CENTRAL MOTIVES ON PARAHORIC FLAG VARIETIES

ROBERT CASS, THIBAUD VAN DEN HOVE, JAKOB SCHOLBACH

ABSTRACT. We construct a refinement of Gaitsgory's central functor for integral motivic sheaves, and show it preserves stratified Tate motives. Towards this end, we develop a reformulation of unipotent motivic nearby cycles, which also works over higher-dimensional bases. We moreover introduce Wakimoto motives and use them to show that our motivic central functor is t-exact. A decategorification of these functors yields a new approach to generic Hecke algebras for general parahorics.

CONTENTS

1. INTRODUCTION

For a split reductive group G, the study of the spherical Hecke algebra and the Iwahori–Hecke algebra are at the center of the Langlands program. These are defined as

$$\mathcal{H}^{\mathrm{sph}} := \mathrm{Fun}_c(G(\mathbf{F}_q[\![t]\!]) \setminus G(\mathbf{F}_q(\!(t)\!))/G(\mathbf{F}_q[\![t]\!]), \mathbf{C}),$$

$$\mathcal{H}^{\mathcal{I}} := \mathrm{Fun}_c(\mathcal{I} \setminus G(\mathbf{F}_q(\!(t)\!))/\mathcal{I}, \mathbf{C}),$$

where $\mathcal{I} \subset G(\mathbf{F}_q[\![t]\!])$ denotes an Iwahori subgroup.

These algebras are categorified by various categories of sheaves on the affine Grassmannian Gr and the affine flag variety Fl, respectively. Moreover, certain properties of Hecke algebras can be upgraded to categorical statements. For example, the commutativity of $\mathcal{H}^{\mathrm{sph}}$ can be refined to the existence of a symmetric monoidal structure on the category $\mathrm{Perv}_{L^+G}(\mathrm{Gr})$ of L^+G-equivariant perverse sheaves on Gr. This property is one of the salient aspects of the geometric Satake equivalence, a cornerstone of the geometric Langlands program, which establishes an equivalence between that category and the representations of the Langlands dual group $\widehat{G}$.

Beginning with Mirković–Vilonen's pioneering paper [MV07], geometric Satake has attracted interest from many authors including [Ric14, Zhu15, Zhu17a, MR18, FS21, XZ22]. These papers treat various contexts, such as Betti sheaves, ℓ-adic sheaves, D-modules, as well as Witt vector and B_{dR}^+-affine Grassmannians. In [CvdHS22], the present authors refined several of these equivalences to a statement for integral motivic sheaves, building upon earlier work for rational motives in [RS21].

Bernstein [Ber84, Theorem 2.13] gave a description of the center of $\mathcal{H}^{\mathcal{I}}$ as

$$Z(\mathcal{H}^{\mathcal{I}}) = \mathcal{H}^{\mathrm{sph}}.$$

Following a suggestion of Beilinson, Gaitsgory [Gai01] achieved a geometrization of this isomorphism, which was improved upon by Zhu [Zhu14]. Namely, there is a geometric object over $\mathbf{A}^1$ whose fibers over $x \neq 0$ are Gr, and whose fiber over 0 is Fl, i.e., a deformation

$$\mathrm{Gr} \rightsquigarrow \mathrm{Fl}. \tag{1.1}$$

Taking nearby cycles for this family then yields the desired geometrization.

The purpose of this paper is to refine Gaitsgory's central functor to the level of integral motivic sheaves. In order to achieve this goal, we use new insights into the geometry of such $\mathbf{A}^1$-deformations and related Beilinson–Drinfeld Grassmannians, and combine these with a much-simplified approach to motivic unipotent nearby cycles.

In the ℓ-adic setting, Gaitsgory's central functor is also a key ingredient in Bezrukavnikov's tamely ramified local Langlands correspondence [AB09, Bez16]. As such, the constructions of the present paper, along with the motivic

Satake equivalence from [CvdHS22], should allow us to enhance [AB09, Bez16] to the motivic setting. In particular, this will eliminate the dependence on ℓ inherent in the use of ℓ-adic cohomology.

1.1. Geometry of Beilinson–Drinfeld Grassmannians.

Let $G/\operatorname{Spec}\mathbf{Z}$ be a split reductive group and let $\mathbf{f}$ be a facet in the Bruhat-Tits building of G contained in the closure of a standard alcove $\mathbf{a}_0$. Let $\mathrm{Fl}_\mathbf{f}$ be the partial affine flag variety associated to the corresponding parahoric $\mathbf{Z}[\![t]\!]$-model $G_\mathbf{f}$ of G (Gr and Fl are special cases). All results below also hold over arbitrary bases; for this introduction we restrict to $\operatorname{Spec}\mathbf{Z}$ for simplicity.

There is a smooth affine group scheme $\mathcal{G}_\mathbf{f}/\mathbf{A}^1_\mathbf{Z}$ which generically agrees with G, and whose fiber over the completed local ring at 0 agrees with the parahoric $G_\mathbf{f}$. The associated quotient $\mathrm{Gr}_{\mathcal{G}_\mathbf{f}} := L\mathcal{G}_\mathbf{f}/L^+\mathcal{G}_\mathbf{f}$ of the loop group by the positive loop group for $\mathcal{G}_\mathbf{f}$, known as a *Beilinson–Drinfeld Grassmannian*, provides a degeneration from Gr to $\mathrm{Fl}_\mathbf{f}$. The family in (1.1) is the special case when $\mathbf{f}$ is a standard alcove.

The $\mathbf{G}_\mathrm{m}$-fiber $(\mathrm{Gr}_{\mathcal{G}_\mathbf{f}})_\eta \cong \mathrm{Gr}\times\mathbf{G}_\mathrm{m}$ carries the stratification by L^+G-orbits, while the special fiber $(\mathrm{Gr}_{\mathcal{G}_\mathbf{f}})_s = \mathrm{Fl}_\mathbf{f}$ is stratified by $L^+G_\mathbf{f}$-orbits in $\mathrm{Fl}_\mathbf{f}$. These sets of strata form a basis of the spherical Hecke algebra, respectively the Hecke algebra for $\mathbf{f}$.

Previous work shows that categories of stratified Tate motives, i.e., motives whose restriction to each stratum lies in the subcategory generated by the "constant" motive $\mathbf{Z}$ and its Tate twists, are a small, but highly interesting subcategory of the category of all motivic sheaves, e.g. [RS21, CvdHS22]. The following theorem ensures this approach is again applicable to $\mathrm{Gr}_{\mathcal{G}_\mathbf{f}}$, cf. Theorem 4.37. Joint with Theorem 1.3(2), it ensures that unipotent nearby cycles preserve stratified Tate motives.

Theorem 1.1. *The above stratifications of* $(\mathrm{Gr}_{\mathcal{G}_\mathbf{f}})_\eta$ *and* $(\mathrm{Gr}_{\mathcal{G}_\mathbf{f}})_s$ *determine a universally anti-effective Whitney–Tate stratification of* $\mathrm{Gr}_{\mathcal{G}_\mathbf{f}}$, *in the sense of* [CvdHS22, Definition 2.6].

Anti-effectivity ensures that no positive Tate twists arise, which will be useful when considering generic Hecke algebras by decategorifying the motivic central functor. The key to proving Whitney–Tateness is the compatibility of push-pull functors from $\mathbf{G}_\mathrm{m}$ to the special fiber of $\mathbf{A}^1$ with hyperbolic localization. By additionally using that certain families of constant term functors preserve and reflect Tateness, we can reduce to the case where $G = T$ is a torus, in which case the degeneration is trivial.

1.2. Revisiting unipotent nearby cycles.

As was mentioned above, Gaitsgory's work crucially hinges on the unipotent nearby cycles functor (in the context of ℓ-adic sheaves). Such a functor has been developed by Ayoub for motivic categories [Ayo07a, Ayo07b]. Taking our cue from Campbell's work [Cam18], we construct this functor independently, using an ∞-categorical and much simplified approach. This is also necessary in view of the need of at least the rudiments of nearby cycles over 2-dimensional bases.

In the definition below, S is a connected scheme that is smooth of finite type over a Dedekind ring or a field. For a scheme of finite type X/S, we have the category $\mathrm{DM}(X)$ of integral motivic sheaves as constructed by Spitzweck [Spi18] and the full subcategory $\mathrm{DTM}(X)$ of Tate motives.

Definition 1.2. For a scheme $X \to \mathbf{A}^1_S$, we consider the generic and special fibers

$$X_\eta := X \times_{\mathbf{A}^1} \mathbf{G}_\mathrm{m} \xrightarrow{j} X \xleftarrow{i} X_s := X \times_{\mathbf{A}^1,0} S.$$

The unipotent nearby cycles functor is

$$\Upsilon : \mathrm{DM}(X_\eta) \to \mathrm{DTM}(\mathbf{G}_{\mathrm{m},S}) \otimes_{\mathrm{DTM}(S)} \mathrm{DM}(X_\eta) \xrightarrow{1^*\otimes i^*j_*} \mathrm{Nilp}_\mathbf{Q}\,\mathrm{DM}(X_s).$$

Here $\otimes$ denotes Lurie's tensor product of presentable stable ∞-categories. The first functor exhibits a natural action of the motivic cohomology of $\mathbf{G}_{\mathrm{m},S}$ (relative to S) on any motivic sheaf on X_η, akin to the natural R-action on any quasi-coherent sheaf $\mathcal{F}$ on a scheme $Z/\operatorname{Spec} R$; see the beginning of Section 3 for more details. Objects in the category $\mathrm{Nilp}_\mathbf{Q}\,\mathrm{DM}(X_s)$ are pairs consisting of some $M \in \mathrm{DM}(X_s)$ and a locally nilpotent map $\phi : M \otimes \mathbf{Q} \to M \otimes \mathbf{Q}(-1)$. This map ϕ can be thought of as the "logarithm of monodromy". We stress, however, that no logarithm (of any sort) appears in this construction; instead this operator arises as the action of the Koszul dual of the cohomology ring of $\mathbf{G}_\mathrm{m}$. The appearance of rational coefficients at this point is caused by the non-formality of the motive of $\mathbf{G}_\mathrm{m}$: while the motive $\mathrm{M}(\mathbf{G}_\mathrm{m})$ is $\mathbf{Z}\oplus\mathbf{Z}(1)[1]$, its natural coalgebra structure is square-zero (on the summand $\mathbf{Z}(1)[1]$) only after passing to rational coefficients. By contrast, for reduced motives as introduced in [ES23], our construction gives a similarly defined functor Υ which carries a "logarithm of monodromy" map already integrally.

Theorem 1.3. *The unipotent nearby cycles functor* Υ *enjoys the following properties.*

(1) *The functor* Υ *is naturally functorial (in a highly structured manner) with respect to smooth pullbacks and proper pushforwards. In addition, it is lax compatible with exterior products (cf. Theorem 3.2).*

(2) *If X carries a Whitney–Tate stratification (compatible with the strata $\mathbf{G}_\mathrm{m}$ and 0 in $\mathbf{A}^1$), then Υ preserves stratified Tate motives (cf. Proposition 3.16).*

(3) If $\mathbf{G}_m$ (regarded as a constant group scheme over $\mathbf{A}^1$) acts on X, with fixed point locus $X^0/\mathbf{A}^1$, then $\Upsilon_{X/\mathbf{A}^1}$ (applied to $\mathbf{G}_m$-monodromic motivic sheaves) is compatible with $\Upsilon_{X^0/\mathbf{A}^1}$ under hyperbolic localization (cf. Proposition 3.17).

Statement (2) is crucial to be able to use Υ in the context of Beilinson–Drinfeld Grassmannians, while (3) is the key to actual computations, since it allows us to eventually replace G by a maximal torus. The ℓ-adic version of (3) appeared in [Ric19] and has already seen many applications, such as [HR21]. We refer to Section 3 for further results, including the setup of (unipotent) nearby cycles for families over $\mathbf{A}^n$.

1.3. **The central functor.** We construct the central functor in Definition 4.15 as

$$\mathsf{Z}_{\mathbf{f}} \colon \mathrm{DM}(L^+G \backslash \mathrm{Gr}) \to \mathrm{Nilp}_{\mathbf{Q}}\, \mathrm{DM}(L^+G_{\mathbf{f}} \backslash \mathrm{Fl}_{\mathbf{f}}), \quad \mathsf{Z}_{\mathbf{f}}(\mathcal{F}) = \Upsilon(\mathcal{F} \boxtimes \mathbf{Z}_{\mathbf{G}_m}),$$

where Υ denotes the (unipotent) nearby cycles functor for the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}$ introduced above (or, more precisely its extension to L^+G-equivariant sheaves developed in Section 3.11).

Its main properties are as follows, cf. Theorem 4.38 and Theorem 4.41.

Theorem 1.4. *Let* $\mathcal{F}, \mathcal{F}' \in \mathrm{DTM}(L^+G \backslash \mathrm{Gr})$ *and* $\mathcal{M} \in \mathrm{DTM}(L^+G_{\mathbf{f}} \backslash \mathrm{Fl}_{\mathbf{f}})$.

(1) *The functor* $\mathsf{Z}_{\mathbf{f}}$ *preserves (anti-effective) stratified Tate motives, and hence restricts to a functor*

$$\mathsf{Z}_{\mathbf{f}} \colon \mathrm{DTM}(L^+G \backslash \mathrm{Gr}_G)^{\mathrm{(anti)}} \to \mathrm{DTM}(L^+G_{\mathbf{f}} \backslash \mathrm{Fl}_{\mathbf{f}})^{\mathrm{(anti)}}.$$

(2) *The functor* $\mathsf{Z}_{\mathbf{f}}$ *takes values in central sheaves with respect to the convolution product* $\star$ *in the sense that there are isomorphisms*

$$\mathsf{Z}_{\mathbf{f}}(\mathcal{F}) \star \mathcal{M} \cong \mathcal{M} \star \mathsf{Z}_{\mathbf{f}}(\mathcal{F}).$$

(3) *There is an isomorphism*

$$\mathsf{Z}_{\mathbf{f}}(\mathcal{F}) \star \mathsf{Z}_{\mathbf{f}}(\mathcal{F}') \cong \mathsf{Z}_{\mathbf{f}}(\mathcal{F} \star \mathcal{F}').$$

At least on the level of the homotopy categories, these isomorphisms endow $\mathsf{Z}_{\mathbf{f}}$ *with the structure of a monoidal functor to the Drinfeld center,*

$$\mathrm{Ho}(\mathrm{DTM}(L^+G \backslash \mathrm{Gr})^{\mathrm{(anti)}}) \to \mathcal{Z}(\mathrm{Ho}(\mathrm{DTM}(L^+G_{\mathbf{f}} \backslash \mathrm{Fl}_{\mathbf{f}})^{\mathrm{(anti)}})).$$

The proof follows similar reasoning as in the work of Achar–Riche [AR]. To state further properties of $\mathsf{Z}_{\mathbf{f}}$, we recall the abelian category of mixed Tate motives $\mathrm{MTM}(L^+G_{\mathbf{f}} \backslash \mathrm{Fl}_{\mathbf{f}})$, which arises as the heart of a t-structure on $\mathrm{DTM}(L^+G_{\mathbf{f}} \backslash \mathrm{Fl}_{\mathbf{f}})$. At least with $\mathbf{Q}$-coefficients, a compact object lies in the heart if and only if its Betti realization is a perverse sheaf. The main result of [CvdHS22] identifies $\mathrm{MTM}(L^+G \backslash \mathrm{Gr}_G)$ with a motivic category of integral representations of $\hat{G}$. For the restriction of $\mathsf{Z}_{\mathbf{f}}$ to $\mathrm{MTM}(L^+G \backslash \mathrm{Gr}_G)$, we upgrade Theorem 1.4 with a compatibility between the centrality and monoidality isomorphisms (Theorem 4.49), as well as a compatibility with the commutativity constraint (Proposition 4.47) coming from fusion [CvdHS22, §5.3]. The former compatibility uses nearby cycles over a 2-dimensional base, which was one of our motivations to revisit the nearby cycles functor. Upgrading the monoidality of $\mathsf{Z}_{\mathbf{f}}$ to an ∞-categorical statement does not seem to require new geometric ideas, but rather an ∞-categorical study of the compatibility of Koszul dualities for varying (co)algebras, which we defer to another paper.

1.4. **Wakimoto motives.** One property of $\mathsf{Z}_{\mathbf{f}}$ that is not yet addressed in Theorem 1.4 is t-exactness. For ℓ-adic and Betti sheaves, t-exactness for the perverse t-structure is a general property of nearby cycles. Such a general result is currently not available for motivic sheaves, but we are able to prove the t-exactness of $\mathsf{Z}_{\mathbf{f}}$ by other methods. The t-exactness when $\mathbf{f} = \mathbf{f}_0$ is hyperspecial follows from Proposition 4.46. On the opposite end, when $G_{\mathbf{a}_0} = \mathcal{I}$ is an Iwahori group scheme, we adapt arguments of [ALWY23], by using a motivic refinement of Wakimoto sheaves as follows.

Classically, Wakimoto sheaves are perverse sheaves introduced by Mirković to categorify a maximal commutative subalgebra of $H^{\mathcal{I}}$. Geometrically, this manifests by the fact that every perverse central sheaf on Fl admits a filtration by Wakimoto sheaves. Motivically, we then construct a Wakimoto functor $\mathbf{J}_\mu^B \colon \mathrm{DTM}(\mathcal{I} \backslash \mathrm{Spec}\, \mathbf{Z}) \to \mathrm{DTM}(\mathcal{I} \backslash \mathrm{Fl})$, cf. [AB09], where $\mu \in X_*(T)$ and $T \subset B$ are a maximal torus and Borel, and prove it is t-exact. We also use the t-exact constant term functor $\mathrm{CT}_{B^-}[-\deg] := \oplus \mathrm{CT}_{B^-}^\mu[-\langle 2\rho, \mu \rangle]$ on $\mathrm{DTM}(L^+G \backslash \mathrm{Gr})$ from [CvdHS22, Definition 5.2], where B^- is the opposite Borel and 2ρ is the sum of the positive roots, which under the Satake equivalence corresponds to the restriction functor $\mathrm{Rep}_{\hat{G}} \to \mathrm{Rep}_{\hat{T}}$.

Theorem 1.5. *For any* $\mathcal{F} \in \mathrm{DTM}(L^+G \backslash \mathrm{Gr})$, *the central motive* $\mathsf{Z}_{\mathbf{a}_0}(\mathcal{F})$ *admits a (necessarily unique)* $X_*(T)$-*filtration with graded pieces given by* $\mathbf{J}_\mu^B(\mathrm{CT}_{B^-}^\mu(\mathcal{F}))(-\langle 2\rho, \mu \rangle)[-\langle 2\rho, \mu \rangle]$ *(cf. Theorem 5.24). In particular,* $\mathsf{Z}_{\mathbf{a}_0}$ *is t-exact.*

This t-exactness will be a crucial ingredient in a motivic refinement of [AB09, Bez16], as alluded to above. Along the way we must also give alternative proofs of two related t-exactness results in Proposition 5.2 (cf. [EK19, Lemma 4.21]) and Lemma 5.4 (cf. [AR16, Proposition 4.6]), whose analogues in [AR, ALWY23] are proved using Artin vanishing. Finally, following a suggestion of Achar and discussions with Lourenço, we use the above Wakimoto filtration to show that the central functor is t-exact for general facets in Theorem 5.30.

1.5. **Generic Hecke algebras.** In [CvdHS22, Definition 6.35], the present authors defined the generic spherical Hecke algebra $\mathcal{H}^{\mathrm{sph}}(\mathbf{q})$. This is a $\mathbf{Z}[\mathbf{q}]$-algebra, where $\mathbf{q}$ is a formal variable, which specializes to the spherical Hecke algebra of functions on $G(\mathbf{F}_q((t)))$ under the map $\mathbf{q} \mapsto q$. The motivic Satake equivalence gives an isomorphism between $\mathcal{H}^{\mathrm{sph}}(\mathbf{q})$ and the representation ring of a certain Vinberg monoid for $\hat{G}$ appearing in Zhu's work [Zhu20], and hence with the Grothendieck ring of a suitable category of anti-effective motives on Gr. On the other hand, Vignéras has defined the generic Iwahori–Hecke algebra $\mathcal{H}^{\mathcal{I}}(\mathbf{q})$ [Vig06]. This can also be realized as the Grothendieck ring of a suitable category of anti-effective motives on the full affine flag variety, cf. Proposition 6.3.

These definitions of generic Hecke algebras (which also appear in [PS23] for $G = \mathrm{GL}_2$) rely on combinatorics: Kazhdan–Lusztig theory for $\mathcal{H}^{\mathrm{sph}}$, and the Iwahori–Matsumoto presentation for $\mathcal{H}^{\mathcal{I}}$. However, the description in terms of Grothendieck rings suggests that, for general facet $\mathbf{f}$, we can *define* the generic parahoric Hecke algebra $\mathcal{H}^{\mathbf{f}}(\mathbf{q})$ at $G_{\mathbf{f}}$-level as the Grothendieck ring $K_0(\mathrm{DTM}(L^+G_{\mathbf{f}} \backslash \mathrm{Fl}_{\mathbf{f}})^{\mathrm{lc,anti}})$, cf. Definition 6.1 for details. These are $\mathbf{Z}[\mathbf{q}]$-algebras which specialize to the usual parahoric Hecke algebras by Proposition 6.4. Moreover, we can again decategorify $\mathsf{Z}_{\mathbf{f}}$ to relate $\mathcal{H}^{\mathrm{sph}}(\mathbf{q})$ to the center of $\mathcal{H}^{\mathbf{f}}(\mathbf{q})$. The situation can be summarized as follows:

$$
\begin{array}{ccc}
\mathrm{DTM}(L^+G \backslash \mathrm{Gr})^{\mathrm{lc,anti}} & \xrightarrow{\mathsf{Z}_{\mathbf{f}}} & \mathrm{DTM}(L^+G_{\mathbf{f}} \backslash \mathrm{Fl}_{\mathbf{f}})^{\mathrm{lc,anti}} \\
\Big\downarrow{\scriptstyle K_0} & & \Big\downarrow{\scriptstyle K_0} \\
\mathcal{H}^{\mathrm{sph}}(\mathbf{q}) & \longrightarrow & \mathcal{H}^{\mathbf{f}}(\mathbf{q}) \\
\Big\downarrow{\scriptstyle -\otimes_{\mathbf{Z}[\mathbf{q}],\mathbf{q} \mapsto q}\mathbf{Z}} & & \Big\downarrow{\scriptstyle -\otimes_{\mathbf{Z}[\mathbf{q}],\mathbf{q} \mapsto q}\mathbf{Z}} \\
\mathcal{H}^{\mathrm{sph}} & \longrightarrow & \mathcal{H}^{\mathbf{f}},
\end{array}
$$

where the two lower horizontal arrows have central image.

For the particular case $\mathbf{f} = \mathbf{a}_0$, we get a *generic Bernstein isomorphism*, generalizing Bernstein's isomorphism $Z(\mathcal{H}^{\mathcal{I}}) \cong \mathcal{H}^{\mathrm{sph}}$, cf. Theorem 6.9.

Theorem 1.6. *If $\mathbf{f} = \mathbf{a}_0$ is a standard alcove, then decategorifying the central functor $\mathsf{Z}_{\mathbf{a}_0}$ induces an isomorphism between $\mathcal{H}^{\mathrm{sph}}(\mathbf{q})$ and the center of $\mathcal{H}^{\mathcal{I}}(\mathbf{q})$.*

Along the way of this theorem, we also obtain independence-of-ℓ for related results in [AR, §5.3] which used mixed ℓ-adic sheaves.

2. Preliminaries

2.1. **Higher algebra.** In this section, we recall and establish some basic notions around tensor products in the context of ∞-categories and Koszul duality. In Section 2, all our monoidal categories are actually *symmetric* monoidal, so that we often drop the word "symmetric" in the sequel.

2.1.1. *Monoidality and adjoints.* We will work with the following monoidal ∞-categories [BGT13, §3.1]

$$
\mathrm{Cat}^{\mathrm{perf}} \overset{\mathrm{Ind}}{\underset{\cong}{\to}} \mathrm{Pr}^{\mathrm{L}}_{\omega,\mathrm{St}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}. \tag{2.1}
$$

From right to left, we have the category of stable presentable ∞-categories, with colimit-preserving functors; then its (non-full) subcategory of compactly generated stable categories, with functors preserving compact objects [Lur17, Notation 5.3.2.8], and then the ∞-category of small stable idempotent complete ∞-categories with exact functors.

We also consider $\mathrm{Pr}^{\mathrm{LR}}_{\omega,\mathrm{St}}$, the ∞-category of presentable compactly generated stable ∞-categories with functors being *both left and right adjoints*.

