Econ 671 – Final Exam

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CONTENTS

1. PART I – SHORT QUESTIONS (30 POINTS)

1. (**5 points**) Let X, Y and Z be square-integrable random variables. Show that

$$\mathbb{E}\left[(X-\mathbb{E}[X|Y])^2\right]=\mathbb{E}\left[(X-\mathbb{E}[X|Y,Z])^2\right]+\mathbb{E}\left[(\mathbb{E}[X|Y,Z]-\mathbb{E}[X|Y])^2\right].$$

Conclude that $\mathbb{E}[\mathbb{V}[X|Y,Z]]\leq\mathbb{E}[\mathbb{V}[X|Y]]\leq\mathbb{V}[X]$.

2. (**5 points**) Let $X_1,\cdots,X_n$ be i.i.d. with $\mathbb{E}[X_1]=\mu$ and $\mathbb{V}[X_1]=\sigma^2<\infty$, and $x\mapsto h(x)$ (at least) two times differentiable. Providing appropriate regularity conditions, show that (as $n\to\infty$)

$$\mathbb{E}\left[h\left(\bar{X}\right)\right]=h(\mu)+\frac{\sigma^2}{2n}h^{(2)}(\mu)+O\left(n^{-2}\right),$$

where $h^{(k)}(x)=\frac{\mathrm{d}^k}{\mathrm{d}x^k}h(x)$ and $h^{(k)}(\mu)\neq 0$, for $k=1,2,\cdots$.

3. (**20 points**) Let $X_1,\cdots,X_n$ be an i.i.d. sample from $X\sim Uniform(0,\theta)$, $\theta>0$. Recall that the MLE for θ is $X_{(n)}=\max_{1\leq i\leq n}X_i$, a (minimally) sufficient statistic.

   (a) (**5 points**) Show that the c.d.f. of $X_{(n)}$ is $F(x;\theta)=x^n\theta^{-n}\mathbf{1}(x\in(0,\theta))+\mathbf{1}(x>\theta)$. Use this result to verify that $X_{(n)}$ is a complete statistic. Is $X_{(n)}$ a UMVU estimator?

   (b) (**5 points**) For $\alpha\in(0,1)$, find a most powerful α-level test of $H_0:\theta=\theta_0$ vs $H_1:\theta=\theta_1$, with $\theta_1>\theta_0>0$. Is the UMP test based on a UMVU estimator?

   (c) (**5 points**) Derive the power function of the test obtained in part (b) and show that the test is consistent. Sketch a graph of the power function.

   (d) (**5 points**) Find a *uniformly* most powerful α-level test of $H_0:\theta\leq\theta_0$ vs $H_1:\theta>\theta_0$, with $\theta_0>0$.

2. PART II – A LONG QUESTION (40 POINTS)

Let $X_1, \cdots, X_n$ be an i.i.d. sample from $X \sim \mathbb{P}_\theta$, with absolutely continuous c.d.f.

$$F_X(x;\theta) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\theta} \exp\left(-\frac{1}{2}\left(\frac{u-\theta}{\theta}\right)^2\right) \mathrm{d}u,$$

where $\theta \in \Theta = \mathbb{R}_{++}$ is the unknown parameter. (Note that $X \sim \mathscr{N}\left(\theta, \theta^2\right)$.) Recall that $\Phi(z) = \int_{-\infty}^{z} \phi(u)\mathrm{d}u$, where $\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right)$.

1. (**5 points**) Verify that $-z\phi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\phi(z)$, and show that for $Z \sim \mathscr{N}(0,1)$,

$$\mathbb{E}[Z^{2k}] = (2k-1)\cdot(2k-3)\cdot\cdots\cdot 3\cdot 1 = \frac{(2k)!}{2^k k!}, \qquad k \in \mathbb{N}.$$

   Combine this result with the fact that $X = \theta + \theta Z$ to verify that $\mathbb{E}[X] = \theta$, $\mathbb{E}[X^2] = 2\theta^2$, $\mathbb{E}[X^3] = 4\theta^3$ and $\mathbb{E}[X^4] = 10\theta^4$.

2. (**5 points**) Show that $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $T_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ are sufficient statistics.

3. (**5 points**) Show that the Maximum Likelihood Estimator, denoted $\hat{\theta}_n$, is given by

$$\hat{\theta}_n = \frac{1}{2}\sqrt{S_n^2 + 4T_n} - \frac{1}{2}S_n.$$

4. (**5 points**) Is $\hat{\theta}_n$ a consistent estimator of θ?

5. (**5 points**) Show that

$$\sqrt{n}\left(\begin{bmatrix} S_n \\ T_n \end{bmatrix} - \begin{bmatrix} \theta \\ 2\theta^2 \end{bmatrix}\right) \xrightarrow{d} \mathscr{N}\left(0, \begin{bmatrix} \theta^2 & 2\theta^3 \\ 2\theta^3 & 6\theta^4 \end{bmatrix}\right).$$

   and verify that $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} \mathscr{N}\left(0, \frac{\theta^2}{3}\right)$.

6. (**5 points**) Find the Cramer-Rao Lower Bound for θ, denoted by $\mathrm{CRLB}(\theta)$. Is $\hat{\theta}_n$ an asymptotically efficient estimator?

7. (**5 points**) Construct an α-level (asymptotic) confidence interval for θ. Is this confidence interval always valid?

8. (**5 points**) Recall that the Cramer-Rao Lower Bounds for the mean and variance of a normal distribution $\mathscr{N}\left(\mu, \sigma^2\right)$, say, are given by $\mathrm{CRLB}(\mu) = \sigma^2/n$ and $\mathrm{CRLB}(\sigma) =$

$\sigma^2/(2n)$, respectively. In this model, for example, two "natural" estimators of θ are $\bar{X}$ and $\hat{\sigma}$, where

$$\bar{X} = S_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad \text{and} \qquad \hat{\sigma}^2 = T_n - S_n^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2.$$

(a) (**2 points**) Verify that

$$\sqrt{n}\left(\bar{X}-\theta\right) \xrightarrow{d} \mathcal{N}\left(0,\theta^2\right), \qquad \text{and} \qquad \sqrt{n}\left(\hat{\sigma}-\theta\right) \xrightarrow{d} \mathcal{N}\left(0,\frac{\theta^2}{2}\right),$$

which implies that they achieve the "Cramer-Rao Lower Bounds" for the mean and variance, respectively.

(b) (**3 points**) However, in this model, $\text{CRLB}(\theta) < \theta^2/n$ and $\text{CRLB}(\theta) < \theta^2/(2n)$. Explain intuitively this result and also discuss the difference between (asymptotically) efficient estimation and "robust" estimation in the context of this model.