased and derive its variance $\mathbb{V}_\theta[\bar{\theta}_{\lambda,n}]$ (as a function of λ).

(b) (**5 points**) Prove the "if" direction by setting $\lambda = 1$.

(c) (**5 points**) Prove the "only if" direction by minimizing $\mathbb{V}_\theta[\bar{\theta}_{\lambda,n}]$ over λ.

2. Part II – A Long Question (35 points)

Let $X_1, \cdots, X_n$ be a random sample from $X \sim \mathbb{P}_\theta$, with absolutely continuous c.d.f.

$$F_X(x;\theta) = \left(a + bx^{-1/\theta}\right) \mathbf{1}(x > 1),$$

where $\theta \in \Theta = (0,1)$ is an unknown parameter, and a and b are (fixed) constants.

1. (**5 points**) Show that $a = 1$ and $b = -1$.
2. (**5 points**) Find a density of $\mathbb{P}_\theta$, denoted $f_X(\cdot;\theta)$. Does $\mathbb{P}_\theta$ belong to the exponential family?
3. (**5 points**) Show that $\mathbb{E}_\theta[X_i] = 1/(1-\theta)$ and use this fact to derive a method of moments estimator for θ, denoted $\hat{\theta}_X^{MM}$. Verify that this estimator is downward biased.
4. (**5 points**) Find the log-likelihood function and show that the maximum likelihood estimator of θ is given by
$$\hat{\theta}^{ML} = \frac{1}{n}\sum_{i=1}^{n} \log(X_i).$$
5. (**5 points**) Let $Y = \log(X)$. Find the c.d.f. of Y, denoted $F_Y(\cdot;\theta)$, and verify that a density of Y is given by
$$f_Y(y;\theta) = \frac{1}{\theta}\exp\left(-\frac{y}{\theta}\right)\mathbf{1}(y > 0).$$
6. (**5 points**) Show that $\mathbb{E}_\theta[Y] = \theta$ and use this fact to derive a method moments estimator of θ, denoted $\hat{\theta}_Y^{MM}$. Is this estimator unbiased?
7. (**5 points**) Compute the Cramér-Rao bound (on the variance of unbiased estimators of θ). Is the MLE estimator UMVU?