Recall Lurie's tensor product on Pr^{L} and on its subcategory $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ [Lur17, §4.8.1, Proposition 4.8.2.18].

Lemma 2.1. *The categories* $\mathrm{Pr}^{\mathrm{L}}_{\omega,\mathrm{St}} \subset \mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ *and* $\mathrm{Pr}^{\mathrm{LR}}_{\omega,\mathrm{St}} \subset \mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ *are monoidal subcategories. Furthermore, with respect to these monoidal structures, there is an equivalence of monoidal categories given by passing to right adjoints:*

$$\mathrm{Pr}^{\mathrm{L}}_{\omega,\mathrm{St}} \xrightarrow{\cong} (\mathrm{Pr}^{\mathrm{LR}}_{\omega,\mathrm{St}})^{\mathrm{op}}.$$

Proof. The condition of being compactly generated is preserved under Lurie's tensor product [Lur17, Lemma 5.3.2.11].

Let $f_i : C_i \to C_i'$ be finitely many colimit-preserving left adjoints between stable compactly generated categories. By our assumptions, the right adjoints f_i^R also preserve colimits, and therefore are *left* adjoints as well. Thus, we can consider the functor $\bigotimes_i (f_i^R)$. Let $f := \bigotimes f_i$. There is a natural map $\bigotimes f_i^R \to f^R$. To see it is an isomorphism let $c_i \in C_i$ be arbitrary objects. The elementary tensors $\boxtimes c_i$ generate the category $\bigotimes C_i$. Thus it suffices to note that

$$\mathrm{Maps}_{\bigotimes C_i}(\boxtimes c_i, \boxtimes f_i^R(c_i')) = \mathrm{Maps}_{\bigotimes C_i}(\boxtimes c_i, f^R(\boxtimes c_i')).$$

Indeed, by [GR17, Proposition I.7.4.2], both sides are canonically identified with $\bigotimes_i \mathrm{Maps}_{C_i}(f_i(c_i), c_i')$.

Therefore, the equivalence $(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}})^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{R}}_{\mathrm{St}}$ given by the adjoint functor theorem restricts to an equivalence of ∞-categories as indicated.

Let $\mathrm{Cat}^{\mathrm{L}}_{\mathrm{St}}$ (resp. $\mathrm{Cat}^{\mathrm{R}}_{\mathrm{St}}$) be the (non-full) subcategory of $\mathrm{Cat}_{\mathrm{St}}$ (stable ∞-categories and exact functors) where only left (resp. right adjoints) are allowed. The cartesian monoidal structure on $\mathrm{Cat}_{\mathrm{St}}(\subset \mathrm{Cat}_\infty)$ [Lur17, Theorem 1.1.4.4] respects these subcategories. We indicate these by $\mathrm{Cat}^{\mathrm{L/R},\times}_{\mathrm{St}}$ (even though on these subcategories it is no longer cartesian). We have an equivalence $\mathrm{Cat}^{\mathrm{L}}_\infty \cong (\mathrm{Cat}^{\mathrm{R}}_\infty)^{\mathrm{op}}$ given by the unicity of adjoints [Lur09, Proposition 5.2.6.2]. According to the construction of this equivalence, this equivalence is monoidal. This respects the (monoidal) "St"-subcategories, giving rise to the commutative triangle at the right below:

$$
\begin{array}{ccccc}
\mathrm{Pr}^{\mathrm{L},\otimes}_{\omega,\mathrm{St}} & \longrightarrow & \mathrm{Cat}^{\mathrm{L},\times}_{\mathrm{St}} & \longrightarrow & \mathrm{Fin}_* \\
\downarrow & & \downarrow{\scriptstyle\cong} & \nearrow & \\
(\mathrm{Pr}^{\mathrm{LR},\otimes}_{\omega,\mathrm{St}})^{\mathrm{op}} & \longrightarrow & (\mathrm{Cat}^{\mathrm{R},\times}_{\mathrm{St}})^{\mathrm{op}} & &
\end{array}
$$

The left horizontal maps are the canonical inclusions [Lur17, Notation 4.8.1.2, Proposition 4.8.1.15]. The left vertical arrow exists, and is an equivalence in each fiber over $\langle n \rangle \in \mathrm{Fin}_*$, by the above (applied to each f_i individually). It is symmetric monoidal by the isomorphisms $(\bigotimes f_i)^R = \bigotimes (f_i^R)$. $\qquad\square$

Recall that a $\otimes$-category D is *rigid* if it is stable, and compactly generated by a monoidal subcategory $D_0 \subset D$ in which every object is dualizable. I.e., $D = \mathrm{Ind}(D_0)$, and $(-)^\vee : D_0 \xrightarrow{\cong} D_0^{\mathrm{op}}$ is an equivalence. The Ind-completion of this equivalence gives an equivalence $D^\vee = D$, where $\vee$ refers to the dualization with respect to the tensor product in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ [GR17, Ch. I, Proposition 7.3.5].

Lemma 2.2. *For a rigid $\otimes$-category D, there are equivalences*

$$\mathrm{Mod}_D(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}) = \mathrm{Mod}_{D^\vee}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}) = \mathrm{coMod}_D(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}),$$

which are given by the identity on the level of the underlying objects in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$.

Proof. The first equivalence arises from the isomorphism (of commutative algebra objects) $D \cong D^\vee$. The second is a generality about (co)modules over dualizable objects: if d is a dualizable object in a symmetric monoidal ∞-category C, the forgetful functor $\mathrm{coMod}_d C \to C$ admits a *left* adjoint, given by $d^\vee \otimes -$. It is therefore monadic, and thus admits a functor to $\mathrm{Mod}_{d^\vee} C$, which is an equivalence. $\qquad\square$

Remark 2.3. Let C be an object in $\mathrm{Mod}_D(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}})$, with action map $a : D \otimes C \to C$. Under the above equivalence, it corresponds to the D-comodule whose coaction is given by a^R, the right adjoint of a [GR17, Lemma I.9.3.2]. Furthermore, we have $a = \mu \otimes_D \mathrm{id}$, for the multiplication $\mu : D \otimes D \to D$. Using again that passing to right adjoints is compatible with $\otimes$, we obtain the following description of the D-coaction on C:

$$a^R = \mu^R \otimes_D \mathrm{id}. \tag{2.2}$$

Lemma 2.4. *For a rigid category D, the right adjoint of the multiplication map, $\mu^R : D \to D \otimes D$, is a lax (symmetric) monoidal functor. In particular, there is a natural transformation like so:*

$$
\begin{array}{ccc}
D \otimes D & \xrightarrow{\mu^R \otimes \mu^R} & D^2 \otimes D^2 \\
\downarrow{\scriptstyle\mu} & \swarrow & \downarrow{\scriptstyle\mu_{D^2}} \\
D & \xrightarrow{\mu^R} & D^2.
\end{array}
$$

Proof. The multiplication μ is monoidal, so its right adjoint is lax monoidal. $\qquad\square$

Corollary 2.5. *Let D be rigid and let*

$$
U : \mathrm{Mod}_D(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}) \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}
$$

be the forgetful functor. According to the above, its left adjoint and its right adjoint agree up to equivalence, and are denoted by L. The unit map (for it being the right adjoint)

$$
\alpha : \mathrm{id} \to LU
$$

is then a natural transformation of lax monoidal endofunctors of $\mathrm{Mod}_D(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}})$.

Proof. The map α is given by applying $\mathrm{id} \otimes_D -$ to $\mu^R : D \to D \otimes D$. $\qquad\square$

Lemma 2.6. [BZFN10, Proposition 4.1] *For a symmetric monoidal presentable ∞-category C, a commutative algebra object $A \in \mathrm{CAlg}(C)$ and a C-module M (in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$), there is an equivalence*

$$
\mathrm{Mod}_A(C) \otimes_C M \xrightarrow{\cong} \mathrm{Mod}_A(M). \tag{2.3}
$$

2.1.2. *Koszul duality.* The following discussion of an element of Koszul duality will later be used in order to establish the monodromy map on unipotent nearby cycles.

Let $\mathrm{Ab}^{\mathbf{Z}}$ denote the (ordinary) category of $\mathbf{Z}$-graded abelian groups, and $\mathrm{Ab}^{\mathbf{Z}}_{\mathrm{free}} \subset \mathrm{Ab}^{\mathbf{Z}}$ its full subcategory of degreewise free abelian groups. We also use $\mathrm{Ch}^{\mathbf{Z}}$, the category of unbounded chain complexes of $\mathbf{Z}$-graded abelian groups. These categories are symmetric monoidal for the usual (underived) tensor product of graded objects, given by Day convolution along $\mathbf{Z}$. Finally, let $\mathrm{D}^{\mathbf{Z}} := \mathrm{D}(\mathrm{Ab}^{\mathbf{Z}})$ be the derived ∞-category of $\mathbf{Z}$-graded abelian groups [Lur17, Definition 1.3.5.8]. This is a symmetric monoidal ∞-category when endowed with the derived tensor product [Lur17, Remark 7.1.2.12], and again Day convolution along $\mathbf{Z}$. The composite $\mathrm{Ab}^{\mathbf{Z}}_{\mathrm{free}} \subset \mathrm{Ab}^{\mathbf{Z}} \subset \mathrm{D}^{\mathbf{Z}}$ is symmetric monoidal with respect to these tensor structures.

Let $\Sigma := \bigoplus_{n \geq 0} \mathbf{Z}(n) \in \mathrm{Ab}^{\mathbf{Z}}_{\mathrm{free}}$. It is the free commutative algebra generated by $\mathbf{Z}(1)$ (i.e. $\mathbf{Z}$ placed in graded degree -1), in the abelian category $\mathrm{Ab}^{\mathbf{Z}}$. Endowed with the counit being the natural augmentation map $\Sigma \to \mathbf{Z}$ and the comultiplication

$$
\begin{aligned}
\nabla_\Sigma : \Sigma &\to \Sigma \otimes \Sigma, \\
\mathbf{Z}(1) \ni t &\mapsto t \otimes 1 + 1 \otimes t,
\end{aligned} \tag{2.4}
$$

Σ is a bicommutative (commutative and cocommutative) bialgebra object in $\mathrm{Ab}^{\mathbf{Z}}_{\mathrm{free}}$. Using the symmetric monoidal functor to $\mathrm{D}^{\mathbf{Z}}$, we will also regard Σ as a bicommutative bialgebra object in $\mathrm{D}^{\mathbf{Z}}$.

Note that Σ is the free associative algebra in $\mathrm{D}^{\mathbf{Z}}$ generated by $\mathbf{Z}(1)$. The map $\Sigma \to \mathrm{Sym}_{\mathrm{D}^{\mathbf{Z}}} \mathbf{Z}(1) = \bigoplus_{n \geq 0} \mathbf{Z}(n)/S_n$ to the free commutative algebra on $\mathbf{Z}(1)$, taken in $\mathrm{D}^{\mathbf{Z}}$, is only an isomorphism after taking 0-th cohomology, or after tensoring with $\mathbf{Q}$. Indeed, the *derived* coinvariants of the trivial action of S_n on $\mathbf{Z}(n)$ have torsion cohomologies in positive cohomological degrees.

Since Σ is a commutative algebra in $\mathrm{D}^{\mathbf{Z}}$, the ∞-category of left Σ-modules $\mathrm{Mod}_\Sigma(\mathrm{D}^{\mathbf{Z}})$ carries the derived tensor product $- \otimes_\Sigma -$. In the sequel, however, we need a symmetric monoidal structure whose tensor product is the derived tensor product over $\mathbf{Z}$ instead. To formally construct that symmetric monoidal structure, we will present $\mathrm{Mod}_\Sigma(\mathrm{D}^{\mathbf{Z}})$ as the ∞-category underlying a symmetric monoidal model category whose construction is as follows: $\mathrm{Ch}^{\mathbf{Z}}$ carries the projective model structure, e.g. [Lur17, Proposition 7.1.2.8]. Being degreewise free, Σ is cofibrant in $\mathrm{Ch}^{\mathbf{Z}}$, so the model structure on $\mathrm{Mod}_\Sigma(\mathrm{Ch}^{\mathbf{Z}})$ right-induced from the forgetful functor $U : \mathrm{Mod}_\Sigma(\mathrm{Ch}^{\mathbf{Z}}) \to \mathrm{Ch}^{\mathbf{Z}}$ exists [SS00, Remark 4.2]. We indicate it by $\mathrm{Mod}_\Sigma(\mathrm{Ch}^{\mathbf{Z}})^{\mathrm{r}}$. Its weak equivalences and fibrations are created by U; in particular weak equivalences are (Σ-module) maps which are quasi-isomorphisms in each graded degree. Its generating (acyclic) cofibrations are of the form $\Sigma(k) \otimes c$, where c is a generating (acyclic) cofibration in $\mathrm{Ch}(\mathrm{Ab})$ (for those, see, e.g. [Hov99, Definition 2.3.3]).

On $\mathrm{Mod}_\Sigma(\mathrm{Ch}^{\mathbf{Z}})$, we now also consider the model structure *left-induced* along the forgetful functor U to C, i.e., U creates *co*fibrations and weak equivalences. We denote it by $\mathrm{Mod}_\Sigma(\mathrm{Ch}^{\mathbf{Z}})^{\mathrm{l}}$. To ensure its existence we use [GKR20, Corollary 2.7]: if $f \in \mathrm{Mod}_\Sigma(\mathrm{Ch}^{\mathbf{Z}})$ is a map having the right lifting property with respect to all maps $g \in \mathrm{Mod}_\Sigma(\mathrm{Ch}^{\mathbf{Z}})$ such that $U(g)$ is a cofibration in $\mathrm{Ch}^{\mathbf{Z}}$, we need to show f is a weak equivalence. Indeed, for any cofibration h in

$\mathrm{Ch}^{\mathbf{Z}}$, the map $g := \Sigma \otimes_{\mathbf{Z}} h$ is such that $U(g)$ is a cofibration, since Σ is cofibrant in $\mathrm{Ch}^{\mathbf{Z}}$. Such maps g generate the cofibrations in $\mathrm{Mod}_\Sigma(\mathrm{Ch}^{\mathbf{Z}})^{\mathrm{r}}$. Maps having the right lifting property against such g are acyclic fibrations in $\mathrm{Mod}_\Sigma(\mathrm{Ch}^{\mathbf{Z}})^{\mathrm{r}}$, in particular these are degreewise quasi-isomorphisms. This confirms the existence of the left-induced model structure.

Given that the ∞-category underlying $\mathrm{Ch}^{\mathbf{Z}}$ is $\mathrm{D}^{\mathbf{Z}}$, and given [Lur17, 4.3.3.17], the ∞-category $\mathrm{Mod}_\Sigma(\mathrm{D}^{\mathbf{Z}})$ is equivalent to the ∞-category underlying $\mathrm{Mod}_\Sigma(\mathrm{Ch}^{\mathbf{Z}})^{\mathrm{l}}$ or the Quillen equivalent $\mathrm{Mod}_\Sigma(\mathrm{Ch}^{\mathbf{Z}})^{\mathrm{r}}$.

We regard $\mathrm{Mod}_\Sigma(\mathrm{Ch}^{\mathbf{Z}})$ as a symmetric monoidal category whose tensor product is computed on the underlying objects in $\mathrm{Ch}^{\mathbf{Z}}$, endowed with the Σ-action

$$\Sigma \otimes (M \otimes N) \overset{\nabla_\Sigma}{\to} \Sigma \otimes \Sigma \otimes M \otimes N \overset{\mathrm{act}_M \otimes \mathrm{act}_N}{\to} M \otimes N.$$

Equivalently, expressing the Σ-action on M as a map $e_M : M(1) \to M$, the map $(M \otimes N)(1) \to M \otimes N$ is the sum $e_M \otimes \mathrm{id}_N + \mathrm{id}_M \otimes e_N$. With this monoidal structure, $\mathrm{Mod}_\Sigma(\mathrm{Ch}^{\mathbf{Z}})^{\mathrm{l}}$ is a symmetric monoidal model category since both (acyclic) cofibrations and the tensor product are created by the forgetful functor U.

Definition 2.7. We denote by $(\mathrm{Mod}_\Sigma(\mathrm{D}^{\mathbf{Z}}), \otimes_{\mathbf{Z}})$ the underlying symmetric monoidal ∞-category [Lur17, Definition 4.1.7.6] of $\mathrm{Mod}_\Sigma(\mathrm{Ch}^{\mathbf{Z}})^{\mathrm{l}}$.

Remark 2.8. By the setup, the forgetful functor $(\mathrm{Mod}_\Sigma(\mathrm{D}^{\mathbf{Z}}), \otimes_{\mathbf{Z}}) \to \mathrm{D}^{\mathbf{Z}}$ is symmetric monoidal, i.e., the (derived) tensor product of Σ-modules is computed on the underlying objects in $\mathrm{D}^{\mathbf{Z}}$. In particular, $\mathbf{Z}$ (not Σ) is the monoidal unit. We expect that the symmetric monoidal ∞-category structure constructed above is compatible with the one constructed by Beardsley [Bea23, Theorem 3.18] in much greater generality, but we have not investigated this in detail.

We let

$$\Lambda := \mathbf{Z} \oplus \mathbf{Z}(-1)[-1] \in \mathrm{Ch}^{\mathbf{Z}}.$$

Endowed with the multiplication such that $\mathbf{Z}(-1)[-1]$ is square zero, it is a cdga, i.e., a commutative algebra object in $\mathrm{Ch}^{\mathbf{Z}}$. It is cofibrant in $\mathrm{Ch}^{\mathbf{Z}}$. Parallely to $\mathrm{Mod}_\Sigma(\mathrm{Ch}^{\mathbf{Z}})^{\mathrm{r}}$, we have the right-induced model structure $\mathrm{Mod}_\Lambda(\mathrm{Ch}^{\mathbf{Z}})^{\mathrm{r}}$, which is a symmetric monoidal model category with respect to the (usual) tensor product $- \otimes_\Lambda -$. Its underlying symmetric monoidal ∞-category is equivalent to $\mathrm{Mod}_\Lambda(\mathrm{D}^{\mathbf{Z}})$ (with its usual derived tensor product over Λ) by applying [Lur17, Theorem 7.1.2.13] to Λ.

The augmentation maps turn $\mathbf{Z}$ into a Σ-Λ-bimodule (in $\mathrm{Ch}^{\mathbf{Z}}$). Moreover, the resulting functor

$$\mathbf{Z} \otimes_\Lambda - : \mathrm{Mod}_\Lambda(\mathrm{Ch}^{\mathbf{Z}})^{\mathrm{r}} \to \mathrm{Mod}_\Sigma(\mathrm{Ch}^{\mathbf{Z}})^{\mathrm{l}} \tag{2.5}$$

is a symmetric monoidal functor left Quillen functor, i.e., preserves the monoidal unit and tensor products up to isomorphism, and preserves (acyclic) cofibrations. Its right adjoint is the (underived) functor $\underline{\mathrm{Hom}}_\Sigma(\mathbf{Z}, -)$.

Definition and Lemma 2.9. The derived tensor product functor

$$\mathbf{Z} \otimes_\Lambda - : (\mathrm{Mod}_\Lambda(\mathrm{D}^{\mathbf{Z}}), \otimes_\Lambda) \to (\mathrm{Mod}_\Sigma(\mathrm{D}^{\mathbf{Z}}), \otimes_{\mathbf{Z}}), \tag{2.6}$$

is fully faithful and symmetric monoidal. Therefore, if we write Nilp for the essential image of the left adjoint, we obtain an equivalence (of presentably symmetric monoidal ∞-categories)

$$(\mathrm{Mod}_\Lambda(\mathrm{D}^{\mathbf{Z}}), \otimes_\Lambda) \overset{\cong}{\to} \mathrm{Nilp} \subset (\mathrm{Mod}_\Sigma(\mathrm{D}^{\mathbf{Z}}), \otimes_{\mathbf{Z}}). \tag{2.7}$$

Henceforth, we abbreviate $\mathrm{Mod}_\Sigma := \mathrm{Mod}_\Sigma(\mathrm{D}^{\mathbf{Z}})$ and $\mathrm{Mod}_\Lambda := \mathrm{Mod}_\Lambda(\mathrm{D}^{\mathbf{Z}})$, with the symmetric monoidal ∞-category structures constructed above.

Proof. We obtain the functor (2.6) from the left Quillen functor (2.5) by passing to the underlying ∞-categories. The latter, and therefore also the former, functor is symmetric monoidal by the above discussion.

The usual short exact sequence in $\mathrm{Mod}_\Sigma(\mathrm{Ab}^{\mathbf{Z}})$

$$0 \to \Sigma(1) \to \Sigma \to \mathbf{Z} \to 0, \tag{2.8}$$

is a resolution by cofibrant Σ-modules. It shows that $\mathbf{Z}$ is compact in $\mathrm{Mod}_\Sigma(\mathrm{D}^{\mathbf{Z}})$, so that the derived right adjoint $\underline{\mathrm{Hom}}_\Sigma(\mathbf{Z}, -)$ (the derived inner Hom in $\mathrm{Mod}_\Sigma(\mathrm{D}^{\mathbf{Z}})$) preserves colimits. It also implies that the unit map $\Lambda \to \underline{\mathrm{Hom}}_\Sigma(\mathbf{Z}, \mathbf{Z})$ for the adjunction given by (2.6) is a quasi-isomorphism. This implies that this unit map is a quasi-isomorphism for all objects in $\mathrm{Mod}_\Lambda(\mathrm{D}^{\mathbf{Z}})$. $\qquad\square$

Remark 2.10. For $\mathbf{Q}$-linear (but otherwise arbitrary) categories, a similar monoidal Koszul-type equivalence appears in [BGV23, Proposition 2.32], which in turn refines non-monoidal results due to Raskin and Barkan (see op. cit. for references).

The objects of $\mathrm{Mod}_\Sigma(\mathrm{D}^{\mathbf{Z}})$ can be described as pairs $(c \in \mathrm{D}(\mathrm{Ab}^{\mathbf{Z}}), \phi : c(1) \to c)$. Being equivalent to $\mathrm{Mod}_\Lambda(\mathrm{D}^{\mathbf{Z}})$, the full subcategory $\mathrm{Nilp} \subset \mathrm{Mod}_\Sigma(\mathrm{D}^{\mathbf{Z}})$ is compactly generated by $\mathbf{Z}(k)$, $k \in \mathbf{Z}$, regarded as a trivial Σ-module (i.e., Σ acts via its augmentation). Thus, for a compact object $(c, \phi) \in \mathrm{Nilp}$ (which is a retract of a finite colimit of such

generators), ϕ is nilpotent. An arbitrary object $(c, \phi) \in \mathrm{Nilp}$ is a filtered colimit of compact objects, so one may think of ϕ as being locally nilpotent.

If we replace $\mathrm{D}(\mathrm{Ab}^{\mathbf{Z}})$ by $\mathrm{D}(\mathrm{Ab})$ (i.e., suppress all gradings), then Nilp can be characterized geometrically as the full subcategory the derived category of quasi-coherent sheaves on $\mathbf{A}_{\mathbf{Z}}^1 = \mathrm{Spec}\,\mathbf{Z}[t]$ spanned by objects supported at 0, i.e., annihilated by inverting t, cf. [DG02, Theorem 2.1]. From this perspective, the monoidal structure on Nilp is the convolution along the addition on $\mathbf{A}^1$.

2.2. Motives.

Notation 2.11. Throughout we fix a connected base scheme S that is smooth of finite type over a Dedekind ring or a field. Unless specifically mentioned otherwise, all schemes are supposed to be of finite type over S. We often abbreviate $\mathbf{G}_{\mathrm{m},S}$ by $\mathbf{G}_{\mathrm{m}}$ etc.

For a scheme X/S, we denote the stable presentable (symmetric) monoidal ∞-category of motives (or motivic sheaves) with integral coefficients by $\mathrm{DM}(X)$. The corresponding category with rational coefficients is denoted by $\mathrm{DM}(X, \mathbf{Q})$.

In the generality of Notation 2.11, its construction is due to Spitzweck [Spi18] as the category of modules over a certain E_∞-ring spectrum in the stable $\mathbf{A}^1$-homotopy category $\mathrm{SH}(X)$. If S is smooth over a field, that spectrum is (isomorphic to) Voevodsky's spectrum representing motivic cohomology (with integral coefficients).

2.2.1. *Six functors.* The six functors on motives can be packaged into a lax (symmetric) monoidal functor

$$\mathrm{DM} := \mathrm{DM}_!^* : \mathrm{Corr} \to \mathrm{Pr}_{\mathrm{St}}^{\mathrm{L}} \tag{2.9}$$

Its domain is the ∞-category of correspondences. Its objects are (finite type S-)schemes. Informally, its morphisms are correspondences of schemes over S, i.e., diagrams of the following form:

$$Y \xleftarrow{g} Z \xrightarrow{f} X. \tag{2.10}$$

The above functor sends $X \mapsto \mathrm{DM}(X)$, while a correspondence as above is mapped to $g_! f^*$. For $X, X'/S$, the map $(X, X') \to X \times_S X'$ in the symmetric monoidal ∞-category Corr is sent to the exterior product functor $\mathrm{DM}(X) \otimes \mathrm{DM}(X') \xrightarrow{\boxtimes} \mathrm{DM}(X \times_S X')$. Part of the lax monoidality of the functor is the assertion that $\boxtimes$ is compatible with *-pullback and !-pushforward. We refer to [GR17, LZ11, Man22] for in-depth discussions of the category of correspondences, to [Kha16, Hoy17] for its application in the context of motivic sheaves, and to [RS21, §A] for an exposition also including the lax monoidality of DM.

Any S-scheme X is a coalgebra object in Corr (the comultiplication being the diagonal $X \to X \times_S X$). The lax monoidality of DM in (2.9) then endows $\mathrm{DM}(X)$ with the structure of a symmetric monoidal ∞-category. In addition, for any scheme T/S (e.g., $T = \mathbf{G}_{\mathrm{m},S}$), any T-scheme $f : X \to T$ is naturally a coalgebra object over T (by means of the graph $\Gamma_f : X \to X \times_S T$). The lax monoidal functor above leads naturally to a lax symmetric monoidal functor

$$\mathrm{DM}_!^* : \mathrm{Corr}_T \to \mathrm{Mod}_{\mathrm{DM}(T)} \mathrm{Pr}_{\mathrm{St}}^{\mathrm{L}}. \tag{2.11}$$

Here Mod denotes the indicated category of modules with respect to the Lurie tensor product in $\mathrm{Pr}_{\mathrm{St}}^{\mathrm{L}}$, cf. Section 2.1.1.

We will also use the ∞-category $\mathrm{DM}(X)$ of motives on an ind-scheme X (such as the affine flag variety), as defined and discussed in [RS20, §2.4].

2.2.2. *Stratified Tate motives.* Recall from [RS20, CvdHS22] the following basic notation related to stratified Tate motives: The presentable stable subcategory $\mathrm{DTM}(X) \subset \mathrm{DM}(X)$ generated by $\mathbf{Z}(n)$, $n \in \mathbf{Z}$ is called the category of *Tate motives*. If we only allow $n \leq 0$, we speak of *anti-effective Tate motives*; the category is denoted by $\mathrm{DTM}(X)^{\mathrm{anti}}$.

For a stratified (ind-)scheme $\iota : X^\dagger = \bigsqcup_{w \in W} X_w \to X$, we denote by $\mathrm{DTM}(X, X^\dagger)^{(\mathrm{anti})} := \{M \in \mathrm{DM}(X), \iota^* M \in \mathrm{DTM}(X^\dagger)^{(\mathrm{anti})}\}$ the category of (anti-effective) stratified Tate motives. We will only use this category if the stratification is *(anti-effective) Whitney–Tate*, which means that $\iota^* \iota_*$ preserves $\mathrm{DTM}(X^\dagger)^{(\mathrm{anti})}$. If ι is clear from the context, we also denote this category by $\mathrm{DTM}(X)^{(\mathrm{anti})}$. We call the objects therein *(anti-effective) stratified Tate motives*. If furthermore the base change map [CvdHS22, Eqn. (2.4)] is an isomorphism then we say the stratification is *universally* Whitney–Tate.

2.2.3. *Reduced motives.* For X/S, the category $\mathrm{DM}(X)$ is a module over $\mathrm{DTM}(S)$, i.e., is endowed with an action of the motivic cohomology of S. The action of the higher motivic cohomology of S often has no representation-theoretic significance. This led to the introduction of reduced motives in [ES23]. By definition, the category of reduced Tate motives over S is $\mathrm{DTM}_{\mathrm{r}}(S) = \mathrm{D}(\mathrm{Ab}^{\mathbf{Z}})$, the derived ∞-category of $\mathbf{Z}$-graded abelian groups. There is a canonical reduction functor $\mathrm{r} : \mathrm{DTM}(S) \to \mathrm{DTM}_{\mathrm{r}}(S)$ considered in op. cit. This functor can be thought of as

modding out the higher motivic cohomology of S. It admits a section $i : \mathrm{DTM_r}(S) \to \mathrm{DTM}(S)$, sending $\mathbf{Z}(k)$ ($\mathbf{Z}$ in graded degree k) to the k-fold Tate twist $\mathbf{Z}(k)$.

For X/S, the ∞-category of reduced Tate motives is defined as $\mathrm{DTM_r}(X) := \mathrm{DTM}(X) \otimes_{\mathrm{DTM}(S),\mathrm{r}} \mathrm{DTM_r}(S)$ and similarly $\mathrm{DM_r}(X) := \mathrm{DM}(X) \otimes_{\mathrm{DTM}(S)} \mathrm{DTM_r}(S)$. The formalism of reduced motives enjoys the same six functor properties as DM.

We let $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{gr}} := \mathrm{Mod}_{\mathrm{D}(\mathrm{Ab}\mathbf{z})}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}})$ be the ∞-category of stable presentable $\mathbf{Z}$-linear $\mathbf{Z}$-graded categories (with functors preserving that structure). Using the natural restriction functor along i, we will sometimes consider $\mathrm{DM}^*_!$ as the lax symmetric monoidal functor

$$\mathrm{DM}^*_! : \mathrm{Corr} \to \mathrm{Mod}_{\mathrm{DM}(S)}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}) \xrightarrow{\mathrm{res}} \mathrm{Mod}_{\mathrm{DTM}(S)}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}) \xrightarrow{\mathrm{res}_i} \mathrm{Mod}_{\mathrm{D}(\mathrm{Ab}\mathbf{z})}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}). \tag{2.12}$$

2.2.4. *Mixed Tate motives.* As in [CvdHS22, §2.3], we will make use of motivic t-structures. We consider the t-structure on $\mathrm{DTM}(S)$ whose aisle $\mathrm{DTM}(S)^{\leq 0}$ is generated under colimits by the objects $\mathbf{Z}(k)$, $k \in \mathbf{Z}$. Recall that if S satisfies the Beilinson–Soulé conjecture (e.g., $S = \mathrm{Spec}\,\mathbf{Z}$), these generators lie in the heart of the t-structure. *Whenever we consider mixed Tate motives, we will assume this condition to hold.*

If X is an ind-scheme with an admissible Whitney–Tate stratification (for the definition of admissibility see [CvdHS22, Definition 2.9], for example this holds if the strata are isomorphic to some $\mathbf{G}^n_{\mathrm{m}} \times_S \mathbf{A}^m$), then $\mathrm{DTM}(X, X^\dagger)$ carries a t-structure whose aisle $\mathrm{DTM}(X, X^\dagger)^{\leq 0}$ is generated by the objects $\iota_{w!}\mathbf{Z}_{X_w}(k)[\dim_S X_w]$, for all strata $\iota_w : X_w \to X$ and $k \in \mathbf{Z}$. The heart of this t-structure is denoted by $\mathrm{MTM}(X) := \mathrm{MTM}(X, X^\dagger)$. It is generated under extensions by intersection motives, as detailed in [CvdHS22, Lemma 2.15].

The same notions exist for categories of (stratified) reduced Tate motives, except that the Beilinson–Soulé condition on S is not needed.

2.2.5. *Equivariant motives.* In order to consider equivariant motives on affine flag varieties, we recall the general formalism of equivariant motives, following [RS20, CvdHS22]. Namely, via Kan extensions we can extend DM to a functor

$$\mathrm{DM} : (\mathrm{PreStk}_S)^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}},$$

cf. [CvdHS22, §2.1.5]. Here PreStk_S is the ∞-category of prestacks, i.e., presheaves of anima on the category of affine (but not necessarily finite type) S-schemes. For a group prestack G acting on an ind-scheme X, the category of equivariant motives on X is by definition $\mathrm{DM}(G\backslash X)$, where $G\backslash X$ is the prestack quotient, i.e., the colimit of the cosimplicial diagram $\dots\ G \times X \rightrightarrows X$.

If X is equipped with a Whitney–Tate stratification, the category of G-equivariant stratified Tate motives on X is defined as

$$\mathrm{DTM}_G(X) := \mathrm{DM}(G\backslash X) \times_{\mathrm{DM}(X)} \mathrm{DTM}(X).$$

In other words, they are those equivariant motives which are stratified Tate after forgetting the equivariance. Sometimes, we also write $\mathrm{DTM}(G\backslash X) = \mathrm{DTM}_G(X)$, but we emphasize that this really depends on the presentation, rather than just the prestack $G\backslash X$. Under certain assumptions on the G-action on X, cf. [CvdHS22, Lemma 2.24], there is moreover a (necessarily unique) t-structure on $\mathrm{DTM}_G(X)$ such that the forgetful functor $u^! : \mathrm{DTM}_G(X) \to \mathrm{DTM}(X)$ is t-exact. (As in Section 2.2.4, we need to assume S satisfies the Beilinson-Soulé vanishing conjecture, if we consider non-reduced motives.) We denote its heart by $\mathrm{MTM}_G(X) = \mathrm{MTM}(G\backslash X)$.

2.2.6. *Grothendieck groups.* Eyeing applications to Iwahori–Hecke algebras in Section 6, we provide some general computations of Grothendieck groups of categories of Tate motives, including the equivariant and stratified case. To this end, we use the slice filtration [HK06]: if $\mathrm{DTM}(X)^{(\geq n)}$ denotes the full stable presentable subcategory generated by $\mathbf{Z}_X(m)$, $m \geq n$, the inclusion into $\mathrm{DTM}(X)$ admits a right adjoint $\nu^{\geq n}$. The kernel of this functor is denoted by $\mathrm{DTM}(X)^{(\leq n-1)}$. We write $\mathrm{DTM}(X)^{(=0)} := \mathrm{DTM}(X)^{(\leq 0)} \cap \mathrm{DTM}(X)^{(\geq 0)}$ for the "heart" of the slice structure.

We begin with unstratified (anti-effective) Tate motives, where the superscript c stands for compact objects.

Lemma 2.12. *Let X/S be a smooth connected scheme. Then the following maps given by $\mathbf{q} \mapsto \mathbf{Z}(-1)$ are ring isomorphisms:*

$$\mathbf{Z}[\mathbf{q}] \to K_0(\mathrm{DTM}(X)^{\mathrm{anti,c}}),$$
$$\mathbf{Z}[\mathbf{q}^{\pm 1}] \to K_0(\mathrm{DTM}(X)^{\mathrm{c}}).$$

Proof. According to [Spi16, Theorem 3], the group $\mathrm{Hom}_{\mathrm{DTM}(X)}(\mathbf{Z}, \mathbf{Z}(r)[n])$ vanishes for $r < 0$ and $n \in \mathbf{Z}$ and also for $r = 0$, $n \neq 0$. For $r = n = 0$, it is isomorphic to $\mathbf{Z}$, since X is connected. Since $\mathrm{DTM}(X)$ is compactly generated by $\mathbf{Z}(m)$, and

$$\nu^{\geq n}\mathbf{Z}(m) = \left\{ \begin{array}{cc} 0 & m < n \\ \mathbf{Z}(m) & m \geq n, \end{array} \right.$$

we obtain that $\nu^{\geq n}$ preserves compact objects. In addition, the cofiber of the natural map $\nu^{\geq n}M \to M$ lies in the stable presentable subcategory generated by $\mathbf{Z}(s)$, $s < n$, so that $\mathrm{DTM}(X)^{(\leq n-1)}$ is the stable presentable subcategory generated by $\mathbf{Z}(s)$, $s < n$. Finally, any compact object $F \in \mathrm{DTM}(X)^c$ lies in some $\mathrm{DTM}(X)^{(\geq -n)}$, and is such that $\nu^{\geq n}F = 0$ for $n \gg 0$. Thus, F is obtained in finitely many steps, using cofiber sequences, from Tate twists of compact objects in $\langle \mathbf{Z} \rangle \subset \mathrm{DTM}(X)$. The subcategory spanned by such compact objects is equivalent to $\mathrm{Perf}_{\mathbf{Z}}$, cf. [HK06, Proposition 4.5], whose K_0-group is isomorphic to $\mathbf{Z}$. $\square$

Lemma 2.12 and the localization fiber sequence $j_! j^* \to \mathrm{id} \to i_! i^*$ (for complementary open and closed embeddings j and i, respectively) imply the following result.

Corollary 2.13. *If $X^\dagger = \bigsqcup_{w \in W} X_w \to X$ is an (anti-effective) Whitney–Tate stratified ind-scheme whose strata X_w are smooth over S and connected, then*

$$K_0(\mathrm{DTM}(X, X^\dagger)^{\mathrm{anti,c}}) = \bigoplus_{w \in W} \mathbf{Z}[\mathbf{q}],$$

$$K_0(\mathrm{DTM}(X, X^\dagger)^{\mathrm{c}}) = \bigoplus_{w \in W} \mathbf{Z}[\mathbf{q}^{\pm 1}].$$

Recall the following notion from [CvdHS22, Definition 5.49]: if H is a smooth group scheme acting on an ind-scheme X, we say that a motive $\mathcal{F} \in \mathrm{DM}_H(X)$ is *locally compact* if its underlying motive $u^! \mathcal{F} \in \mathrm{DM}(X)$ is compact. We denote the full subcategory of locally compact equivariant Tate motives by $\mathrm{DTM}_H(X)^{\mathrm{lc}}$.

Lemma 2.14. *Suppose H is a smooth, fiberwise connected group scheme over S. If H acts on a smooth S-scheme X, the forgetful functor induces an isomorphism for the Grothendieck groups of equivariant vs. non-equivariant (unstratified) Tate motives:*

$$K_0(\mathrm{DTM}_H(X)^{\mathrm{lc,(anti)}}) \xrightarrow{\cong} K_0(\mathrm{DTM}(X)^{\mathrm{c,(anti)}}).$$

Proof. Note first that relative purity (using smoothness of H) gives an equivalence

$$\mathrm{DTM}_H(X) = \lim(\mathrm{DTM}(X) \underset{p^*}{\overset{a^*}{\rightrightarrows}} \mathrm{DTM}(X \times_S H) \overset{\to}{\underset{\to}{\rightrightarrows}} \dots). \tag{2.13}$$

It therefore makes sense to consider anti-effective equivariant Tate motives to begin with. The functors a^* and p^* are "slice-exact", i.e., preserve the subcategories $\mathrm{DTM}(-)^{(\geq n)}$ and $\mathrm{DTM}(-)^{(\leq n-1)}$ and thus also the heart $\mathrm{DTM}(-)^{(=0)}$. Both a and p have a section, so that the functors are conservative; and therefore also detect the property of being in either subcategory. Therefore, (2.13) holds mutatis mutandis for these three subcategories.

By [Sta24, Tags 0385, 0378], $\pi_0(X) = \pi_0(X \times_S H^{\times n})$ for each $n \geq 0$. This gives equivalences $\mathrm{DTM}(X \times H^n)^{\mathrm{c,(=0)}} = \mathrm{DTM}(X)^{\mathrm{c,(=0)}}$. Passing to the limit gives an equivalence $\mathrm{DTM}_H(X)^{\mathrm{lc,(=0)}} = \mathrm{DTM}(X)^{\mathrm{c,(=0)}}$. This implies our claim by functoriality of the slice filtration. $\square$

Corollary 2.15. *Suppose an ind-scheme X carries an (anti-effective) Whitney–Tate stratification $X^\dagger = \bigsqcup_{w \in W} X_w$ whose strata X_w are smooth. Let a pro-algebraic group H act on X in such a way that each stratum X_w is preserved by the H-action, and such that the H-action induces an isomorphism $X_w = (H/H_w)_{\mathrm{Zar}}$, where $H_w \subset H$ is an extension of a split pro-unipotent group by a connected split reductive group. Then the forgetful functors again induce isomorphism on Grothendieck groups:*

$$K_0(\mathrm{DTM}_H(X, X^\dagger)^{\mathrm{lc,(anti)}}) \xrightarrow{\cong} K_0(\mathrm{DTM}(X, X^\dagger)^{\mathrm{c,(anti)}}).$$

Proof. Localization reduces the case of X/H to the case X_w/H. By [RS20, Proposition 3.1.23], [RS] there is an equivalence $\mathrm{DTM}_H(X_w) = \mathrm{DTM}_{H_w}(S)$, so that we can apply Lemma 2.14. $\square$

Remark 2.16. In any of the situations above, the rationalization functor induces an equivalence on Grothendieck rings. Indeed, if X/S is smooth connected, we still have an isomorphism $\mathbf{Z}[\mathbf{q}] \cong K_0(\mathrm{DTM}(X, \mathbf{Q})^{\mathrm{anti,c}})$ as in Lemma 2.12, and this carries over to equivariant and/or stratified Tate motives.

2.2.7. *Tate motives on $\mathbf{G}_\mathrm{m}$.* Our approach to (unipotent) nearby cycles rests on applying some higher categorical algebra to various categories of motivic sheaves. This is facilitated by a module-theoretic interpretation of the category of Tate motives on $\mathbf{G}_\mathrm{m}$ that we study in this section. It is worth noting that the multiplication on $\mathbf{G}_\mathrm{m}$ plays no role at this point.

Let $p : \mathbf{G}_{\mathrm{m},S} \to S$ be the structural map. A basic, but key object, is the "cohomology" of $\mathbf{G}_\mathrm{m}$,

$$\Lambda := p_* p^* \mathbf{Z}. \tag{2.14}$$

It is a commutative algebra object in $\mathrm{DM}(S)$ whose underlying object in $\mathrm{DM}(S)$ is $\mathbf{Z} \oplus \mathbf{Z}(-1)[-1]$. In particular, it lies in the subcategory $\mathrm{DTM}(S)^{\mathrm{anti}}$. The counit map

$$p^* \Lambda = p^* p_* p^* \mathbf{Z} \to p^* \mathbf{Z} = \mathbf{Z}_{\mathbf{G}_\mathrm{m}}$$

is a map of commutative algebras in that category. The point $S \xrightarrow{1} \mathbf{G}_{\mathrm{m},S}$ equips Λ with an augmentation, aug : $\Lambda \to \mathbf{Z}$. The following result appears independently in [BGV23, Proposition 2.53].

Lemma 2.17. *The restriction of p_* to $\mathrm{DTM}(\mathbf{G}_{\mathrm{m}})$ is monadic and therefore gives rise to an equivalence*

$$\mathrm{DTM}(\mathbf{G}_{\mathrm{m}}) \cong \mathrm{Mod}_\Lambda(\mathrm{DTM}(S)) \tag{2.15}$$

under which p_* corresponds to the functor forgetting the Λ-action. The same statement also holds for reduced and / or anti-effective Tate motives.

Proof. The functor p_*, restricted to $\mathrm{DTM}_{(\mathrm{r})}(\mathbf{G}_{\mathrm{m}}) \to \mathrm{DTM}_{(\mathrm{r})}(S)$, is conservative since $p^* \mathrm{DTM}_{(\mathrm{r})}(S)$ generates $\mathrm{DTM}_{(\mathrm{r})}(\mathbf{G}_{\mathrm{m}})$. By general theory, p_* preserves colimits so that p_* is monadic. For anti-effective motives, the same proof works, since p^* and p_* preserve anti-effective motives. $\qquad\square$

Recall that for any object M in a stable presentable symmetric monoidal ∞-category C, there is the split square zero extension, which is an object in $\mathrm{CAlg}(C)$ whose underlying object in C is isomorphic to $1 \oplus M$ [Lur17, §7.3.4].

Lemma 2.18. *There is an isomorphism in $\mathrm{CAlg}(\mathrm{DTM}(S, \mathbf{Q}))$*

$$\Lambda \otimes \mathbf{Q} = \mathbf{Q} \oplus \mathbf{Q}(-1)[-1].$$

Similarly, if we let $\Lambda_{\mathrm{r}} = p_* p^* \mathbf{Z} \in \mathrm{DTM}_{\mathrm{r}}(S)(= \mathrm{D}(\mathrm{Ab}^{\mathbf{Z}}))$ be the dual of the reduced motive of $\mathbf{G}_{\mathrm{m}}$, there is an isomorphism in $\mathrm{CAlg}(\mathrm{DTM}_{\mathrm{r}}(S))$:

$$\Lambda_{\mathrm{r}} = \mathbf{Z} \oplus \mathbf{Z}(-1)[-1]. \tag{2.16}$$

Proof. The statement with rational coefficients holds by [BGV23, Proposition 2.60] which constructs isomorphisms between the free symmetric algebra $\mathrm{Sym}(\mathbf{Q}(-1)[-1])$ and both terms. We also refer to [AEWH15, Theorem 7.1.1] for the identification of the Hopf algebra structure on $\mathrm{M}(\mathbf{G}_{\mathrm{m}})$ (again with rational coefficients).

For the statement for reduced motives (with integral coefficients), note that both objects lie in the full subcategory spanned by $\mathbf{Z}(-n)[-n]$ for $n \geq 0$, which is an *ordinary* symmetric monoidal category. Let $I = \mathbf{Z}(-1)[-1]$ be the the augmentation ideal of Λ_{r}. The space of maps $\mathrm{Maps}_{\mathrm{DTM}_{\mathrm{r}}(S)}(I^{\otimes n}, I^{\otimes m}) = \mathrm{Maps}_{\mathrm{D}(\mathrm{Ab}^{\mathbf{Z}})}(\mathbf{Z}(-n)[-n], \mathbf{Z}(-m)[-m])$ is trivial for any $n \neq m$ and is isomorphic to $\mathbf{Z}$ for $n = m$. Therefore, the multiplication maps are all null-homotopic. The switching maps (for the symmetric monoidal structure) $I \otimes I \xrightarrow{\cong} I \otimes I$ are the identity both for Λ_{r} (since this is true for the unreduced object Λ), and also for the square zero extension. $\qquad\square$

Remark 2.19. The analogous statement for DM and integral or $\mathbf{F}_p$-coefficients fails. In this sense the passage to rational coefficients in the above formality result [BGV23, Proposition 2.60] is optimal. To see this note that the multiplication $\Lambda \otimes \Lambda \to \Lambda$ is dual to the comultiplication on $\mathrm{M}(\mathbf{G}_{\mathrm{m}}) = \Lambda^\vee$, given by the diagonal $\mathrm{M}(\mathbf{G}_{\mathrm{m}}) \xrightarrow{\Delta_{\mathbf{G}_{\mathrm{m}}}} \mathrm{M}(\mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}}) = \mathrm{M}(\mathbf{G}_{\mathrm{m}}) \otimes \mathrm{M}(\mathbf{G}_{\mathrm{m}})$. The restriction of this map to the direct summands $\mathbf{Z}(1)[1] \to \mathbf{Z}(2)[2]$ is nonzero since applying $\mathrm{Hom}_{\mathrm{DM}(\mathrm{Spec}\,\mathbf{Q})}(\mathbf{Z}, -)$ gives the non-zero map $K_1^M(\mathbf{Q}) = \mathbf{Q}^\times \to K_2^M(\mathbf{Q})$, $[a] \mapsto [a] \cdot [a]$. (Instead, the Steinberg relation implies $[a] \cdot [-a] = 0 \in K_2^M(\mathbf{Q})$.) Inverting 2 in the coefficients, we do have $\mathrm{Maps}_{\mathrm{DM}(\mathrm{Spec}\,\mathbf{Z})}(\mathbf{Z}[\frac{1}{2}](1)[1], \mathbf{Z}[\frac{1}{2}](2)[2]) = 0$, but higher order multiplications still form an obstruction to $\mathrm{M}(\mathbf{G}_{\mathrm{m}}) \otimes \mathbf{Z}[\frac{1}{2}]$ being the square zero coalgebra (or $\Lambda[\frac{1}{2}]$ the square zero algebra). In fact, the Betti realization functor ρ_{B} is symmetric monoidal and therefore preserves square zero extensions. Now $\rho_{\mathrm{B}}(\Lambda \otimes \mathbf{F}_p) = C^*(\mathbf{G}_{\mathrm{m}}^{\mathrm{an}}, \mathbf{F}_p) = C^*(S^1, \mathbf{F}_p) \in \mathrm{CAlg}(\mathrm{D}(\mathrm{Mod}_{\mathbf{F}_p}))$. We claim it is not isomorphic, as a commutative algebra object, to the square zero extension $\mathbf{F}_p \oplus \mathbf{F}_p(-1)[-1]$. Indeed, any such isomorphism would be compatible with the Steenrod operations, but $P^0 = \mathrm{id}$ on $C^*(S^1, \mathbf{F}_p)$ [May70, Proposition 8.1], while $P^0 = 0$ on the (strictly commutative) cdga $\mathbf{F}_p \oplus \mathbf{F}_p[-1]$. Thus, there is no isomorphism, in $\mathrm{CAlg}(\mathrm{DM}(\mathrm{Spec}\,\mathbf{Z}, \mathbf{F}_p))$, between $\Lambda \otimes \mathbf{F}_p$ and the split square zero extension $\mathbf{F}_p \oplus \mathbf{F}_p(-1)[-1]$.

Corollary 2.20. *There are equivalences (of commutative algebra objects in $\mathrm{Mod}_{\mathrm{D}(\mathrm{Ab}^{\mathbf{Z}})}(\mathrm{Pr}_{\mathrm{St}}^{\mathrm{L}})$)*

$$\mathrm{DTM}(\mathbf{G}_{\mathrm{m}}, \mathbf{Q}) = \mathrm{DTM}_{\mathrm{r}}(\mathbf{G}_{\mathrm{m}}) \otimes_{\mathrm{DTM}_{\mathrm{r}}(S)} \mathrm{DTM}(S, \mathbf{Q}).$$
$$\mathrm{DTM}_{\mathrm{r}}(\mathbf{G}_{\mathrm{m}}) = \mathrm{Nilp}.$$

Proof. The statement about $\mathrm{DTM}_{\mathrm{r}}(\mathbf{G}_{\mathrm{m}})$ follows from the construction of Nilp in Definition and Lemma 2.9 and from (2.16). To deduce the first statement, we consider the symmetric monoidal functor $i : \mathrm{DTM}_{\mathrm{r}}(S) = \mathrm{D}(\mathrm{Ab}^{\mathbf{Z}}) \to \mathrm{DTM}(S, \mathbf{Q})$ given by $\mathbf{Z}(k) \mapsto \mathbf{Q}(k)$. It maps the split square zero extension $\Lambda_{\mathrm{r}} = \mathbf{Z} \oplus \mathbf{Z}(-1)[-1]$ to the split square zero extension $\mathbf{Q} \oplus \mathbf{Q}(-1)[-1]$, which by Lemma 2.18 is isomorphic to $\Lambda \otimes \mathbf{Q}$. Thus using Lemma 2.17 and Lemma 2.6, we get a chain of equivalences of ∞-categories

$$\begin{aligned}
\mathrm{DTM}(\mathbf{G}_{\mathrm{m}}, \mathbf{Q}) &= \mathrm{Mod}_{\Lambda \otimes \mathbf{Q}}(\mathrm{DTM}(S, \mathbf{Q})) \\
&= \mathrm{Mod}_{i(\Lambda_{\mathrm{r}})}(\mathrm{DTM}(S, \mathbf{Q})) \\
&= \mathrm{Mod}_{\Lambda_{\mathrm{r}}}(\mathrm{DTM}_{\mathrm{r}}(S)) \otimes_{\mathrm{DTM}_{\mathrm{r}}(S)} \mathrm{DTM}(S, \mathbf{Q}).
\end{aligned}$$

$\qquad\square$

Remark 2.21. In other words, Tate motives M on $\mathbf{G}_m$ with rational coefficients (and, likewise, reduced motives with integral coefficients) are equivalent to the datum of their restriction 1^*M together with a locally nilpotent map $(1^*M)(1) \to 1^*M$. Concerning (unreduced) motives with integral coefficients, we still have a natural map $\mathbf{Z}(1) \xrightarrow{\alpha} \underline{\mathrm{Hom}}_\Lambda(\mathbf{Z}, \mathbf{Z})$, and thus a map of E_1-algebras $\Sigma \to \underline{\mathrm{Hom}}_\Lambda(\mathbf{Z}, \mathbf{Z})$. Note that $1^*M = \mathbf{Z} \otimes_\Lambda M$ is naturally a module over $\underline{\mathrm{Hom}}_\Lambda(\mathbf{Z}, \mathbf{Z})$ and therefore by restriction a Σ-module. In other words, 1^*M still comes naturally with the map α, but we have little control over it (cf. Proposition 3.9) and therefore do not use it in this paper.

We end this section by introducing a category that will eventually allow to keep track of some motivic sheaf $M \in \mathrm{DM}(X)$ together with a map $M \otimes \mathbf{Q}(1) \to M \otimes \mathbf{Q}$.

Definition 2.22. Let $\mathrm{Nilp}_{\mathbf{Q}} := \mathrm{D}(\mathrm{Ab}^{\mathbf{Z}}) \times_{\mathrm{D}(\mathrm{Ab}^{\mathbf{Z}}) \otimes \mathbf{Q}} (\mathrm{Nilp} \otimes \mathbf{Q})$, where we write $C \otimes \mathbf{Q} := C \otimes_{\mathrm{D}(\mathrm{Ab})} \mathrm{D}(\mathrm{Mod}_{\mathbf{Q}})$ for the rationalization of any $\mathbf{Z}$-linear stable ∞-category, and the pullback is formed using the forgetful functor $\mathrm{Nilp} \subset \mathrm{Mod}_\Sigma(\mathrm{D}(\mathrm{Ab}^{\mathbf{Z}})) \to \mathrm{D}(\mathrm{Ab}^{\mathbf{Z}})$.

Both Nilp and $\mathrm{Nilp}_{\mathbf{Q}}$ are modules over $\mathrm{D}(\mathrm{Ab}^{\mathbf{Z}})$. For some $C \in \mathrm{Mod}_{\mathrm{D}(\mathrm{Ab}^{\mathbf{Z}})}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}})$ (such as some category of motives, cf. (2.12)), we will abbreviate $\mathrm{Nilp}_{\mathbf{Q}} C := \mathrm{Nilp}_{\mathbf{Q}} \otimes_{\mathrm{D}(\mathrm{Ab}^{\mathbf{Z}})} C$ and $\mathrm{Nilp}\, C := \mathrm{Nilp} \otimes_{\mathrm{D}(\mathrm{Ab}^{\mathbf{Z}})} C$.

Remark 2.23. Thus, $\mathrm{Nilp}_{\mathbf{Q}}$ is a full subcategory of $\mathrm{D}(\mathrm{Ab}^{\mathbf{Z}}) \times_{\mathrm{D}(\mathrm{Mod}_{\mathbf{Q}}^{\mathbf{Z}})} \mathrm{Mod}_{\Sigma \otimes \mathbf{Q}}(\mathrm{D}(\mathrm{Mod}_{\mathbf{Q}}^{\mathbf{Z}}))$; objects in the latter category can be described as $(A \in \mathrm{D}(\mathrm{Ab}^{\mathbf{Z}}), \phi : A \otimes \mathbf{Q}(1) \to A \otimes \mathbf{Q})$. The category $\mathrm{Nilp}_{\mathbf{Q}}$ is compactly generated by $(\mathbf{Z}(k), \mathbf{Q}(k+1) \xrightarrow{0} \mathbf{Q}(k))$, $k \in \mathbf{Z}$. Thus, if C is compactly generated, then $\mathrm{Nilp}_{\mathbf{Q}} C = C \times_{C \otimes \mathbf{Q}} (C \otimes \mathrm{Nilp} \otimes \mathbf{Q})$ and this category is compactly generated as well. If an object $(c \in C, \phi : c \otimes \mathbf{Q}(1) \to c \otimes \mathbf{Q})$ in this category is compact, then the map ϕ is nilpotent. Similar statements hold for $\mathrm{Nilp}\, C$. The natural functor $\mathrm{Nilp} \to \mathrm{Nilp}_{\mathbf{Q}}$ induces an equivalence after rationalization, i.e., $\mathrm{Nilp} \otimes \mathbf{Q} \xrightarrow{\cong} \mathrm{Nilp}_{\mathbf{Q}} \otimes \mathbf{Q}$.

In the setup of the unipotent nearby cycles functor we will use the following functor.

Definition 2.24. The symmetric monoidal functors

$$1^* : \mathrm{DTM}(\mathbf{G}_m) \to \mathrm{DTM}(S),$$

$$\mathrm{DTM}(\mathbf{G}_m) \to \mathrm{DTM}(\mathbf{G}_m, \mathbf{Q}) \overset{2.20}{=} \mathrm{Nilp} \otimes_{\mathrm{D}(\mathrm{Ab}^{\mathbf{Z}})} \mathrm{DTM}(S, \mathbf{Q})$$

induce a symmetric monoidal functor

$$\mathrm{DTM}(\mathbf{G}_m) \to \mathrm{Nilp}_{\mathbf{Q}} \mathrm{DTM}(S).$$

(After passing to rationalizations, this functor recovers the equivalence in Corollary 2.20.)

2.2.8. *Convolution structures.* For a coherent construction of Wakimoto functors (Definition 5.7), we refine the construction of the convolution monoidal structure on categories such as $\mathrm{DM}(\mathcal{I} \backslash LG/\mathcal{I})$ from [RS21, §3] to the level of ∞-categories (as opposed to their homotopy categories). Related statements appear, e.g., in [GR17, Ch. 5, §5] and [ALWY23, §4.2], so we will keep this brief.

First, the restriction of $\mathrm{DM}^!$ to the category $\mathrm{Sch}^{\mathrm{pl}}_S$ of placid schemes (and placid morphisms) is lax symmetric monoidal [RS21, §A.2]. Recall that this encodes the existence of $\boxtimes$ and its compatibility with !-pullback along smooth (and then placid) maps. Second, the Kan extension to placid ind-schemes $\mathrm{IndSch}^{\mathrm{pl}}_S$ [Gai20, §C.1], is therefore again lax symmetric monoidal. Third, the Kan extension to the free completion under sifted colimits is lax symmetric monoidal, see, e.g., the discussion in [Rob14, §3.2]. Denote this by $\mathrm{IndSch}^{\mathrm{pl}}_S(\mathrm{sift})$. Finally, consider the category $\mathrm{Corr}(\mathrm{IndSch}^{\mathrm{pl}}_S(\mathrm{sift}))$ of correspondences, where as "horizontal" maps we only allow maps f such that for any pullback f' of f, the functor $f'^!$ admits a left adjoint, denoted $f'_!$, which is moreover adjointable (i.e., commutes with all !-pullbacks). Then $\mathrm{DM}^!$ extends naturally to a lax symmetric monoidal functor, see, e.g., [Man22, Proposition A.5.10].

These facts can be applied as follows: let L be a group object in the category of placid ind-schemes over S, and fix a subgroup object $L^+ \subset L$ such that the quotient $(L/L^+)_{\mathrm{Nis}}$ is representable by an ind-proper ind-scheme. Then the convolution groupoid $X := L^+ \backslash L/L^+$ is an object in the sifted completion. The multiplication map $m : X \times_{S/L^+} X \to X$ is a pullback of the projection $LG/L^+G \to S$. By assumption, up to sheafification, it is an ind-proper map, so that the same is true for any pullback of it. For ind-proper maps f, the left adjoint $f_!$ of $f^!$ exists and commutes with all !-pullbacks [RS20, Proposition 2.3.3]. Thus, X is a (non-commutative) algebra object in $\mathrm{Corr}(\mathrm{IndSch}^{\mathrm{pl}}_S(\mathrm{sift}))$ [GR17, Part I, Ch. 9, Corollary 4.4.5], so that the (symmetric) monoidal functor $\mathrm{DM}^!$ maps it to an algebra object $\mathrm{DM}(X)$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ or $\mathrm{Mod}_{\mathrm{DM}(S)}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}})$.

3. Unipotent nearby cycles

The following definition of unipotent nearby cycles is a motivic adaptation of the approach in [Cam18, §3], see also [Che23, §2.1]. Throughout, we consider the following geometric situation, where X is a scheme:

$$
\begin{array}{ccccc}
X_\eta & \xrightarrow{\ j\ } & X & \xleftarrow{\ i\ } & X_s \\
\downarrow{\scriptstyle f_\eta} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_s} \\
\mathbf{G}_{\mathrm{m},S} & \longrightarrow & \mathbf{A}^1_S & \xleftarrow{\ 0\ } & S
\end{array}
\tag{3.1}
$$
$$
\mathbf{G}_{\mathrm{m},S} \xrightarrow{\ p\ } S \xleftarrow{q} \ \mathbf{A}^1_S,\ S \xrightarrow{\mathrm{id}} S.
$$

We write $\Gamma_{f_\eta} : X_\eta \to X_\eta \times_S \mathbf{G}_{\mathrm m}$ for the graph of f_η and consider the functor

$$
\mathrm{DTM}(\mathbf{G}_{\mathrm m}) \otimes_{\mathrm{DTM}(S)} \mathrm{DM}(X_\eta) \xrightarrow{\boxtimes} \mathrm{DM}(X_\eta \times \mathbf{G}_{\mathrm m}) \xrightarrow{\Gamma^*_{f_\eta}} \mathrm{DM}(X_\eta).
$$

This composite, which is a map in $\mathrm{Pr}^{\mathrm L}_{\mathrm{St}}$, has a right adjoint, which we denote by $\Gamma_{f_\eta?}$.

Definition 3.1. The *unipotent nearby cycles functor* is the composite

$$
\Upsilon_X : \mathrm{DM}(X_\eta) \xrightarrow{\Gamma_{f_\eta?}} \mathrm{DTM}(\mathbf{G}_{\mathrm m}) \otimes_{\mathrm{DTM}(S)} \mathrm{DM}(X_\eta) \xrightarrow{1^* \otimes i^* j_*} \mathrm{DM}(X_s).
$$

We now make this definition more explicit, using the equivalence $\mathrm{DTM}(\mathbf{G}_{\mathrm m}) \otimes_{\mathrm{DTM}(S)} \mathrm{DM}(X_\eta) = \mathrm{Mod}_\Lambda(\mathrm{DM}(X_\eta))$, the category of Λ-modules (where Λ is pulled back from S to X_η along $p f_\eta$) in $\mathrm{DM}(X_\eta)$. This equivalence results from Lemma 2.6 and Lemma 2.17. Here and throughout, all tensor products of categories of (Tate) motives are carried out in $\mathrm{Mod}_{\mathrm{DTM}(S)}(\mathrm{Pr}^{\mathrm L}_{\mathrm{St}})$. I.e., unless otherwise mentioned, $C \otimes D$ is a shorthand for $C \otimes_{\mathrm{DTM}(S)} D$.

Every object $\mathcal{F} \in \mathrm{DM}(X_\eta)$ becomes naturally a Λ-module: its Λ-action is just the map

$$
f^*_\eta p^* \Lambda \otimes \mathcal{F} = f^*_\eta p^* p_* \mathbf{Z} \otimes \mathcal{F} \xrightarrow{adj.} f^*_\eta \mathbf{Z} \otimes \mathcal{F} = \mathcal{F}.
$$

The functor $\Gamma_{f_\eta?}$ agrees with the resulting natural functor $\mathrm{DM}(X_\eta) \to \mathrm{Mod}_\Lambda(\mathrm{DM}(X_\eta))$, since their left adjoints, given by $\Gamma^*_{f_\eta}$ and $- \otimes_\Lambda \mathbf{Z}$ agree. Here the tensor product is formed using the counit map $\Lambda(:= p^*\Lambda) = p^* p_* \mathbf{Z}_{\mathbf{G}_{\mathrm m}} \to \mathbf{Z}_{\mathbf{G}_{\mathrm m}}$. The standard functor $i^* j_*$ extends naturally to a functor $\mathrm{Mod}_\Lambda(\mathrm{DM}(X_\eta)) \to \mathrm{Mod}_\Lambda(\mathrm{DM}(X_s))$ (in a way that is compatible with forgetting the Λ-action). This can be identified with $\mathrm{id}_{\mathrm{DTM}(\mathbf{G}_{\mathrm m})} \otimes i^* j_*$. In other words, that functor is just keeping track of the extra Λ-action, but otherwise is the standard $i^* j_*$. Finally, the functor $1^* \otimes \mathrm{id}_{\mathrm{DM}(X_s)}$ identifies with $- \otimes_\Lambda \mathbf{Z} : \mathrm{Mod}_\Lambda(\mathrm{DM}(X_s)) \to \mathrm{DM}(X_s)$, where the tensor product is formed using the augmentation map, which geometrically corresponds to pullback along $1 \in \mathbf{G}_{\mathrm m}$.

The fiber sequence

$$
I := \mathbf{Z}(-1)[-1] \xrightarrow{\iota} \Lambda \xrightarrow{\mathrm{aug}} \mathbf{Z},
\tag{3.2}
$$

gives rise to fiber sequences

$$
I^{\otimes n+1} \xrightarrow{\iota \otimes \mathrm{id}} \Lambda \otimes I^{\otimes n} = \Lambda(-n)[-n] \xrightarrow{\mathrm{aug} \otimes \mathrm{id}} I^{\otimes n}.
$$

This gives a "resolution" of $\mathbf{Z}$ by free Λ-modules:

$$
\mathrm{colim}\,(\ldots \Lambda(-2)[-2] \to \Lambda(-1)[-1] \to \Lambda) \xrightarrow{\cong} \mathbf{Z}.
\tag{3.3}
$$

Using this resolution, we obtain the following explicit formula for Υ:

$$
\Upsilon(M) = \mathrm{colim}\,(\ldots \to i^* j_* M(-2)[-2] \to i^* j_* M(-1)[-1] \to i^* j_* M).
\tag{3.4}
$$

By the construction of (3.3), the maps in this diagram are obtained from the map $\mathbf{Z}(-1)[-1] \otimes M \to \Lambda \otimes M \to M$ induced by the Λ-action on M.

3.1. **Functoriality, Künneth formula and monodromy.** Our goal in this section is to enhance this definition in the following ways: we establish the compatibility of Υ with proper pushforward, smooth pullback and its lax compatibility with exterior products. We do this in a highly structured way, having as a payoff an effortless approach to, say, equivariant nearby cycles, cf. Corollary 3.30. We also enhance Υ by the additional datum of a monodromy map, which works integrally for reduced motives, but requires rational coefficients for (unreduced) motives.

We will cast Υ as a natural transformation between two functors, the domain of which is the following lax monoidal functor

$$
\mathrm{DM}_\eta : \mathrm{Corr}_{\mathbf{A}^1} \xrightarrow{j^*} \mathrm{Corr}_{\mathbf{G}_{\mathrm m}} \xrightarrow{\mathrm{DM}^*_!} \mathrm{Mod}_{\mathrm{DTM}(\mathbf{G}_{\mathrm m})} \mathrm{Pr}^{\mathrm L}_{\mathrm{St}} \xrightarrow{U} \mathrm{Mod}_{\mathrm{DTM}(S)}(\mathrm{Pr}^{\mathrm L}_{\mathrm{St}}).
$$

It sends $X/\mathbf{A}^1$ to $\mathrm{DM}(X_\eta)$. Its lax monoidality expresses the maps

$$
\boxtimes_{\mathbf{G}_{\mathrm m}} : \mathrm{DM}(X_{1\eta}) \otimes \mathrm{DM}(X_{2\eta}) \to \mathrm{DM}(X_{1\eta} \times_{\mathbf{G}_{\mathrm m}} X_{2\eta})
$$

and their functoriality with respect to !-pushforward and *-pullback.

As for the codomain of Υ, recall from Section 2.1.2 and Definition 2.22 the category $\mathrm{Nilp}_{\mathbf{Q}}$. It is a commutative algebra object in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{gr}} := \mathrm{Mod}_{\mathrm{D(Ab\mathbf{Z})}}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}})$. Objects in the category $\mathrm{Nilp}_{\mathbf{Q}}\,\mathrm{DM}(Y) := \mathrm{Nilp}_{\mathbf{Q}} \otimes_{\mathrm{D(Ab\mathbf{Z})}}\mathrm{DM}(Y)$ are pairs $(M \in \mathrm{DM}(Y), \phi : M(1) \otimes \mathbf{Q} \to M \otimes \mathbf{Q})$, where ϕ is a locally nilpotent map of the rationalized motives, as is explained in Remark 2.10 and Remark 2.23. The codomain of Υ is now the lax monoidal functor

$$\mathrm{Nilp}_{\mathbf{Q}}\,\mathrm{DM}_s : \mathrm{Corr}_{\mathbf{A}^1} \xrightarrow{i^*} \mathrm{Corr} \xrightarrow{\mathrm{DM}^*_!} \mathrm{Pr}^{\mathrm{L}}_{\mathrm{gr}} \xrightarrow{\mathrm{Nilp}_{\mathbf{Q}} \otimes_{\mathrm{D(Ab\mathbf{Z})}}-} \mathrm{Pr}^{\mathrm{L}}_{\mathrm{gr}}.$$

It maps $X/\mathbf{A}^1$ to $\mathrm{Nilp}_{\mathbf{Q}}\,\mathrm{DM}(X_s)$. The lax monoidality of this records the functors

$$\boxtimes : \mathrm{Nilp}_{\mathbf{Q}}\,\mathrm{DM}(X_{1s}) \otimes \mathrm{Nilp}_{\mathbf{Q}}\,\mathrm{DM}(X_{2s}) \to \mathrm{Nilp}_{\mathbf{Q}}\,\mathrm{DM}(X_{1s} \times X_{2s}),$$

sending $(M_1, \phi_1), (M_2, \phi_2)$ to $(M_1 \boxtimes M_2, \phi_1 \boxtimes \mathrm{id} + \mathrm{id} \boxtimes \phi_2)$.

We consider the wide subcategory $\mathrm{Corr}^{\mathrm{pr,sm}}_{\mathbf{A}^1} \subset \mathrm{Corr}_{\mathbf{A}^1}$ in which we require the correspondences in (2.10) to consist of a smooth map f and a proper map g. Thus, the restriction $\mathrm{DM}^*_!|_{\mathrm{Corr}^{\mathrm{pr,sm}}_{\mathbf{A}^1}}$ only records *-pullback along smooth maps and !-pushforward along proper maps.

The following theorem will be proven in Section 3.3.

Theorem 3.2. *There is a natural transformation of lax monoidal functors* $\mathrm{Corr}^{\mathrm{pr,sm}}_{\mathbf{A}^1} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{gr}}$:

$$\Upsilon : \mathrm{DM}_\eta \to \mathrm{Nilp}_{\mathbf{Q}}\,\mathrm{DM}_s,$$

whose evaluation at a scheme $X/\mathbf{A}^1$, *composed with the forgetful functor* $\mathrm{Nilp}_{\mathbf{Q}}\,\mathrm{DM}(X_s) \to \mathrm{DM}(X_s)$, *is the functor* Υ *in Definition* 3.1.

Remark 3.3. We explain the content of Theorem 3.2 by probing Υ in various ways.

(1) Evaluating Υ on some scheme $X/\mathbf{A}^1$, we have a colimit-preserving functor

$$\Upsilon_X : \mathrm{DM}(X_\eta) \to \mathrm{Nilp}_{\mathbf{Q}}\,\mathrm{DM}(X_s).$$

That is, for $M \in \mathrm{DM}(X_\eta)$, $\Upsilon(M) \in \mathrm{DM}(X_s)$ is naturally equipped with a locally nilpotent morphism $\Upsilon(M)(1) \otimes \mathbf{Q} \to \Upsilon(M) \otimes \mathbf{Q}$, that we refer to as the *monodromy*. In Proposition 3.9 we will compute its cofiber to be $i^* j_* M(1)[1] \otimes \mathbf{Q}$.

(2) Evaluating Υ at a correspondence $Y \xleftarrow{g} Z \xrightarrow{f} X$ (of schemes over $\mathbf{A}^1$), with f smooth and g proper, we have a natural isomorphism

$$g_{s!} f_s^* \Upsilon_X \xrightarrow{\cong} \Upsilon_Y g_{\eta!} f_\eta^* \tag{3.5}$$

or, more diagramatically, a commutative diagram

$$
\begin{array}{ccc}
\mathrm{DM}(X_\eta) & \xrightarrow{\;\Upsilon_X\;} & \mathrm{Nilp}_{\mathbf{Q}}\,\mathrm{DM}(X_s) \\
{\scriptstyle g_{\eta!} f_\eta^*}\big\downarrow & & \big\downarrow{\scriptstyle g_{s!} f_s^*} \\
\mathrm{DM}(Y_\eta) & \xrightarrow{\;\Upsilon_Y\;} & \mathrm{Nilp}_{\mathbf{Q}}\,\mathrm{DM}(Y_s).
\end{array}
$$

Here g_η is the pullback of g along $\mathbf{G}_m \subset \mathbf{A}^1$ etc. In other words, the formation of (unipotent) nearby cycles, including the monodromy, is compatible with proper pushforward and smooth pullback.

For g non-proper or f non-smooth, we still have a map in (3.5), but it need not be an isomorphism, cf. Remark 3.8.

(3) For a pair X_1, X_2 of schemes over $\mathbf{A}^1$, there is a morphism

$$(X_1, X_2) \to X_1 \times_{\mathbf{A}^1} X_2$$

in (the monoidal ∞-category) $\mathrm{Corr}_{\mathbf{A}^1}$. The lax monoidality of Υ then gives a natural transformation in the diagram

$$
\begin{array}{ccc}
\mathrm{DM}(X_{1\eta}) \otimes \mathrm{DM}(X_{2\eta}) & \xrightarrow{\;\Upsilon_{X_1} \otimes \Upsilon_{X_2}\;} & \mathrm{Nilp}_{\mathbf{Q}}\,\mathrm{DM}(X_{1s}) \otimes \mathrm{Nilp}_{\mathbf{Q}}\,\mathrm{DM}(X_{2s}) \\
{\scriptstyle \boxtimes_{\mathbf{G}_m}}\big\downarrow & & \big\downarrow{\scriptstyle \boxtimes} \\
\mathrm{DM}(X_{1\eta} \times_{\mathbf{G}_m} X_{2\eta}) & \xrightarrow[\;\Upsilon_{X_1 \times_{\mathbf{A}^1} X_2}\;]{} & \mathrm{Nilp}_{\mathbf{Q}}\,\mathrm{DM}(X_{1s} \times X_{2s}).
\end{array}
$$

Thus, for $M_i \in \mathrm{DM}(X_{i\eta})$, there is a natural map

$$\kappa_{(X_1,X_2)} : \Upsilon_{X_1}(M_1) \boxtimes \Upsilon_{X_2}(M_2) \to \Upsilon_{X_1 \times_{\mathbf{A}^1} X_2}(M_1 \boxtimes_{\mathbf{G}_m} M_2).$$

We call this the *Künneth map* for unipotent nearby cycles. (In general it is not an isomorphism, but see Theorem 4.37(3) for a case where it is.) This map is functorial and compatible with arbitrary colimits in both M_i. It is compatible with the monodromy maps in the sense that it is a map of (locally nilpotent)

14

Σ-modules, where Σ acts on the source of κ via the comultiplication of the bialgebra Σ, cf. (2.4). Further, κ is compatible with functoriality in $\mathrm{Corr}_{\mathbf{A}^1}^{\mathrm{pr,sm}}$. For example, for a pair of smooth maps $f_k : Z_k \to X_k$, there is a commutative square in $\mathrm{Corr}_{\mathbf{A}^1}$

$$
\begin{array}{ccc}
(Z_1, Z_2) & \longrightarrow & Z_1 \times_{\mathbf{A}^1} Z_2 \\
{\scriptstyle (f_1, f_2)} \downarrow & & \downarrow {\scriptstyle f_1 \times_{\mathbf{A}^1} f_2} \\
(X_1, X_2) & \longrightarrow & X_1 \times_{\mathbf{A}^1} X_2.
\end{array}
$$

Then

$$
(f_{1s} \times f_{2s})^* \kappa_{(X_1, X_2)} (M_1, M_2) = \kappa_{(Z_1, Z_2)} \left((f_{1\eta})^* M_1, (f_{2\eta})^* M_2 \right).
$$

A similar formula holds for !-pushforwards along proper maps.

(4) For reduced motives, we have a similarly defined functor

$$
\Upsilon : \mathrm{DM}_{\mathrm{r}, \eta} \to \mathrm{Nilp}\, \mathrm{DM}_{\mathrm{r}, s},
$$

i.e., the monodromy map ϕ exists already integrally. Indeed, the point of using $\mathrm{Nilp}_{\mathbf{Q}}$ above is the usage of Definition 2.24 (which goes back to Lemma 2.18), which already holds integrally for reduced motives. All properties of Υ proved in this section (for regular motives) then hold verbatim for reduced motives.

Corollary 3.4. *For fixed* $X/\mathbf{A}^1$, $\Upsilon : \mathrm{DM}(X_\eta) \to \mathrm{Nilp}_{\mathbf{Q}}\, \mathrm{DM}(X_s)$ *is a lax symmetric monoidal functor.*

Proof. The symmetric monoidal structure on individual categories $\mathrm{DM}(Y)$ arises by appending Δ^* (for $Y \xrightarrow{\Delta} Y \times Y$) to $\boxtimes$. The monoidal structure is then given by the composite:

$$
\begin{aligned}
\Upsilon(-) \otimes \Upsilon(-) &= \Delta_{X_s}^* (\Upsilon(-) \boxtimes \Upsilon(-)) \\
&\xrightarrow{*} \Upsilon(\Delta_{X_\eta}^* (- \boxtimes -)) \\
&= \Upsilon(- \otimes -).
\end{aligned}
$$

The map labelled "$*$" exists by the lax compatibility of Υ with respect to (non-smooth) pullback, cf. Remark 3.8. Alternatively, using Zariski descent to reduce to the case of separated X, one can use the compatibility of Υ with pushforward along the proper map Δ. $\qquad\square$

3.2. **Further remarks.** The above definition is related to various works: a precise comparison with Ayoub's unipotent nearby cycles appears below in Section 3.9. The approach in Definition 3.1 can be regarded as a motivic refinement of Campbell's approach to Beilinson's unipotent nearby cycles [Cam18]. It also bears some similarity with the one in [Ayo15, Scholie 1.3.26], but is simpler in that it avoids the usage of rigid analytic motives. It also shares a kinship with the recent work of Binda–Gallauer–Vezzani [BGV23], who have independently carved out the close relation between unipotent nearby cycles (in the context of rigid-analytic motives) and Koszul duality.

As was highlighted to us by Vezzani, the existence of an integral nearby cycles functor (without monodromy), and the existence of a monodromy map after passing to rational coefficients matches the outcome of the approach in Ayoub's work [Ayo07a, §3.6] as well as the situation for rigid analytic motives in the work of Binda–Gallauer–Vezzani [BGV23], with the salient point being [AGV22, Theorem 3.3.3]. An integral monodromy map for *reduced* motives is discussed in Remark 3.3(4).

If we consider $\Upsilon^! := (1^* \otimes i^! j_!) \circ \Gamma_{f_\eta?}$ instead of Υ, we get $\Upsilon^! = \Upsilon[1]$.

An interesting variant is to consider other categories in place of $\mathrm{DTM}(\mathbf{G}_{\mathrm{m}})$. For example, one can consider the functor

$$
\mathrm{DM}(X_\eta) \to \mathrm{DATM}(\mathbf{G}_{\mathrm{m}}) \otimes_{\mathrm{DTM}(S)} \mathrm{DM}(X_\eta) \xrightarrow{1^* \otimes i^* j_*} \mathrm{DM}(X_s),
$$

where $\mathrm{DATM}(\mathbf{G}_{\mathrm{m}})$ denotes the subcategory of $\mathrm{DM}(\mathbf{G}_{\mathrm{m}})$ spanned by $f_{n*} \mathbf{Z}(n)$, for $f_n : \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$ being the n-th power map. Over a field of characteristic 0, does this functor agree with Ayoub's full nearby cycles functor? Other interesting variants might be to allow $f_* \mathbf{Z}(n)$ for all finite étale f, or even replace $\mathrm{DTM}(\mathbf{G}_{\mathrm{m}})$ by the subcategory $\mathrm{DM}(\mathbf{G}_{\mathrm{m}})^{\mathrm{dualizable}} \subset \mathrm{DM}(\mathbf{G}_{\mathrm{m}})$ of dualizable objects.

3.3. **Proof of Theorem** 3.2. The following diagram gives an overview of the functors and natural transformations appearing in the construction of Υ:

$$
\begin{array}{ccccccc}
\mathrm{Corr}_{\mathbf{A}^1}^{\mathrm{sm,pr}} & \xrightarrow{j^*} & \mathrm{Corr}_{\mathbf{G}_{\mathrm{m}}}^{\mathrm{sm,pr}} & \xrightarrow{\mathrm{DM}} & \mathrm{Mod}_{\mathrm{DTM}(\mathbf{G}_{\mathrm{m}})}(\mathrm{Pr}_{\mathrm{St}}^{\mathrm{L}}) & \xrightarrow{F} & \mathrm{Mod}_{\mathrm{Nilp}_{\mathbf{Q}}\, \mathrm{DTM}(S)}(\mathrm{Pr}_{\mathrm{St}}^{\mathrm{L}}) \\
& {\scriptstyle i^*} \searrow & {\scriptstyle i^* j_*} \nearrow & & {\scriptstyle U} \downarrow\,)\!)\alpha)\, {\scriptstyle L} & & {\scriptstyle U'} \downarrow\,)\, {\scriptstyle L'} \\
& \mathrm{Corr}^{\mathrm{sm,pr}} & \xrightarrow{\mathrm{DM}} & \mathrm{Mod}_{\mathrm{DTM}(S)}(\mathrm{Pr}_{\mathrm{St}}^{\mathrm{L}}) & = = = & \mathrm{Mod}_{\mathrm{DTM}(S)}(\mathrm{Pr}_{\mathrm{St}}^{\mathrm{L}}).
\end{array}
$$

Here i^* and j^* at the left are the pullbacks along the respective inclusions; DM is the functor in (2.11). We have abbreviated $F := \mathrm{Nilp}_{\mathbf{Q}}\, \mathrm{DTM}(S) \otimes_{\mathrm{DTM}(\mathbf{G}_{\mathrm{m}})} -$, where the tensor product is formed using the functor in Definition 2.24. The functors U and U' are the forgetful functors, whose left adjoints are given by $L := \mathrm{DTM}(\mathbf{G}_{\mathrm{m}}) \otimes_{\mathrm{DTM}(S)} -$ and $L' := \mathrm{Nilp}_{\mathbf{Q}}\, \mathrm{DTM}(S) \otimes_{\mathrm{DTM}(S)} -$. In this diagram, i^*, j^*, F, L and L' are monoidal and all other functors are lax monoidal. As before, we abbreviate $\mathrm{DM}_\eta := \mathrm{DM} \circ j^*$, $\mathrm{DM}_s = \mathrm{DM} \circ i^*$.

We now discuss the natural transformations appearing in this diagram, including their lax monoidality. The simplest part is the natural transformation of lax monoidal endofunctors on $\mathrm{Mod}_{\mathrm{DTM}(S)}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}})$, $UL \to U'L'$. It arises from $L' = FL$, so if G denotes the right adjoint of F (which is again a forgetful functor), the unit map $\mathrm{id} \to GF$ gives rise to

$$UL \to U'L' = UGFL.$$

For some $C \in \mathrm{Mod}_{\mathrm{DTM}(S)}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}})$ (such as $C = \mathrm{DM}(X_\eta)$), the evaluation on C is the functor $\mathrm{DTM}(\mathbf{G}_{\mathrm{m}}) \otimes_{\mathrm{DTM}(S)} C \to \mathrm{Nilp}_{\mathbf{Q}}\, \mathrm{DTM}(S) \otimes_{\mathrm{DTM}(S)} C = \mathrm{Nilp}_{\mathbf{Q}} \otimes_{\mathrm{D}(\mathrm{Ab}^{\mathbf{z}})} C$ that on the first factor sends $M \in \mathrm{DTM}(\mathbf{G}_{\mathrm{m}})$ to the pair consisting of $1^*M \in \mathrm{DTM}(S)$ and $\phi : M \otimes \mathbf{Q}(1) \to M \otimes \mathbf{Q}$, and on the second factor is id_C.

Lemma 3.5. *There is a natural transformation*

$$\alpha : \mathrm{DM}_\eta \to LU\mathrm{DM}_\eta$$

of lax monoidal functors $\mathrm{Corr}_{\mathbf{A}^1} \to \mathrm{Mod}_{\mathrm{DTM}(\mathbf{G}_{\mathrm{m}})}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}})$, *obtained from the counit* $LU \to \mathrm{id}$ *of the above adjunction by passing to right adjoints.*

Proof. The categories $\mathrm{DTM}(\mathbf{G}_{\mathrm{m}})$ and $\mathrm{DTM}(S)$ are compactly generated by the objects $\mathbf{Z}(n)$, and therefore rigid. By Lemma 2.2, the right and left adjoint functor (denoted L) of $U : \mathrm{Mod}_{\mathrm{DTM}(\mathbf{G}_{\mathrm{m}})}\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}} \to \mathrm{Mod}_{\mathrm{DTM}(S)}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}})$ agree up to equivalence. The unit map $\mathrm{id} \to LU$ is a natural map of lax symmetric monoidal functors, and therefore so is its composition with DM_η. $\square$

Remark 3.6. The functor $\Gamma_{f_\eta?}$ has the following properties:

- For any scheme $X/\mathbf{A}^1$, the evaluation $\alpha(X)$ is the right adjoint

$$\Gamma_{f_\eta?} : \mathrm{DM}(X_\eta) \to \mathrm{DTM}(\mathbf{G}_{\mathrm{m}}) \otimes_{\mathrm{DTM}(S)} \mathrm{DM}(X_\eta)$$

  of the functor $\mathrm{DTM}(\mathbf{G}_{\mathrm{m}}) \otimes \mathrm{DM}(X_\eta) \overset{\boxtimes}{\to} \mathrm{DM}(\mathbf{G}_{\mathrm{m}} \times_S X_\eta) \overset{\Gamma^*_{f_\eta}}{\to} \mathrm{DM}(X_\eta)$. Indeed, this latter functor encodes the action of $\mathrm{DTM}(\mathbf{G}_{\mathrm{m}})$ on $\mathrm{DM}(X_\eta)$.

- We can disentangle the rôle of $\mathbf{G}_{\mathrm{m}}$ and X_η using the formula (2.2):

$$\Gamma_{f_\eta?} = \Delta_? \otimes \mathrm{DM}(X_\eta), \tag{3.6}$$

  where $\Delta_?$ is the right adjoint of the $\mathrm{DTM}(\mathbf{G}_{\mathrm{m}})$-action on itself, which is more concretely given by

$$\mathrm{DTM}(\mathbf{G}_{\mathrm{m}}) \otimes_{\mathrm{DTM}(S)} \mathrm{DTM}(\mathbf{G}_{\mathrm{m}}) \overset{\boxtimes}{\underset{\cong}{\to}} \mathrm{DTM}(\mathbf{G}^2_{\mathrm{m}}) \overset{\Delta^*}{\to} \mathrm{DTM}(\mathbf{G}_{\mathrm{m}}).$$

  The first functor $\boxtimes$ is an equivalence by Lemma 2.17 and Lemma 2.6. Therefore, $\Delta_?$ is essentially the right adjoint of Δ^*. The functor $\Delta_?$ is *distinct* from the right adjoint Δ_* of $\Delta^* : \mathrm{DM}(\mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}}) \to \mathrm{DM}(\mathbf{G}_{\mathrm{m}})$. Instead, the inclusion $\mathrm{DTM}(\mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}}) \subset \mathrm{DM}(\mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}})$ admits a right adjoint R, and $\Delta_? = R \circ \Delta_*$. The functor R was studied by Totaro [Tot18].

  More generally, if X_η carries a universal Whitney–Tate stratification (in the sense of [CvdHS22, Definition 2.6]), then one can show an equivalence

$$\mathrm{DTM}(\mathbf{G}_{\mathrm{m}}) \otimes_{\mathrm{DTM}(S)} \mathrm{DTM}(X_\eta) \overset{\boxtimes}{\underset{\cong}{\to}} \mathrm{DTM}(\mathbf{G}_{\mathrm{m}} \times_S X_\eta).$$

  Thus, in this case $\Gamma_{f_\eta?}$ can be computed on $\mathrm{DTM}(X_\eta)$ as $\Gamma_{f_\eta?} = R \circ \Gamma_{f_\eta*}$, where R is again a right adjoint to the inclusion $\mathrm{DTM}(X_\eta) \subset \mathrm{DM}(X_\eta)$ of stratified Tate motives.

- Among other things, Lemma 3.5 asserts that $\Gamma_{f_\eta?}$ is compatible with *-pullbacks and !-pushforwards (along the X-direction), as well as with exterior products in the following sense. For two schemes $f_k : X_k \to \mathbf{A}^1$, the following diagram commutes, where we abbreviate $D := \mathrm{DTM}_{\mathrm{r}}(\mathbf{G}_{\mathrm{m}})$:

$$\begin{array}{ccc}
\mathrm{DM}(X_{1\eta}) \otimes \mathrm{DM}(X_{2\eta}) & \xrightarrow{\boxtimes_{\mathbf{G}_{\mathrm{m}}}} & \mathrm{DM}(X_{1\eta} \times_{\mathbf{G}_{\mathrm{m}}} X_{2\eta}) \\
\downarrow{\scriptstyle \Gamma_{f_{1\eta}?} \otimes \Gamma_{f_{2\eta}?}} & & \downarrow{\scriptstyle \Gamma_{(f_1 \times f_2)_\eta?}} \\
(D \otimes \mathrm{DM}(X_{1\eta})) \otimes_D (D \otimes \mathrm{DM}(X_{2\eta})) & \xrightarrow{\boxtimes_{\mathbf{G}_{\mathrm{m}}}} & D \otimes \mathrm{DM}(X_{1\eta} \times_{\mathbf{G}_{\mathrm{m}}} X_{2\eta}).
\end{array}$$

Lemma 3.7. *There is a natural transformation*

$$i^*j_* : U\mathrm{DM}_\eta \to \mathrm{DM}_s$$

of lax monoidal functors $\mathrm{Corr}^{\mathrm{sm,pr}}_{\mathbf{A}^1} \to \mathrm{Mod}_{\mathrm{DTM}(S)}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}})$.

Proof. The functor $i^* : \mathrm{DM} \to \mathrm{DM}_s$ and $j^* : \mathrm{DM} \to \mathrm{DM}_\eta$ are clearly transformations of lax monoidal functors.

By construction, the restriction of $\mathrm{DM}^*_!$ to $\mathrm{Corr}^{\mathrm{sm,pr}}_{\mathbf{A}^1}$ is taking values in the full $\otimes$-subcategory $\mathrm{Pr}^{\mathrm{LR}}_{\omega,\mathrm{St}} \subset \mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ (cf. Lemma 2.1), since the categories $\mathrm{DM}(-)$ are compactly generated, and the functors f^* (for f smooth) and $g_!$ (for g proper) admit left adjoints, namely $f_\sharp$ and g^*, respectively. Thus, $\mathrm{DM}^*_!$ is a lax $\otimes$-functor $\mathrm{Corr}^{\mathrm{sm,pr}}_{\mathbf{A}^1} \to \mathrm{Pr}^{\mathrm{LR}}_{\omega,\mathrm{St}}$. Appending the $\otimes$-equivalence $\mathrm{Pr}^{\mathrm{LR}}_{\omega,\mathrm{St}} \cong (\mathrm{Pr}^{\mathrm{L}}_{\omega,\mathrm{St}})^{\mathrm{op}}$ obtained by passing to left adjoints (Lemma 2.1), we obtain a monoidal functor denoted by

$$\mathrm{DM}^*_\sharp : \mathrm{Corr}^{\mathrm{sm,pr}}_{\mathbf{A}^1} \to (\mathrm{Pr}^{\mathrm{L}}_{\omega,\mathrm{St}})^{\mathrm{op}}.$$

It encodes $\sharp$-pushforwards along smooth maps f and $*$-pullbacks along proper maps g.

Precomposition with $j_\sharp j^* \to \mathrm{id}$, which is a transformation of symmetric monoidal endofunctors on $\mathrm{Corr}_{\mathbf{A}^1}$ (its evaluation on $X/\mathbf{A}^1$ is the map $j : X_\eta \to X$) induces the natural transformation of lax $\otimes$-functors $j^* : \mathrm{DM}^*_\sharp \to \mathrm{DM}^*_{\eta,\sharp}$. Passing back to right adjoints, we obtain the requested transformation, of lax $\otimes$-functors, $j_* : \mathrm{DM}^*_{\eta!} \to \mathrm{DM}^*_!$. $\square$

This accomplishes the construction of unipotent nearby cycles in the form stated in Theorem 3.2.

Remark 3.8. The restriction to $\mathrm{Corr}^{\mathrm{sm,pr}}_{\mathbf{A}^1} \subset \mathrm{Corr}_{\mathbf{A}^1}$ is only necessary to ensure that $j_* : \mathrm{DM}_\eta \to \mathrm{DM}$ is functorial. Without that restriction, j_* and therefore also Υ is a *lax* natural transformation of functors $\mathrm{Fun}(\mathrm{Corr}_{\mathbf{A}^1}, \mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}})$ obtained from the (non-lax) natural transformation j^* by passing to adjoints, cf. e.g. [Hau21].

3.4. **Monodromy triangle.** The following fiber sequence has been achieved for motives with rational coefficients by Ayoub in [Ayo07b, Theorem 4.28]. The approach taken here using Koszul duality highlights the fact that the relevance of rational coefficients is related to the non-formality of the motive of $\mathbf{G}_\mathrm{m}$, cf. Remark 2.19, Remark 2.21. Recall the object $\Sigma = \bigoplus_{n \geq 0} \mathbf{Z}(n)$ from Section 2.1.2.

Proposition 3.9. *Let $X/\mathbf{A}^1$, $M \in \mathrm{DM}(X_\eta, \mathbf{Q})$ or $M \in \mathrm{DM}_\mathrm{r}(X_\eta)$. There is a functorial isomorphism*

$$\Upsilon(M) \otimes_\Sigma \mathbf{Z} = i^* j_* M(1)[1]. \tag{3.7}$$

In other words, there is a cofiber sequence (in $\mathrm{DM}(X_s, \mathbf{Q})$, resp. $\mathrm{DM}_\mathrm{r}(X_s)$),

$$i^* j_* M \to \Upsilon(M) \to \Upsilon(M)(-1). \tag{3.8}$$

Proof. Indeed, the cofiber sequence (2.8) shows that $\mathbf{Z} \otimes_\Sigma \mathbf{Z}$ is the split square zero extension $\mathbf{Z} \oplus \mathbf{Z}(1)[1]$. For reduced motives (and likewise after passing to rational coefficients for regular motives), this identifies with $\Lambda(1)[1]$, by Lemma 2.18. This yields

$$\begin{aligned}
\Upsilon(M) \otimes_\Sigma \mathbf{Z} &= (i^* j_* M \otimes_\Lambda \mathbf{Z}) \otimes_\Sigma \mathbf{Z} \\
&= i^* j_* M \otimes_\Lambda \Lambda(1)[1].
\end{aligned}$$

The cofiber sequence is then an immediate consequence. $\square$

Remark 3.10. Proposition 3.9 shows that $i^* j_*$ can be recovered from Υ (while the converse holds by definition of Υ). This implies that Beilinson's glueing theorem for perverse sheaves (e.g. [Mor18, Theorem 8.1]) can be carried out motivically as soon as one has a motivic t-structure, e.g., for mixed stratified Tate motives over an appropriately stratified scheme, cf. [CvdHS22, Lemma 2.15].

Remark 3.11. Both maps in (3.8) exist already integrally, but we only claim it is a cofiber sequence for rational motives (or reduced motives). For the latter map, see Remark 2.21. The map $\alpha : i^* j_* M \to \Upsilon(M)$ arises from applying $i^* j_* M \otimes_\Lambda -$ to the unit map $\Lambda = p_* p^* \mathbf{Z} \to p_* 1_* 1^* p^* \mathbf{Z} = \mathbf{Z}$, where $S \xrightarrow{1} \mathbf{G}_{\mathrm{m},S} \xrightarrow{p} S$. We say that $\Upsilon(M)$ has *trivial monodromy* if α splits. By functoriality, this condition on M is preserved under proper pushforward and smooth pullback.

3.5. **Computation for trivial families.**

Lemma 3.12. *For the trivial family $f = \mathrm{id}_{\mathbf{A}^1}$ we have $\Upsilon_{\mathrm{id}}(\mathbf{Z}) = \mathbf{Z}$, and the monodromy is trivial (in the sense of Remark 3.11).*

Proof. We use the following diagram, in which the functors Δ^*, $\mathrm{id} \otimes j^*$ and $\mathrm{id} \otimes q^*$ are the left adjoints of the adjacent functors in the opposite direction. All tensor products are over $\mathrm{DTM}(S)$, and $\mathrm{DTM}(\mathbf{A}^1)$ denotes stratified Tate motives with respect to the stratification given by $\mathbf{G}_\mathrm{m} \sqcup 0$. Note that q_* maps these motives to $\mathrm{DTM}(S)$, so $\mathrm{id} \otimes q_*$ below takes values in $\mathrm{DTM}(\mathbf{G}_\mathrm{m}) = \mathrm{DTM}(\mathbf{G}_\mathrm{m}) \otimes \mathrm{DTM}(S)$.

$$\begin{array}{ccccc}
\mathrm{DTM}(\mathbf{G}_\mathrm{m}) \otimes \mathrm{DTM}(\mathbf{G}_\mathrm{m}) & \underset{\mathrm{id} \otimes j_*}{\overset{\mathrm{id} \otimes j^*}{\longleftarrow}} & \mathrm{DTM}(\mathbf{G}_\mathrm{m}) \otimes \mathrm{DTM}(\mathbf{A}^1) & \xrightarrow{\mathrm{id} \otimes i^*} & \mathrm{DTM}(\mathbf{G}_\mathrm{m}) \\
\Delta^* \Big\Updownarrow \Delta_? & & \mathrm{id} \otimes q^* \Big\Updownarrow \mathrm{id} \otimes q_* & & \Big\downarrow 1^* \\
\mathrm{DTM}(\mathbf{G}_\mathrm{m}) & = & \mathrm{DTM}(\mathbf{G}_\mathrm{m}) & \xrightarrow{1^*} & \mathrm{DTM}(S).
\end{array} \tag{3.9}$$

At the left half, we have $(\mathrm{id} \otimes q_*)(\mathrm{id} \otimes j_*)\Delta_? = \mathrm{id}_{\mathrm{DTM}(\mathbf{G}_{\mathrm{m}})}$ since it is right adjoint to $\Delta^*(\mathrm{id} \otimes j^*)(\mathrm{id} \otimes q^*) = (q \circ (\mathrm{id} \times j) \circ \Delta)^* = \mathrm{id}^*_{\mathbf{G}_{\mathrm{m}}}$.

The unit map

$$q_* \to q_* i_* i^* = i^* \tag{3.10}$$

is an isomorphism when evaluated on objects in $\mathrm{DTM}(\mathbf{A}^1)$: it is enough to check this for $i_*\mathbf{Z}$, which is clear, and for $\mathbf{Z}$, where it follows from homotopy invariance, i.e., $q_* q^* = \mathrm{id}$.

The triviality of the monodromy also follows from this: under the equivalence $\mathrm{DTM}(\mathbf{G}_{\mathrm{m}}) = \mathrm{Mod}_\Lambda$, $i^* j_* \mathbf{Z}$ corresponds to Λ, and the unit map in Remark 3.11 is the augmentation map $\Lambda \to \mathbf{Z}$, which is split by the unit map for Λ. $\qquad\square$

Remark 3.13. The importance of the isomorphism $q_*\mathbf{Z} = i^*\mathbf{Z}$ is hard to overstate. First, it is also the basis of the same result in Ayoub's approach [Ayo07b, Proposition 4.9] (or [Ayo07a, Proposition 3.4.9]). It is also the core computation in the proof of hyperbolic localization, cf. [Ric19, Corollary 2.9] or [DG14, Proposition 3.2.2]. The full force of $\mathbf{A}^1$-invariance is actually not needed, instead it suffices to use that the pullback along the stack quotients $\overline{q} : \mathbf{A}^1/\mathbf{G}_{\mathrm{m}} \to B\mathbf{G}_{\mathrm{m}}(:= S/\mathbf{G}_{\mathrm{m}})$ is fully faithful. This should prove useful in applying the above paradigm to prismatic cohomology, cf. [KP21, Lemma 5.3.2], and also to the motivic spaces considered in [AHI23].

Corollary 3.14. *Suppose $f : X = Y \times \mathbf{A}^1 \to \mathbf{A}^1$ is a trivial family. Let $M \in \mathrm{DM}(Y)$ be such that the natural map*

$$M \boxtimes j_*\mathbf{Z} \to (\mathrm{id}_Y \times j)_*(M \boxtimes \mathbf{Z}) \tag{3.11}$$

is an isomorphism. Then there is a natural isomorphism (and the monodromy is trivial)

$$\Upsilon(M \boxtimes \mathbf{Z}_{\mathbf{G}_{\mathrm{m}}}) = M.$$

Proof. We have the following diagram, where the bottom right horizontal functor is $\mathrm{id} \otimes (i \times \mathrm{id}^*)(j \times \mathrm{id})_*$. (Recall the convention that $\otimes$ is meant to be over $\mathrm{DTM}(S)$.)

$$
\begin{array}{ccccc}
\mathrm{DTM}(\mathbf{G}_{\mathrm{m}}) \otimes \mathrm{DM}(Y) & \xrightarrow{\Delta_? \otimes \mathrm{id}} & \mathrm{DTM}(\mathbf{G}_{\mathrm{m}})^{\otimes 2} \otimes \mathrm{DM}(Y) & \xrightarrow{\mathrm{id} \otimes i^* j_* \otimes \mathrm{id}} & \mathrm{DTM}(\mathbf{G}_{\mathrm{m}}) \otimes \mathrm{DM}(Y) \\
\downarrow{\scriptstyle \boxtimes} & & \downarrow{\scriptstyle \mathrm{id} \otimes \boxtimes} & & \Vert \\
\mathrm{DM}(\mathbf{G}_{\mathrm{m}} \times Y) & \xrightarrow{(\Delta \times \mathrm{id}_Y)_?} & \mathrm{DTM}(\mathbf{G}_{\mathrm{m}}) \otimes \mathrm{DM}(\mathbf{G}_{\mathrm{m}} \times Y) & \longrightarrow & \mathrm{DTM}(\mathbf{G}_{\mathrm{m}}) \otimes \mathrm{DM}(Y).
\end{array}
$$

The left square commutes since the left vertical $\boxtimes$-functor is a map of $\mathrm{DTM}(\mathbf{G}_{\mathrm{m}})$-modules, and thus also a map of $\mathrm{DTM}(\mathbf{G}_{\mathrm{m}})$-comodules (Lemma 2.2). The two left horizontal arrows are just these coaction maps.

The indicated natural transformation at the right stems from the commutativity of i^* with $\boxtimes$ (which holds for any motive in $\mathrm{DM}(\mathbf{A}^1) \otimes \mathrm{DM}(Y)$), and the map (3.11) (which in general need not be an isomorphism).

If we append $1^* \otimes \mathrm{id}$ to the top row, we get $\Upsilon_{\mathrm{id}} \otimes \mathrm{id}_{\mathrm{DM}(Y)}$. Using $\Upsilon(\mathbf{Z}_{\mathbf{G}_{\mathrm{m}}}) = \mathbf{Z}$ (Lemma 3.12) our claim follows. $\qquad\square$

Remark 3.15. If the base scheme S is a field of exponential characteristic n, then (3.11) (and also (3.19) below) is an isomorphism for arbitrary motives with $\mathbf{Z}[1/n]$-coefficients [JY21a, Theorem 2.4.6].

The motivation for the tailored statement above is that we will perform the construction of central motives over $S = \mathrm{Spec}\,\mathbf{Z}$, where that property does not hold in general.

We also note that Jin–Yang [JY21b] take a similar approach by singling out a ULA-ness condition that ensures compatibility of tame nearby cycles with exterior products.

3.6. Preservation of Tate motives.

Proposition 3.16. *Suppose $X/\mathbf{A}^1$ carries an (anti-effective) Whitney–Tate stratification that is compatible with the stratification of $\mathbf{A}^1_S$ by $\mathbf{G}_{\mathrm{m},S} \sqcup S$. Then Υ preserves (anti-effective) stratified Tate motives, i.e., it restricts to a functor*

$$\Upsilon : \mathrm{DTM}(X_\eta, X_\eta^+)^{(\mathrm{anti})} \to \mathrm{DTM}(X_s, X_s^+)^{(\mathrm{anti})}.$$

Proof. By (3.6), we can compute $\Gamma_{f_\eta?}$ as

$$\Delta_? \otimes \mathrm{id} : \mathrm{DTM}(\mathbf{G}_{\mathrm{m}}) \otimes_{\mathrm{DTM}(\mathbf{G}_{\mathrm{m}})} \mathrm{DM}(X_\eta) \to \mathrm{DTM}(\mathbf{G}_{\mathrm{m}}^2) \otimes_{\mathrm{DTM}(\mathbf{G}_{\mathrm{m}})} \mathrm{DM}(X_\eta),$$

where $\Delta_?$ is the right adjoint of Δ^* (restricted to Tate motives!). In particular, $\Gamma_{f_\eta?}$ maps $\mathrm{DTM}(X_\eta)^{(\mathrm{anti})}$ to the full subcategory $\mathrm{DTM}(\mathbf{G}_{\mathrm{m}}) \otimes \mathrm{DTM}(X_\eta)^{(\mathrm{anti})}$. Since $i^* j_*$ preserves (anti-effective) Tate motives by assumption, and $1^* : \mathrm{DTM}(\mathbf{G}_{\mathrm{m}}) \to \mathrm{DTM}(S)$ also does, this shows that Υ preserves (anti-effective) Tate motives. $\qquad\square$

3.7. **Compatibility with hyperbolic localization.** We address the compatibility of unipotent nearby cycles with hyperbolic localization. Given a scheme $X/\mathbf{A}^1$ endowed with a Nisnevich-locally linearizable $\mathbf{G}_{\mathrm{m}}$-action (where $\mathbf{A}^1$ is acted upon trivially), we consider the hyperbolic localization diagram (e.g., [DG14, Ric19])

$$X^0 \xleftarrow{p^\pm} X^\pm \xrightarrow{q^\pm} X \tag{3.12}$$
$$\begin{array}{ccc} & f^0 \searrow \quad \downarrow f^\pm \quad \swarrow f & \\ & \mathbf{A}^1. & \end{array}$$

Here $X^+ = \mathrm{Maps}^{\mathbf{G}_{\mathrm{m}}}(\mathbf{A}^1, X)$ are the attractors of the $\mathbf{G}_{\mathrm{m}}$-action, $X^- = \mathrm{Maps}^{\mathbf{G}_{\mathrm{m}}}(\mathbf{A}^1_{\mathrm{anti}}, X)$ are the repellers and $X^0 = \mathrm{Maps}^{\mathbf{G}_{\mathrm{m}}}(S, X)$ are the fixed points of the $\mathbf{G}_{\mathrm{m}}$-action. We will write subscripts s and η for pullbacks along $0 \subset \mathbf{A}^1$ and $\mathbf{G}_{\mathrm{m}} \subset \mathbf{A}^1$ and superscripts 0 and $+$ for the corresponding objects on the level of fixed points and attractors. Note that $(X_\eta)^+ = X^+ \times_{\mathbf{A}^1} \mathbf{G}_{\mathrm{m}}$ etc.

The subcategory $\mathrm{DM}(X_\eta)^{\mathbf{G}_{\mathrm{m}}-\mathrm{mono}} \subset \mathrm{DM}(X_\eta)$ of $\mathbf{G}_{\mathrm{m}}$-*monodromic motives* is defined to be the presentable subcategory generated by $\mathbf{G}_{\mathrm{m}}$-equivariant motives, i.e., by the image of the forgetful functor $\mathrm{DM}(X_\eta/\mathbf{G}_{\mathrm{m}}) \to \mathrm{DM}(X_\eta)$ (see (3.21) for the category of equivariant motives). Braden's hyperbolic localization asserts that the natural map

$$q_*^- p^{-!} \to q_!^+ p^{+*} \ (\in \mathrm{Fun}(\mathrm{DM}(X), \mathrm{DM}(X^0))), \tag{3.13}$$

is an isomorphism when restricted to $\mathrm{DM}(X)^{\mathbf{G}_{\mathrm{m}}-\mathrm{mono}}$ (cf. [CvdHS22, Proposition 2.5] for this motivic statement and the references there for precursor statements for ℓ-adic sheaves and D-modules).

Proposition 3.17. *Unipotent nearby cycles commute with hyperbolic localization in the sense that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{DM}(X_\eta)^{\mathbf{G}_{\mathrm{m}}-\mathrm{mono}} & \xrightarrow{\Upsilon_f} & \mathrm{Nilp}_{\mathbf{Q}} \, \mathrm{DM}(X_s) \\ {\scriptstyle p_{\eta!}^+ q_\eta^{+*}} \downarrow & & \downarrow {\scriptstyle p_{s!}^+ q_s^{+*}} \\ \mathrm{DM}(X_\eta^0) & \xrightarrow{\Upsilon_{f^0}} & \mathrm{Nilp}_{\mathbf{Q}} \, \mathrm{DM}(X_s^0). \end{array}$$

Proof. Similarly to the proof of Proposition 3.16, the functor $\Gamma_{f_\eta?} = \Delta_? \otimes \mathrm{id}$ maps $\mathrm{DM}(X_\eta)^{\mathbf{G}_{\mathrm{m}}-\mathrm{mono}}$ to $\mathrm{DTM}(\mathbf{G}_{\mathrm{m}}) \otimes \mathrm{DM}(X_\eta)^{\mathbf{G}_{\mathrm{m}}-\mathrm{mono}}$. On this category, Braden's theorem ensures that $\mathrm{id}_{\mathrm{DTM}(\mathbf{G}_{\mathrm{m}})} \otimes i^* j_*$ commutes with the hyperbolic localization functors in (3.13). And so does, for trivial reasons, 1^*. $\qquad \square$

3.8. **Compatibility with realization.** Suppose given a *realization functor*, i.e., a natural transformation

$$\rho : \mathrm{DM} \to D$$

(of lax symmetric monoidal functors $\mathrm{Corr} \to \mathrm{Pr}_{\mathrm{St}}^{\mathrm{L}}$; note this forces ρ to be compatible with $*$-pullbacks, $!$-pushforwards, and exterior products). Then the construction of Υ can be repeated verbatim for D if we replace $\mathrm{DTM}(X)$ by the subcategory $DT(X) \subset D(X)$ generated by the objects $\rho(\mathbf{Z}(n))$.

Lemma 3.18. *Suppose ρ is right adjointable, i.e., that it commutes with $*$-pushforwards and $!$-pullbacks. Then, for $X/\mathbf{A}^1$, there is a natural commutative diagram*

$$\begin{array}{ccc} \mathrm{DM}(X_\eta) & \xrightarrow{\Upsilon} & \mathrm{DM}(X_s) \\ \rho \downarrow & & \downarrow \rho \\ D(X_\eta) & \xrightarrow{\Upsilon} & D(X_s). \end{array}$$

Proof. We will prove that each functor in the composite

$$\Upsilon_X : \mathrm{DM}(X_\eta) \xrightarrow{\Gamma_{f_\eta?}} \mathrm{DTM}(\mathbf{G}_{\mathrm{m}}) \otimes_{\mathrm{DTM}(S)} \mathrm{DM}(X_\eta) \xrightarrow{1^* \otimes i^* j_*} \mathrm{DM}(X_s)$$

is compatible with ρ. This holds by the naturality and right adjointability of ρ for $1^* \otimes i^* j_*$. It remains to check that

$$\begin{array}{ccc} \mathrm{DTM}(\mathbf{G}_{\mathrm{m}}^2) & \xrightarrow{\Delta^*} & \mathrm{DTM}(\mathbf{G}_{\mathrm{m}}) \\ \rho \downarrow & & \downarrow \rho \\ DT(\mathbf{G}_{\mathrm{m}}^2) & \xrightarrow{\Delta^*} & DT(\mathbf{G}_{\mathrm{m}}) \end{array}$$

is right adjointable, i.e., that $\Delta_?$ commutes with ρ. This implies our claim since $\Gamma_{f_\eta?} = \mathrm{id} \otimes \Delta_?$. Using Lemma 2.17 (and the identical statement for DT), this means that

$$
\begin{array}{ccc}
\mathrm{Mod}_{\Lambda^{\otimes 2}}(\mathrm{DTM}(S)) & \xleftarrow{\ \ \mathrm{res}\ \ } & \mathrm{Mod}_\Lambda(\mathrm{DTM}(S)) \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\
\mathrm{Mod}_{\rho(\Lambda^{\otimes 2})}(DT(S)) & \xleftarrow{\ \ \mathrm{res}\ \ } & \mathrm{Mod}_{\rho(\Lambda)}(DT(S))
\end{array}
$$

should commute. This is the case as can be seen after applying the conservative forgetful functor to $DT(S)$. $\qquad\square$

Example 3.19. According to [Ayo10, Théorème 3.7], [Ayo14, §6], and [ES23, Proposition 3.9], respectively, examples of such realization functors are the Betti and ℓ-adic realization functor and the reduction functor $\mathrm{DM} \to \mathrm{DM_r}$.

Beilinson's approach to unipotent nearby cycles (see, e.g., [Mor18, Corollary 4.3] for a recent exposition) is

$$
\Upsilon^B(F) := \operatorname*{colim}_n i^* j_*(L_n \otimes F),
$$

where $L_n = \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}(-n)$, where $\mathbf{Z}(1)$ is the local system on $\mathbf{G}_{\mathrm{m}}^{\mathrm{an}}$ associated to $\pi_1(\mathbf{G}_{\mathrm{m}}^{\mathrm{an}})$. We have a natural isomorphism

$$
L_n = \operatorname{colim}(\mathbf{Z}(-n)[-n] \to \ldots \to \mathbf{Z}(-1)[-1] \to \mathbf{Z})
$$

so that the definition of Υ (for sheaves in the analytic topology), cf. (3.4) agrees with Υ^B.

In the context of analytic sheaves, it is known that Υ^B is a direct summand of the full (i.e., not necessarily unipotent) nearby cycles functor, and is therefore t-exact and preserves constructible sheaves. Without using a notion of full motivic nearby cycles, we will prove such properties for Tate motives in the special situation of the deformation of, say, $\mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbf{G}_{\mathrm{m}}$ to Fl in Proposition 4.46 and Theorem 4.37.

3.9. **Comparison with Ayoub's approach.** We now show that our definition of Υ agrees with the one due to Ayoub [Ayo07a], [Ayo14, §10]. Ayoub's construction is based on the geometric bar construction associated to the diagram

$$
\begin{array}{ccc}
& & Y := \mathbf{G}_{\mathrm{m}} \qquad\qquad\qquad\qquad (3.14) \\
& & \downarrow{\scriptstyle \mathrm{id} \times 1} \\
X := \mathbf{G}_{\mathrm{m}} & \xrightarrow{\ \Delta\ } & B := \mathbf{G}_{\mathrm{m}}^2.
\end{array}
$$

We denote this geometric bar construction, which is a cosimplicial $\mathbf{G}_{\mathrm{m}}$-scheme, by $A : \Delta \to \mathrm{Sch}_{\mathbf{G}_{\mathrm{m}}}$. In low degrees, it is given by

$$
\mathbf{G}_{\mathrm{m}} \underset{\mathrm{id} \times 1}{\overset{\Delta}{\rightrightarrows}} \mathbf{G}_{\mathrm{m}}^2 \underset{\mathrm{id} \times \mathrm{id} \times 1}{\overset{\Delta \times \mathrm{id}}{\underset{\mathrm{id} \times \Delta}{\rightrightarrows}}} \mathbf{G}_{\mathrm{m}}^3 \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} \cdots
$$

We refer to, say, [Ayo07a, Lemme 3.4.1] for its full definition. Below, we will use that the terms in this cosimplicial scheme are $\mathbf{G}_{\mathrm{m}}$-isomorphic to schemes of the form $\mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}}^n$. We also have the functor

$$
\Gamma : \mathrm{Sch}_{\mathbf{G}_{\mathrm{m}}}^{\mathrm{op}} \to \mathrm{DM}(\mathbf{G}_{\mathrm{m}}), (X \xrightarrow{f} \mathbf{G}_{\mathrm{m}}) \mapsto f_* f^* \mathbf{Z}. \qquad\qquad (3.15)
$$

Definition 3.20. Ayoub's unipotent nearby cycles functor is defined as

$$
\Upsilon^A : \mathrm{DM}(X_\eta) \to \mathrm{DM}(X_s), M \mapsto \Upsilon^A(M) := i^* j_*(\mathscr{U} \otimes M),
$$

where

$$
\mathscr{U} := \operatorname{colim}(\Delta^{\mathrm{op}} \xrightarrow{A} \mathrm{Sch}_{\mathbf{G}_{\mathrm{m}}}^{\mathrm{op}} \xrightarrow{\Gamma} \mathrm{DM}(\mathbf{G}_{\mathrm{m}})).
$$

Remark 3.21. The above presentation of $\mathscr{U}$, which is somewhat easier to digest than Ayoub's original approach using categories of motives over diagrams of schemes, can be found in [JY21b, §2.2].

Previously to this definition, Ayoub had proposed another, mildly different definition [Ayo07b, Definition 4.3],

$$
\Upsilon^{A'}(M) := i^* j_* \underline{\mathrm{Hom}}(\mathscr{U}', M),
$$

where $\mathscr{U}' := \lim(\Delta \xrightarrow{A} \mathrm{Sch}_{\mathbf{G}_{\mathrm{m}}} \xrightarrow{\Gamma} \mathrm{DM}(\mathbf{G}_{\mathrm{m}}) \xrightarrow{(-)^\vee} \mathrm{DM}(\mathbf{G}_{\mathrm{m}})^{\mathrm{op}})$. That is, instead of tensoring termwise with $f_{n*} f_n^* \mathbf{Z}$ (for $f_n : A_n \to \mathbf{G}_{\mathrm{m}}$ the structural map in the n-th term in the cosimplicial scheme given by the bar construction) one can also take $\underline{\mathrm{Hom}}(\lim f_{n!} f_n^! \mathbf{Z}, -)$. Given that $\mathscr{U}'$ is the dual of $\mathscr{U}$ we have a natural map $\Upsilon^A \to \Upsilon^{A'}$. However, since $\mathscr{U}$ is not dualizable (unlike the individual terms in that diagram), it is not an isomorphism. From a structural point of view, the former functor is preferable since it preserves colimits.

Proposition 3.22. *Ayoub's unipotent nearby cycle is isomorphic to the one defined in Definition 3.1:*

$$
\Upsilon^A = \Upsilon.
$$

Proof. Applying $- \otimes_{\mathrm{DTM}(\mathbf{G}_\mathrm{m})} \mathrm{DM}(X_\eta)$ to the functors

$$\mathrm{DTM}(\mathbf{G}_\mathrm{m}) \overset{\Delta_?}{\to} \mathrm{DTM}(\mathbf{G}_\mathrm{m}^2) \overset{(\mathrm{id},1)^*}{\to} \mathrm{DTM}(\mathbf{G}_\mathrm{m}), \tag{3.16}$$

gives us, according to (3.6),

$$\mathrm{DM}(X_\eta) \overset{\Gamma_{f_\eta}?}{\to} \mathrm{DTM}(\mathbf{G}_\mathrm{m}) \otimes_{\mathrm{DTM}(S)} \mathrm{DM}(X_\eta) \overset{1^* \otimes \mathrm{id}}{\to} \mathrm{DM}(X_\eta).$$

By definition, Υ is the composition of this with $i^* j_*$. Thus, it suffices to prove that the functor $(\mathrm{id}, 1)^* \Delta_?$ agrees with $- \otimes \mathscr{U}$.

Let us write $C^\otimes := (\mathrm{Sch}_{\mathbf{G}_\mathrm{m}}^\mathrm{op})^\otimes$ for the symmetric ordinary monoidal category of $\mathbf{G}_\mathrm{m}$-schemes (with their fiber product over $\mathbf{G}_\mathrm{m}$). We regard any $\mathbf{G}_\mathrm{m}$-scheme as an algebra-object (in the opposite category) via the diagonal, and X as a B-module via $\Gamma_\Delta : X \to X \times_{\mathbf{G}_\mathrm{m}} B$ etc. In the notation of [Lur17, §4.4.1, 4.4.2], the diagram (3.14) then amounts to a map $\mathrm{Tens}_{[2]}^\otimes \to C^\otimes$. By inspection of the definitions, the composite $\Delta^\mathrm{op} \overset{U}{\to} \mathrm{Tens}_{[2]}^\otimes \to C^\otimes$, with U as in [Lur17, Notation 4.4.2.4], is such that its pushforward along the maps $[m] \overset{\alpha}{\to} [n] \to [1]$ in Fin_* agrees with the m-th, and n-th terms of the geometric bar construction, including the simplicial structure mentioned above.

The functor Γ in (3.15) is lax monoidal. When restricted to the full subcategory $C' \subset \mathrm{Sch}_{\mathbf{G}_\mathrm{m}}$ consisting of $\mathbf{G}_\mathrm{m}$-schemes that are isomorphic to $\mathbf{G}_\mathrm{m} \times \mathbf{G}_\mathrm{m}^n$, the functor is monoidal and produces Tate motives, i.e., we have a monoidal functor

$$\Gamma : (C'^\mathrm{op})^\otimes \to \mathrm{DTM}(\mathbf{G}_\mathrm{m}) \overset{2.17}{=} \mathrm{Mod}_\Lambda(\mathrm{DTM}(S)).$$

Applying Γ to (3.14), gives

$$
\begin{array}{c}
\Lambda \\
\uparrow {\scriptstyle \mathrm{id} \otimes \mathrm{aug}} \\
\Lambda \overset{\mu}{\longleftarrow} \Lambda \otimes \Lambda,
\end{array}
$$

i.e., the multiplication map of the algebra object Λ, and the augmentation map. The colimit of the bar construction $\mathrm{Bar}(\Lambda, \Lambda \otimes \Lambda, \Lambda)$ (formed in $\mathrm{Mod}_\Lambda(\mathrm{DTM}(S))$, using these maps) agrees with $\Lambda \otimes_{\mu, \Lambda \otimes \Lambda, \mathrm{id} \otimes \mathrm{aug}} \Lambda$. This is precisely the image of Λ under the functor (3.16). $\qquad \square$

3.10. **Higher-dimensional unipotent nearby cycles.** Let $n \geq 0$ and consider the following "n-dimensional" variant of Nilp and $\mathrm{Nilp}_\mathbf{Q}$ obtained by redoing the construction of Definition and Lemma 2.9 and Definition 2.22 for $p_*^n p^{n*} \mathbf{Z}$, where $p^n : \mathbf{G}_\mathrm{m}^n \to S$ is the structural map. By the Künneth formula (for Tate motives), this is isomorphic to $\Lambda^{\otimes n} = (p_* p^* \mathbf{Z})^{\otimes n}$. If $\Lambda_\mathrm{r} = \mathbf{Z} \oplus \mathbf{Z}(-1)[-1]$ denotes the square zero extension in $\mathrm{D}(\mathrm{Ab}^\mathbf{Z})$, we again have a fully faithful, symmetric monoidal functor

$$\mathrm{Mod}_{\Lambda_\mathrm{r}^{\otimes n}} \mathrm{D}(\mathrm{Ab}^\mathbf{Z}) \overset{\mathbf{Z} \otimes_{\Lambda_\mathrm{r}^{\otimes n}} -}{\to} \left(\mathrm{Mod}_{(\bigoplus_{k \geq 0} \mathbf{Z}(k))^{\otimes n}}(\mathrm{D}(\mathrm{Ab}^\mathbf{Z})), \otimes_\mathbf{Z} \right)$$

and we define Nilp^n to be the essential image of this functor. As in Definition 2.22, we let $\mathrm{Nilp}_\mathbf{Q}^n := \mathrm{D}(\mathrm{Ab}^\mathbf{Z}) \times_{(\mathrm{D}(\mathrm{Ab}^\mathbf{Z}) \otimes \mathbf{Q})}$ $(\mathrm{Nilp}^n \otimes \mathbf{Q})$. The category $\mathrm{Nilp}_\mathbf{Q}^n$ is linear over $\mathrm{D}(\mathrm{Ab}^\mathbf{Z})$, and for some $\mathrm{D}(\mathrm{Ab}^\mathbf{Z})$-module C, we write $\mathrm{Nilp}_\mathbf{Q}^n C := \mathrm{Nilp}_\mathbf{Q}^n \otimes_{\mathrm{D}(\mathrm{Ab}^\mathbf{Z})} C$ as before. We have a natural symmetric monoidal functor

$$\mathrm{DTM}(\mathbf{G}_\mathrm{m}^n) \to \mathrm{Nilp}_\mathbf{Q}^n \mathrm{DTM}(S),$$

which is an equivalence after passing to rationalizations (cf. Definition 2.24).

The following functor serves as a replacement and far-reaching extension of the constructions in [AR, §9.4].

Definition 3.23. The *n-dimensional unipotent nearby cycles functor*

$$\Upsilon_n := \Upsilon_{X/\mathbf{A}^n} : \mathrm{DM}(X \times_{\mathbf{A}^n} \mathbf{G}_\mathrm{m}^n) \to \mathrm{Nilp}_\mathbf{Q}^n \mathrm{DM}(X \times_{\mathbf{A}^n, 0} S)$$

is obtained by replacing $\mathbf{G}_\mathrm{m}$ by $\mathbf{G}_\mathrm{m}^n$ in Definition 3.1.

This functor has essentially the same good properties. In more detail, Υ_n enjoys the same functoriality as in Theorem 3.2. The formula (3.7) continues to hold as stated, but the cofiber sequence in (3.8) becomes slightly more involved (since a resolution of $\mathbf{Z}$ by free Σ^n-modules is longer). For the computation of $\Upsilon_n(\mathbf{Z}) = \mathbf{Z}$ for the trivial family $\mathrm{id}_{\mathbf{A}^n}$ (Lemma 3.12) note that for any $\mathbf{G}_\mathrm{m}$-equivariant sheaf on $\mathbf{A}^n$, the map in (3.10) is still an isomorphism when applied to $j_* \mathbf{Z}$, where $j : \mathbf{G}_\mathrm{m}^I \times 0^J \to \mathbf{A}^{I \sqcup J}$, as can be seen by applying (3.10) iteratively. The preservation of Tate motives and compatibility with hyperbolic localization can also be proven the exact same way. Finally, as in Remark 3.3(4), the same definition leads to a functor $\Upsilon_n : \mathrm{DM}_\mathrm{r}(X \times_{\mathbf{A}^n} \mathbf{G}_\mathrm{m}^n) \to \mathrm{Nilp}^n \mathrm{DM}_\mathrm{r}(X \times_{\mathbf{A}^n, 0} S)$ between categories of *reduced* motives with an integrally-defined monodromy map.

In this section, we collect a few basic compatibilities of this construction as n varies. For concreteness, we highlight only the relation of Υ_1 vs. Υ_2, which is going to be used in Lemma 4.20.

We consider the diagonal $\mathbf{A}^1 \overset{\Delta}{\to} \mathbf{A}^2$. Taking the pullback to $\mathbf{G}_{\mathrm{m}}^2$, and taking cohomology (as in (2.14)) gives the map $\Lambda_{\mathbf{Q}}^2 = (\operatorname{Sym}\mathbf{Q}(-1)[-1])^{\otimes 2} \to \Lambda_{\mathbf{Q}} = \operatorname{Sym}(\mathbf{Q}(-1)[-1])$. We will use the adjunction (pullback, and forgetful functor, respectively):

$$\Delta^* : \operatorname{Corr}_{\mathbf{A}^2} \rightleftarrows \operatorname{Corr}_{\mathbf{A}^1} : \Delta_*.$$

In the following, we consider the functor (for $X/\mathbf{A}^1$)

$$\operatorname{res}_{\Sigma^2}^{\Sigma} \Upsilon_2 \Delta_* : \operatorname{DM}(X_\eta) \to \operatorname{Nilp}_{\mathbf{Q}} \operatorname{DM}(X_s),$$

which regards X as a scheme over $\mathbf{A}^2$ via the diagonal, then applies 2-dimensional nearby cycles, and then appends the restriction

$$\operatorname{res}_{\Sigma^2}^{\Sigma} : \operatorname{Mod}_{\Sigma_{\mathbf{Q}}^{\otimes 2}}(\operatorname{DTM}(S)) \to \operatorname{Mod}_{\Sigma_{\mathbf{Q}}}(\operatorname{DTM}(S)) \tag{3.17}$$

along the comultiplication in (2.4) (which maps $\operatorname{Nilp}_{\mathbf{Q}}^2$ to $\operatorname{Nilp}_{\mathbf{Q}}$).

Proposition 3.24. *For schemes over $\mathbf{A}^1$, the two-dimensional nearby cycles functors are related to the (one-dimensional) ones by means of a natural transformation*

$$\operatorname{res}_{\Sigma^2}^{\Sigma} \Upsilon_2 \Delta_* \to \Upsilon$$

of functors $\operatorname{Corr}_{\mathbf{A}^1}^{\mathrm{pr,sm}} \to \operatorname{Fun}(\Delta^1, \operatorname{Pr}_{\mathrm{St}}^{\mathrm{L}})$. Thus for any $X/\mathbf{A}^1$, and writing $X' := \Delta_*(X)$ for the same scheme, but now regarded as lying over (the diagonal in) $\mathbf{A}^2$, there is a natural transformation (which is compatible with smooth pullbacks and proper pushforwards):

$$
\begin{array}{ccc}
\operatorname{DM}(X_\eta) & \xrightarrow{\ \Upsilon\ } & \operatorname{Nilp}_{\mathbf{Q}}(\operatorname{DM}(X_s)) \\[2pt]
\Big\| & \nearrow & \Big\uparrow {\scriptstyle \text{res}} \\[2pt]
\operatorname{DM}(X'_\eta) & \xrightarrow{\ \Upsilon_2\ } & \operatorname{Nilp}_{\mathbf{Q}}^2(\operatorname{DM}(X'_s))
\end{array}
$$

Proof. Note that $X_\eta = X'_\eta := X \times_{\mathbf{A}^2} \mathbf{G}_{\mathrm{m}}^2$, $X_s = X'_s := X \times_{\mathbf{A}^2,0} S$. We write $f_\eta^2 : X_\eta \to \mathbf{G}_{\mathrm{m}}^2$ etc. Let again $\Delta_?$ be the right adjoint of $\Delta^* : \operatorname{DTM}(\mathbf{G}_{\mathrm{m}}^2) \to \operatorname{DTM}(\mathbf{G}_{\mathrm{m}})$. By passing to left adjoints, one sees that there is a diagram, where all tensor products are over $\operatorname{DTM}(S)$:

$$
\begin{array}{ccccccc}
\operatorname{DM}(X_\eta) & \xrightarrow{(\Gamma_{f_\eta})_?} & \operatorname{DTM}(\mathbf{G}_{\mathrm{m}}) \otimes \operatorname{DM}(X_\eta) & \xrightarrow{\operatorname{id}\otimes i^* j_*} & \operatorname{DTM}(\mathbf{G}_{\mathrm{m}}) \otimes \operatorname{DM}(X_s) & \xrightarrow[\cong]{1^*\otimes\operatorname{id}} & \operatorname{Nilp}_{\mathbf{Q}} \operatorname{DM}(X_s) \\[2pt]
\Big\| & & \Big\downarrow {\scriptstyle \Delta_?\otimes\operatorname{id}} & & \Big\downarrow {\scriptstyle \Delta_?\otimes\operatorname{id}} & \nearrow & \Big\uparrow {\scriptstyle \text{res}} \\[2pt]
\operatorname{DM}(X'_\eta) & \xrightarrow{(\Gamma_{f_\eta^2})_?} & \operatorname{DTM}(\mathbf{G}_{\mathrm{m}}^2) \otimes \operatorname{DM}(X'_\eta) & \xrightarrow{\operatorname{id}\otimes i^* j_*} & \operatorname{DTM}(\mathbf{G}_{\mathrm{m}}^2) \otimes \operatorname{DM}(X'_s) & \xrightarrow[\cong]{(1,1)^*\otimes\operatorname{id}} & \operatorname{Nilp}_{\mathbf{Q}}^2 \operatorname{DM}(X'_s).
\end{array}
$$

The left square commutes, since their left adjoints commute: $\Gamma_{f_\eta}^* \circ (\Delta^* \otimes \operatorname{id}) = \Gamma_{f_\eta^2}^*$. The middle square commutes trivially. The indicated natural transformation at the right arises again by passing to adjoints, using the identity $1^*\Delta^* = (1,1)^*$. $\qquad\square$

Remark 3.25. The natural transformation at the right is not invertible. Suppressing the $\operatorname{DM}(X_s)$-factor from the notation and forgetting the $\operatorname{Nilp}_{\mathbf{Q}}$-action, the square identifies with

$$
\begin{array}{ccc}
\operatorname{Mod}_\Lambda(\operatorname{DTM}(S)) & \xrightarrow{-\otimes_\Lambda \mathbf{Z}} & \operatorname{DTM}(S) \\[2pt]
\Big\downarrow {\scriptstyle \text{res}} & & \Big\| \\[2pt]
\operatorname{Mod}_{\Lambda^2}(\operatorname{DTM}(S)) & \xrightarrow{-\otimes_{\Lambda^2} \mathbf{Z}} & \operatorname{DTM}(S).
\end{array}
$$

Here res is the restriction functor along the map $\Lambda^2 = \Gamma(\mathbf{G}_{\mathrm{m}}^2) \to \Gamma(\mathbf{G}_{\mathrm{m}}) = \Lambda$ obtained by functoriality of Γ, cf. (3.15).

The following statement is a replacement and enhancement of [AR, Lemma 9.4.11].

Corollary 3.26. *For $X/\mathbf{A}^2$, let us write $\Delta : X' := X \times_{\mathbf{A}^2} \mathbf{A}^1 \to X$ for the pullback along the diagonal etc. For $M \in \operatorname{DM}(X)$, there is a natural map in $\operatorname{Nilp}_{\mathbf{Q}} \operatorname{DM}(X_s)$*

$$\operatorname{res}_{\Sigma^2}^{\Sigma} \Upsilon_2(M) \to \Upsilon(\Delta^* M).$$

It is functorial in X (with respect to smooth-proper correspondences over $\mathbf{A}^2$).

Proof. As before, we write subscripts η and s for the pullbacks along $\mathbf{G}_m^2 \subset \mathbf{A}^2$ and $(0,0) \subset \mathbf{A}^2$. The unit map $M \to \Delta_{\eta*}\Delta_\eta^* M$ yields a map

$$\Upsilon_{X/\mathbf{A}^2}(M) \to \Upsilon_{X/\mathbf{A}^2}(\Delta_{\eta*}\Delta_\eta^* M) = \Delta_{s*}\Upsilon_{X'/\mathbf{A}^2}(\Delta_\eta^* M).$$

The second isomorphism is the compatibility with proper pushforward. The map $\Delta_s : X' \times_{\mathbf{A}^2,(0,0)} S \to X \times_{\mathbf{A}^2,(0,0)} S$ is an isomorphism. We then append the map supplied by Proposition 3.24, $\operatorname{res}_{\Sigma^2}^{\Sigma} \Upsilon_{X'/\mathbf{A}^2}(\Delta_\eta^* M) \to \Upsilon_{X'/\mathbf{A}^1}(\Delta_\eta^* M)$. $\square$

3.10.1. *Compatibility with exterior products.* In order to state the following result, we use the symmetric monoidal functor

$$\operatorname{Nilp}_{\mathbf{Q}} \otimes_{\mathrm{D}(\mathrm{Ab}^{\mathbf{Z}})} \operatorname{Nilp}_{\mathbf{Q}} \to \operatorname{Nilp}_{\mathbf{Q}}^2 := \mathrm{D}(\mathrm{Ab}^{\mathbf{Z}}) \times_{\mathrm{D}(\mathrm{Ab}^{\mathbf{Z}})\otimes\mathbf{Q}} (\operatorname{Nilp}^2 \otimes \mathbf{Q})$$

whose first component is induced by the equivalence $\mathrm{D}(\mathrm{Ab}^{\mathbf{Z}}) \otimes_{\mathrm{D}(\mathrm{Ab}^{\mathbf{Z}})} \mathrm{D}(\mathrm{Ab}^{\mathbf{Z}}) \overset{\cong}{\to} \mathrm{D}(\mathrm{Ab}^{\mathbf{Z}})$ and whose second component is induced by the (rationalization of the) equivalence $\operatorname{Nilp} \otimes_{\mathrm{D}(\mathrm{Ab}^{\mathbf{Z}})} \operatorname{Nilp} = \operatorname{Nilp}^2$ which follows from Definition and Lemma 2.9 and Lemma 2.6. The description of compact generators of $\operatorname{Nilp}_{\mathbf{Q}}$ (which carries over to $\operatorname{Nilp}_{\mathbf{Q}}^2$) in Remark 2.23 implies this functor is an equivalence.

Proposition 3.27. *Let $f_k : X_k \to \mathbf{A}^1$, $k = 1,2$ be two schemes, and $f : X := X_1 \times X_2 \to \mathbf{A}^2$ their product. For $M_k \in \mathrm{DM}(X_{k\eta})$, there is a natural map (in $\operatorname{Nilp}_{\mathbf{Q}}^2 \mathrm{DM}(X_{1s} \times X_{2s})$)*

$$\Upsilon_{X_1/\mathbf{A}^1}(M_1) \boxtimes \Upsilon_{X_2/\mathbf{A}^1}(M_2) \to \Upsilon_{X/\mathbf{A}^2}(M_1 \boxtimes M_2) \tag{3.18}$$

which is an isomorphism if the natural map

$$(j_{1*}M_1) \boxtimes (j_{2*}M_2) \to (j_1 \times j_2)_*(M_1 \boxtimes M_2) \tag{3.19}$$

is an isomorphism (concerning this condition, cf. Remark 3.15).

Proof. We consider the following diagram, where all tensor products are over $\mathrm{DTM}(S)$:

$$
\begin{array}{ccc}
\bigotimes_{k=1}^2 \mathrm{DM}(X_{k\eta}) & \xrightarrow{\Gamma_{f_k?}} \bigotimes_{k=1}^2 (\mathrm{DTM}(\mathbf{G}_m) \otimes \mathrm{DM}(X_{k\eta})) \xrightarrow{1^* \otimes i_k^* j_{k*}} & \bigotimes_{k=1}^2 \operatorname{Nilp}_{\mathbf{Q}} \mathrm{DM}(X_{ks}) \\
\downarrow \boxtimes & \downarrow \boxtimes & \downarrow \boxtimes \\
\mathrm{DM}(X_{1\eta} \times X_{2\eta}) & \xrightarrow{\Gamma_{f?}} \mathrm{DTM}(\mathbf{G}_m^2) \otimes \mathrm{DM}(X_{1\eta} \times X_{2\eta}) \xrightarrow{(1,1)^* \otimes i^* j_*} & \operatorname{Nilp}_{\mathbf{Q}}^2 \mathrm{DM}(X_{1s} \times X_{2s})
\end{array}
$$

The top horizontal composite is $\Upsilon \otimes \Upsilon$. The left horizontal functors are the right adjoints of the action of $\mathrm{DTM}(\mathbf{G}_m)$ on $\mathrm{DM}(X_{k\eta})$ (resp. likewise for $X_\eta \to \mathbf{G}_m^2$). (Therefore, the middle horizontal composite relates closely to the functor considered in [AR, Definition 9.4.1].) These actions are clearly compatible with the indicated $\boxtimes$-functors, which gives rise to the natural transformation at the left.

The left hand vertical $\boxtimes$ is a map of $\mathrm{DTM}(\mathbf{G}_m^2)$-modules, and thus also of comodules (Lemma 2.2). The left hand horizontal functors are just the coaction maps. Thus, the left hand part commutes. The natural transformation in the right hand square arises since $*$-pullbacks are compatible with $\boxtimes$, while $*$-pushforwards are only lax compatible as in (3.19). If that latter map is an isomorphism, then so is (3.18). Indeed, $\Gamma_{f_k?}(M_k)$ is a colimit of objects of the form $\mathbf{Z}_{\mathbf{G}_m}(-i)[-i] \boxtimes M_k$, and j_{k*} and $\boxtimes$ preserve colimits. $\square$

Remark 3.28. Along the lines of the proof above (or the ones in Corollary 3.14), one sees that for Y/S, $X/\mathbf{A}^1$, $N \in \mathrm{DM}(Y)$, $M \in \mathrm{DM}(X_\eta)$ there is a natural map (as usual, compatible with monodromy, smooth pullback and proper pushforward):

$$N \boxtimes \Upsilon_{X/\mathbf{A}^1}(M) \to \Upsilon_{Y \times X/\mathbf{A}^1}(N \boxtimes M). \tag{3.20}$$

3.10.2. *Decomposing higher-dimensional nearby cycles.*

Lemma 3.29. *Let $X/\mathbf{A}^2$, write $X_\eta := X \times_{\mathbf{A}^2} \mathbf{G}_m^2$, and let $X' := X \times_{\mathbf{A}^2} \mathbf{A}^1 \times 0/\mathbf{A}^1$ via the first coordinate in $\mathbf{A}^2$, so that $X_\eta' = X \times_{\mathbf{A}^2} (\mathbf{G}_m \times 0)$, and $X_s' = X \times_{\mathbf{A}^2} (0,0) =: X_s$. Denote by*

$$\Upsilon_2 : \mathrm{DM}(X_\eta) \to \operatorname{Nilp}_{\mathbf{Q}} \mathrm{DM}(X_\eta')$$
$$\Upsilon_1 : \mathrm{DM}(X_\eta') \to \operatorname{Nilp}_{\mathbf{Q}} \mathrm{DM}(X_s')$$

the (one-dimensional) nearby cycles along the second and first coordinate, respectively. Finally, write

$$\Upsilon : \mathrm{DM}(X_\eta) \to \operatorname{Nilp}_{\mathbf{Q}}^2 \mathrm{DM}(X_s)$$

for the two-dimensional nearby cycle functor. There is a natural transformation (of functors $\mathrm{Corr}_{\mathbf{A}^2}^{\mathrm{pr,sm}} \to \mathrm{Fun}(\Delta^1, \mathrm{Pr}_{\mathrm{gr}}^{\mathrm{L}})$)

$$\Upsilon \to \Upsilon_1 \circ \Upsilon_2.$$

23

Proof. For $k = 1, 2$, let Γ_k be the graph of $X_\eta \to \mathbf{G}_m^2 \xrightarrow{p_k} \mathbf{G}_m$, and Γ the graph of $X_\eta \to \mathbf{G}_m^2$. Then

$$\mathrm{DM}(X_\eta) \xrightarrow{\Gamma_{1?}} \mathrm{DTM}(\mathbf{G}_m) \otimes \mathrm{DM}(X_\eta) \xrightarrow{\mathrm{id} \otimes \Gamma_{2?}} \mathrm{DTM}(\mathbf{G}_m)^2 \otimes \mathrm{DM}(X_\eta)$$

agrees with $\Gamma_?$. Indeed their left adjoints agree: $\Gamma^* = \Gamma_1^* (\Gamma_2 \times \mathrm{id})^*$. From here, the proof of [AR, Lemma 9.4.3(2)] carries over. $\qquad\square$

3.11. **Kan extension to ind-schemes and placid prestacks.** Taking advantage of the ∞-categorical setup of Υ, we can now quickly extend its reach to more general objects of relevance in geometric representation theory. As a preliminary observation note from Remark 2.23 that $\mathrm{Nilp}_\mathbf{Q}$ is compactly generated, and therefore a dualizable object in $\mathrm{Pr}_{\mathrm{gr}}^\mathrm{L}$ [GR17, Propositions I.7.3.2 and 9.4.4]. Thus, tensoring with it preserves not only colimits but also limits: $\mathrm{Nilp}_\mathbf{Q}(\lim C_i) = \lim \mathrm{Nilp}_\mathbf{Q}(C_i)$.

We can apply a Kan extension to obtain a unipotent nearby cycles functor for motives on placid prestacks in two steps:

$$
\begin{array}{ccc}
(\mathrm{AffSch}_{\mathbf{A}^1}^{\mathrm{sm}})^{\mathrm{op}} & \longrightarrow \mathrm{Corr}_{\mathbf{A}^1}^{\mathrm{sm,pr}} \xrightarrow{\ \Upsilon\ } \mathrm{Fun}(\Delta^1, \mathrm{Pr}_{\mathrm{gr}}^\mathrm{L})\ . \\
\downarrow & \qquad\qquad \nearrow \\
(\mathrm{AffSch}_{\mathbf{A}^1}^{\kappa,\mathrm{pl}})^{\mathrm{op}} & \quad \mathrm{Lan}\,\Upsilon \\
\downarrow & \quad \mathrm{Ran}(\mathrm{Lan}\,\Upsilon) \\
(\mathrm{PreStk}_{\mathbf{A}^1}^{\mathrm{pl}})^{\mathrm{op}} &
\end{array}
$$

We first form the left Kan extension (denoted by Lan) to the category of κ-small (for some fixed regular cardinal κ) placid affine $\mathbf{A}_S^1$-schemes (i.e., $X = \lim X_i$, with smooth affine transition maps $X_i \to X_j$), so that $\mathrm{DM}(X) = \mathrm{colim}\,\mathrm{DM}(X_i)$, with transition functors given by *-pullback. We then form the right Kan extension to the category of placid prestacks, which is the free cocompletion of $\mathrm{AffSch}_{\mathbf{A}^1}^{\kappa,\mathrm{pl}}$. If X is a finite type $\mathbf{A}_S^1$-scheme and $G/\mathbf{A}^1$ is a pro-smooth algebraic group, then the quotient prestack $(X/G)/\mathbf{A}^1$ is placid and we have the following description of equivariant motives (cf. around [RS20, Lemma 2.2.7]):

$$\mathrm{DM}(X/G) = \lim \left(\mathrm{DM}(X) \underset{a^!}{\overset{p^!}{\rightrightarrows}} \mathrm{DM}(G \times X) \substack{\rightarrow\\\rightarrow\\\rightarrow} \dots \right)$$

$$= \lim \left(\mathrm{DM}(X) \underset{a^*}{\overset{p^*}{\rightrightarrows}} \mathrm{DM}(G \times X) \substack{\rightarrow\\\rightarrow\\\rightarrow} \dots \right). \tag{3.21}$$

Evaluating the functor $\Upsilon := \mathrm{Ran}(\mathrm{Lan}\,\Upsilon)$ at such quotient prestacks gives the following.

Corollary 3.30. *Let X be a finite type $\mathbf{A}_S^1$-scheme, acted upon by a pro-smooth algebraic group $G/\mathbf{A}^1$. Then there is a natural equivariant unipotent nearby cycles functor, also denoted Υ, such that the following diagram commutes:*

$$
\begin{array}{ccc}
\mathrm{DM}(X_\eta/G_\eta) & \xrightarrow{\ \Upsilon\ } & \mathrm{Nilp}_\mathbf{Q}\,\mathrm{DM}(X_s/G_s) \\
\downarrow{\scriptstyle u^!} & & \downarrow{\scriptstyle u^!} \\
\mathrm{DM}(X_\eta) & \xrightarrow{\ \Upsilon\ } & \mathrm{Nilp}_\mathbf{Q}\,\mathrm{DM}(X_s).
\end{array}
$$

In a similar manner, it is possible to do a Kan extension along proper maps (see also [RS20, §2.4]): For example, if $X = \mathrm{colim}_{i \in I} X_i$ is an ind-scheme over $\mathbf{A}^1$, using that Υ commutes with pushforwards along the closed immersions $X_i \to X_j$, we get a natural functor

$$\mathrm{DM}(X_\eta) = \mathrm{colim}\,\mathrm{DM}(X_{i\eta}) \xrightarrow{\mathrm{colim}\,\Upsilon_{X_i}} \mathrm{colim}\,\mathrm{Nilp}_\mathbf{Q}\,\mathrm{DM}(X_{is}) = \mathrm{Nilp}_\mathbf{Q}\,\mathrm{DM}(X_s).$$

This functor is compatible with the one given by Theorem 3.2 under the canonical insertion maps $\mathrm{DM}(X_{i\eta}) \to \mathrm{DM}(X_\eta)$. More formally, the above functor is the evaluation (at X) of the left Kan extension of $\mathrm{Sch}^{\mathrm{pr}} \to \mathrm{Corr}_{\mathbf{A}^1}^{\mathrm{sm,pr}} \xrightarrow{\Upsilon} \mathrm{Fun}(\Delta^1, \mathrm{Pr}_{\mathrm{gr}}^\mathrm{L})$ along the inclusion $\mathrm{Sch}_{\mathbf{A}^1}^{\mathrm{pr}} \to \mathrm{IndSch}^{\mathrm{pr}}$ into the category of ind-schemes.

4. Geometry of Beilinson–Drinfeld Grassmannians

Recall that we fixed a connected base scheme S which is smooth of finite type over a Dedekind ring or a field. For an S-scheme of finite type X, when working with t-structures on $\mathrm{DTM}(X)$ we must further assume that S satisfies the Beilinson–Soulé vanishing condition (cf. Section 4.5.4). For reduced motives this last assumption is not necessary.

4.1. **Affine flag varieties.**

4.1.1. *Group-theoretic notation.* Let G be a split Chevalley group scheme over $\mathbf{Z}$. Fix a maximal torus and Borel subgroup $T \subset B \subset G$. When working over a general base S we will usually write G instead of $G \times_{\mathbf{Z}} S$. Let $X_*(T)$ and $X^*(T)$ be the groups of characters and cocharacters. Let $X_*(T)^+ \subset X_*(T)$ be the submonoid of dominant cocharacters and let R_+ be the positive roots determined by B. This determines a standard apartment $\mathscr{A} = X_*(T) \otimes \mathbf{R}$, and a special facet $\mathbf{f}_0$ containing the origin. We moreover fix a standard alcove $\mathbf{a}_0$, containing $\mathbf{f}_0$ in its closure (see [RS21, §4.1] for more details).

Remark 4.1. Although the Borel B containing T yields a preferred choice of alcove $\mathbf{a}_0$, we do not make this specific choice. Indeed, in the proof of Theorem 5.30, it will be useful to consider different Borels, while keeping the alcove $\mathbf{a}_0$ fixed.

The finite Weyl group and Iwahori–Weyl group are, respectively, $W_0 = N_G(T)/T$ and $W = X_*(T) \rtimes W_0$. To avoid potential confusion, we denote the image of a cocharacter μ under the inclusion $X_*(T) \subset W$ by $t(\mu)$. There is an isomorphism $W \cong N_G(T)(\mathbf{Z}((t)))/T(\mathbf{Z}[\![t]\!])$ which extends the isomorphism $X_*(T) \cong T(\mathbf{Z}((t)))/T(\mathbf{Z}[\![t]\!])$, $\lambda \mapsto \lambda(t)$. Let W_{aff} be the subgroup of W generated by the reflections in the walls of $\mathbf{a}_0$. This is a Coxeter group. If $\Omega \subset W$ is the stabilizer of $\mathbf{a}_0$ we have $W = W_{\mathrm{aff}} \rtimes \Omega$. Declaring elements of Ω to have length 0 makes W a quasi-Coxeter group. We denote the extension of the partial Bruhat order on W_{aff} to W by $\leq$ and the length function by l.

4.1.2. *Loop groups.* Let $S[\![t]\!]$ be the affine S-scheme which represents the functor $\mathrm{AffSch}_S^{\mathrm{op}} \to \mathrm{Set}$, $R \mapsto R[\![t]\!]$ (if S is affine then $S[\![t]\!] = \mathrm{Spec}\, \mathcal{O}_S[\![t]\!]$). For a group scheme H over $S[\![t]\!]$ we have the loop group (resp. positive loop group) functor $LH \colon \mathrm{AffSch}_S^{\mathrm{op}} \to \mathrm{Set}$, $LH(R) = H(R((t)))$ (resp. $L^+H(R) = H(R[\![t]\!])$). Applying this to G viewed as an $S[\![t]\!]$-scheme by base change, the functor LG is an ind-affine S-scheme and L^+G is an S-scheme (typically not of finite type). For each facet $\mathbf{f} \subset \mathscr{A}$ there is a parahoric Bruhat–Tits group scheme $G_{\mathbf{f}}$ constructed in [PZ13, §4.2.2] using $\mathbf{Z}[\![t]\!]$ as the base ring. For general S this is obtained by base change from $\mathbf{Z}[\![t]\!]$. The scheme $G_{\mathbf{f}}$ is smooth, affine, and has geometrically connected fibers. Then $L^+G_{\mathbf{f}} \subset LG$ is a pro-algebraic $\mathbf{Z}$-group scheme with geometrically connected fibers [RS20, Lemma 4.2.4]. We have $L^+G_{\mathbf{f}_0} = L^+G$, and $L^+G_{\mathbf{a}_0}$ is the preimage of B under the projection $L^+G \to G$, $t \mapsto 0$. We denote $\mathcal{I} = L^+G_{\mathbf{a}_0}$ and call it the Iwahori group.

4.1.3. *The partial affine flag variety.*

Definition 4.2. The *partial affine flag variety* $\mathrm{Fl}_{\mathbf{f}}$ associated to a facet $\mathbf{f}$ is the étale-sheafification $(LG/L^+G_{\mathbf{f}})_{\mathrm{ét}}$ of the presheaf quotient of LG by $L^+G_{\mathbf{f}}$.

It is well-known that $\mathrm{Fl}_{\mathbf{f}}$ is represented by an ind-projective S-scheme (see [RS20, §4.2]). Since G is split, it follows from work of Faltings [Fal03, Def. 5 ff.] that the quotient map $LG \to \mathrm{Fl}_{\mathbf{f}}$ admits sections Zariski-locally. In particular, $\mathrm{Fl}_{\mathbf{f}} = (LG/L^+G_{\mathbf{f}})_{\mathrm{Zar}}$. This fact is necessary to define convolution (see §4.2.7) on $\mathrm{DM}(L^+G_{\mathbf{f}} \backslash \mathrm{Fl}_{\mathbf{f}})$ since DM satisfies Nisnevich but not étale descent. It will be convenient to work with étale-sheafifications so that we have access to the general theory of [HR20, §3].

We denote $\mathrm{Fl} = \mathrm{Fl}_{\mathbf{a}_0}$ and $\mathrm{Gr} = \mathrm{Fl}_{\mathbf{f}_0}$, and we call these the (full) affine flag variety and the affine Grassmannian. Sometimes we use a subscript to emphasize the group, e.g. Gr_G. Since S is connected, the connected components of $\mathrm{Fl}_{\mathbf{f}}$ are indexed by $\pi_1(G) = X_*(T)/X_*(T_{\mathrm{sc}})$ where T_{sc} is the preimage of $T \cap G_{\mathrm{der}}$ in G_{sc}.

4.1.4. *Affine Schubert schemes.* Let $W_{\mathbf{f}} := (N_G(T)(\mathbf{Z}((t))) \cap L^+G_{\mathbf{f}}(\mathbf{Z}))/T(\mathbf{Z}[\![t]\!]) \subseteq W$ for a fixed facet $\mathbf{f}$. For another facet $\mathbf{f}' \subset \mathscr{A}$, there is a natural length function and partial order on the double cosets $W_{\mathbf{f}'} \backslash W/W_{\mathbf{f}}$ also denoted by l and $\leq$, cf. [Ric13, Lemma 1.6 ff.]. The Bruhat decomposition implies that left $L^+\mathcal{G}_{\mathbf{f}'}$-orbits in $\mathrm{Fl}_{\mathbf{f}}$ are indexed by $W_{\mathbf{f}'} \backslash W/W_{\mathbf{f}}$. For example, $W_{\mathbf{f}_0} = W_0$ and $W_{\mathbf{a}_0}$ is the trivial group. Thus, $\mathcal{I}$-orbits in Fl are indexed by W and L^+G-orbits in Gr are indexed by $X_*(T)/W_0 \cong X_*(T)^+$. These observations motivate the following definition over any base scheme S (see also [RS20, Definition 4.4.1]).

Definition 4.3. Let $w \in W_{\mathbf{f}'} \backslash W/W_{\mathbf{f}}$. The *affine Schubert scheme* $\mathrm{Fl}_w(\mathbf{f}', \mathbf{f})$ is the scheme-theoretic image of the map
$$L^+G_{\mathbf{f}'} \to \mathrm{Fl}_{\mathbf{f}}, \quad g \mapsto g \cdot \dot{w} \cdot e$$
where $\dot{w} \in LG(S)$ is any representative of w under the map $LG(\mathbf{Z}) \to LG(S)$ and $e \in \mathrm{Fl}_{\mathbf{f}}(S)$ is the basepoint. The $L^+G_{\mathbf{f}'}$-orbit of w in Fl, denoted $\mathrm{Fl}_w^\circ(\mathbf{f}', \mathbf{f})$, is the étale sheaf-theoretic image of the above map.

The following proposition summarizes the properties of affine Schubert schemes.

Proposition 4.4. *The affine Schubert schemes satisfy the following properties.*

(1) *The affine Schubert scheme $\mathrm{Fl}_w(\mathbf{f}', \mathbf{f})$ is a reduced, projective S-scheme which is moreover $L^+G_{\mathbf{f}'}$-stable.*

(2) *The $L^+G_{\mathbf{f}'}$-orbit $\mathrm{Fl}_w^\circ(\mathbf{f}', \mathbf{f}) \subset \mathrm{Fl}_w(\mathbf{f}', \mathbf{f})$ is an open S-subscheme which is smooth, and fiberwise geometrically connected and dense.*

(3) *We have $\mathrm{Fl}_w^\circ(\mathbf{a}_0, \mathbf{f}) \cong \mathbf{A}_S^{l(w)}$ where l is the length function on $W/W_{\mathbf{f}}$.*

(4) If $v \leq w$ with respect to the partial order on $W_{\mathbf{f}'} \backslash W / W_{\mathbf{f}}$ then there is a closed immersion $\mathrm{Fl}_v(\mathbf{f}', \mathbf{f}) \to$ $\mathrm{Fl}_w(\mathbf{f}', \mathbf{f})$. As sets there is a decomposition

$$\mathrm{Fl}_w(\mathbf{f}', \mathbf{f}) = \bigsqcup_{v \leq w} \mathrm{Fl}_v^\circ(\mathbf{f}', \mathbf{f}).$$

Moreover, the reduced locus of $\mathrm{Fl}_{\mathbf{f}}$ is $(\mathrm{Fl}_{\mathbf{f}})_{\mathrm{red}} = \mathrm{colim}_w \mathrm{Fl}_w(\mathbf{f}', \mathbf{f})$.

(5) If $S = k$ is a field, $\mathrm{Fl}_w(\mathbf{f}', \mathbf{f})$ agrees with the usual affine Schubert varieties defined in the literature, e.g. [PR08, Def. 8.3], [Ric13, Def. 2.5].

(6) The formation of $\mathrm{Fl}_w(\mathbf{f}', \mathbf{f})$ commutes with base change along a map $S' \to S$ up to a possible nilpotent thickening.

Proof. All of these facts can be found in [RS20, §4.3, §4.4]. The proofs combine standard techniques from the case where S is a field as in [Ric13, Ric16] with the Bruhat–Tits group schemes $G_{\mathbf{f}}$ from [PZ13]. □

We introduce the following standard notation for the most important affine Schubert schemes in this paper.

$$\mathrm{Gr}^{\leq \mu} = \mathrm{Fl}_\mu(\mathbf{f}_0, \mathbf{f}_0), \quad \mathrm{Gr}^\mu = \mathrm{Fl}_\mu^\circ(\mathbf{f}_0, \mathbf{f}_0), \quad \mu \in X_*(T)^+,$$
$$\mathrm{Fl}^{\leq w} = \mathrm{Fl}_w(\mathbf{a}_0, \mathbf{a}_0), \quad \mathrm{Fl}^w = \mathrm{Fl}_w^\circ(\mathbf{a}_0, \mathbf{a}_0), \quad w \in W.$$

4.1.5. *The admissible locus.* Inside $\mathrm{Fl}_{\mathbf{f}}$ there is another important family of subschemes.

Definition 4.5. Let $\mu \in X_*(T)^+$ and let $W_0(\mu) \subset X_*(T)$ be the W_0-orbit of μ. The *μ-admissible set* is

$$\mathrm{Adm}_{\{\mu\}} = \{w \in W \ : \ w \leq \lambda \text{ for some } \lambda \in W_0(\mu)\}.$$

For a facet $\mathbf{f}$ the *μ-admissible set relative to $\mathbf{f}$* is $\mathrm{Adm}_{\{\mu\}}^{\mathbf{f}} = W_{\mathbf{f}} \backslash \mathrm{Adm}_{\{\mu\}} / W_{\mathbf{f}}$. The *$\mu$-admissible locus $\mathcal{A}^{\mathbf{f}}(\mu)$ relative to $\mathbf{f}$* is the scheme-theoretic image of the immersion $\bigsqcup_{\lambda \in W_0(\mu)} \mathrm{Fl}_\lambda(\mathbf{f}, \mathbf{f}) \to \mathrm{Fl}$. As a set, $\mathcal{A}^{\mathbf{f}}(\mu)$ is the union of the $\mathrm{Fl}_w(\mathbf{f}, \mathbf{f})$ for those $w \in \mathrm{Adm}_{\{\mu\}}^{\mathbf{f}}$.

The admissible loci arise as reduced special fibers of global Schubert schemes in the Beilinson–Drinfeld Grassmannian, which we define next.

4.2. Beilinson–Drinfeld Grassmannians.

4.2.1. *Bruhat–Tits group schemes.* Throughout this paper we let $C = \mathbf{A}_S^1$. For a prestack $Y \to C$, let $Y_s \to S$ be the fiber over the origin and let $Y_\eta \to \mathbf{G}_{\mathrm{m},S}$ be the fiber over the open complement. We define the constant group scheme $G_C = G \times_{\mathbf{Z}} C$ over C. Regarding $T \subset B$ as closed subschemes of G_C supported over the origin, we may form the dilatation $\mathcal{G} = \mathrm{Bl}_B^G G_C$ of G_C in B along $G = (G_C)_s$ in the sense of [MRR23, Definition 2.1]. This is a smooth affine C-group scheme with geometrically connected fibers [MRR23, Theorem 3.2].

Let $\mathcal{O}_s$ be the formal neighborhood of the origin in C (so $\mathcal{O}_s \cong S[\![t]\!]$). Then

$$\mathcal{G}|_{\mathcal{O}_s} = G_{\mathbf{a}_0}, \quad \mathcal{G}_\eta = G \times_{\mathbf{Z}} C_\eta, \quad L^+(\mathcal{G}|_{\mathcal{O}_s}) = \mathcal{I}.$$

Thus, if k is any field, $\mathcal{G}(k[\![t]\!]) \subset G(k[\![t]\!])$ consists of those elements whose reduction modulo t lies in $B(k)$ [MRR23, Example 3.3]. The fiber of $\mathcal{G}$ along $\mathbf{Z}[t] \to k[t]$ is precisely the group scheme used in the construction of central sheaves in [AR, Zhu14] (in the split case), cf. [AR, Remark 2.2.9]. The fiber along $\mathbf{Z}[t] \to \mathbf{Z}_p$, $t \mapsto p$ for p a prime is a parahoric group scheme in mixed characteristic, but we will not need this fact.

More generally, for each facet $\mathbf{f}$ there is a smooth affine group scheme $\mathcal{G}_{\mathbf{f}}$ over C such that $(\mathcal{G}_{\mathbf{f}})_\eta = G \times_{\mathbf{Z}} C_\eta$ and $\mathcal{G}_{\mathbf{f}}|_{\mathcal{O}_s} = G_{\mathbf{f}}$. The group $\mathcal{G}_{\mathbf{f}}$ is characterized uniquely as a particular family of parahorics [Lou23, Theorem 1.3]. The primary objective in *loc. cit.* is to overcome obstacles related to wild ramification, and $\mathcal{G}_{\mathbf{f}}$ can also be constructed by the methods in [BT84] and [PZ13] (these works restrict to local fields but in fact they work over $\mathbf{Z}$ when the group is split). According to our conventions, $\mathcal{G}_{\mathbf{f}_0} = G_C$ and $\mathcal{G}_{\mathbf{a}_0} = \mathcal{G}$.

4.2.2. *Basic properties of Beilinson–Drinfeld Grassmannians.* Let I be a finite nonempty set and let $X = C^I \times_S C \cong C^{I \sqcup \{*\}}$. We have the divisor $D \subset X$ corresponding to the ideal generated by $\prod_{i \in I}(t - x_i)$, where t is the coordinate function on C (the right factor) and the x_i are the coordinates on C^I. For $R \in \mathrm{AffSch}_{C^I}^{\mathrm{op}}$ let C_R, X_R, and D_R be the base changes to R. We denote by $R[\![D]\!] \in \mathrm{AffSch}_S^{\mathrm{op}}$ the ring of functions on the formal neighborhood $\hat{D}_R$ of D_R in X_R. Let $\hat{D}_R^\circ$ be the complement of D_R in $\mathrm{Spec}\, R[\![D]\!]$. This is an affine S-scheme; let $R(\!(D)\!)$ be its ring of functions (see [HR20, §3.1.1] for more details). In the following definition, both $\hat{D}_R$ and $\hat{D}_R^\circ$ are viewed as X-schemes.

Definition 4.6. Let I be a finite nonempty set. Let H be a smooth affine C-group scheme, which we view as an X-group scheme via pullback along the projection $X = C^I \times_S C \to C$. The *loop group relative to I* (resp. *positive loop group relative to I*) is the functor $\mathrm{AffSch}_{C^I}^{\mathrm{op}} \to \mathrm{Set}$,

$$L_I H(R) = H(R(\!(D)\!)), \quad (\text{resp. } L_I^+ = H(R[\![D]\!])).$$

The *Beilinson–Drinfeld Grassmannian of H relative to I* is the étale quotient

$$\mathrm{Gr}_{H,I} = (L_I H / L_I^+ H)_{\text{ét}}.$$

By [HR20, Lemma 3.2], $L_I H$ is an ind-affine C^I-group-scheme and $L_I^+ H$ is an affine, faithfully flat, pro-smooth C^I-group-scheme. When working with equivariant motives it is convenient to work with groups of finite type. Toward this end, for $m \in \mathbb{Z}_{\geq 0}$ we let $R\llbracket D^{(m)} \rrbracket$ be the ring of functions on the m-th infinitesimal neighborhood of D_R in X_R. Let

$$L_I^{+,(m)} H(R) = H(R\llbracket D^{(m)} \rrbracket).$$

Then $L_I^+ H = \lim_m L_I^{+,(m)} H$ and each $L_I^{+,(m)} H$ is a smooth affine C-group scheme. The kernel of $L_I^+ H \to L_I^{+,(0)} H$ is split pro-unipotent by [RS20, Proposition A.9].

If $\mathcal{G}_{\mathbf{f}}$ is the Bruhat–Tits group scheme over C associated to the facet $\mathbf{f}$ as in Section 4.2.1 we have the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I} = (L_I \mathcal{G}_{\mathbf{f}} / L_I^+ \mathcal{G}_{\mathbf{f}})_{\text{ét}}$. If $I = \{*\}$ is a singleton we omit it from the notation, so $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}} = \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},\{*\}}$.

It follows from the definitions that $LG_C = LG \times_S C$ and $L^+ G_C = L^+ G \times_S C$. More generally, $\mathrm{Gr}_{G_C,I}$ is the Beilinson–Drinfeld Grassmannian denoted by $\mathrm{Gr}_{G,I}$ in [CvdHS22], so it is ind-projective. The proof of ind-projectivity for general $\mathcal{G}_{\mathbf{f}}$ follows from work of Pappas–Zhu [PZ13], Haines–Richarz [HR20], and Achar–Riche [AR] (see also the work of Lourenço [Lou23, Theorem 5.1.3] which proves ind-projectivity when $I = \{*\}$); we recall the main points in Proposition 4.10.

We now describe the fibers of $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$. Let $\phi \colon I \twoheadrightarrow J$ be a surjection of non-empty finite sets and let $j \in J \sqcup \{*\}$. We set

$$C^{\phi,j} = \{(x_i) \in C^I \ : \ x_i = x_{i'} \text{ iff } \phi(i) = \phi(i'), \text{ and } x_i = 0 \text{ iff } \phi(i) = j\}.$$

Here exactly $|\phi^{-1}(j)|$ coordinates in $C^{\phi,j}$ are zero if $j \in J$, and every coordinate in $C^{\phi,*}$ is nonzero. By letting ϕ and j vary, we have a stratification of C^I by the locally closed subschemes $C^{\phi,j}$. Then

$$\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I} \times_{C^I} C^{\phi,j} = \begin{cases} \mathrm{Gr}^{|J|-1} \times \mathrm{Fl}_{\mathbf{f}} \times C^{\phi,j} & j \in J \\ \mathrm{Gr}^{|J|} \times C^{\phi,j} & j = *. \end{cases}$$

This follows from the fact that $L_I^+ \mathcal{G}_{\mathbf{f}} \times_{C^I} C^{\phi,j} = (L^+ G)^{|J|-1} \times L^+ \mathcal{G}_{\mathbf{f}} \times C^{\phi,j}$ if $j \in J$ and $L_I^+ \mathcal{G}_{\mathbf{f}} \times_{C^I} C^{\phi,j} = (L^+ G)^{|J|} \times C^{\phi,j}$ if $j = *$, and a similar description of the fibers of $L_I \mathcal{G}_{\mathbf{f}}$. In particular,

$$(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_\eta = \mathrm{Gr} \times C_\eta, \quad (\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_s = \mathrm{Fl}_{\mathbf{f}} \times C_s \cong \mathrm{Fl}_{\mathbf{f}}.$$

In order to prove properties of the central functor we will need to consider the restriction of $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$ to certain copies of C. The notation in the following definition is borrowed from [Zhu14, AR].

Definition 4.7. Let H be a smooth affine C-group scheme, and let $C \cong C \times \{0\} \subset C^2$ be the closed subscheme obtained by setting $x_2 = 0$. We define the following three functors $\mathrm{AffSch}_C^{\mathrm{op}} \to \mathrm{Set}$,

$$LH^{\mathrm{BD}} = (L_{\{1,2\}} H)\big|_{C \times \{0\}}, \quad L^+ H^{\mathrm{BD}} = (L_{\{1,2\}}^+ H)\big|_{C \times \{0\}},$$

and

$$\mathrm{Gr}_H^{\mathrm{BD}} = (LH^{\mathrm{BD}} / L^+ H^{\mathrm{BD}})_{\text{ét}} = \mathrm{Gr}_{H,\{1,2\}}\big|_{C \times \{0\}}.$$

All representability results which will be proved in Proposition 4.10 apply to $\mathrm{Gr}_H^{\mathrm{BD}}$ by base change. If $H = \mathcal{G}_{\mathbf{f}}$ for a facet $\mathbf{f}$ then by the above there are canonical isomorphisms

$$(L^+ \mathcal{G}_{\mathbf{f}}^{\mathrm{BD}})_\eta = L^+ G \times L^+ \mathcal{G}_{\mathbf{f}} \times C_\eta \qquad\qquad (\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{\mathrm{BD}})_\eta = \mathrm{Gr} \times \mathrm{Fl}_{\mathbf{f}} \times C_\eta,$$

$$(L^+ \mathcal{G}_{\mathbf{f}}^{\mathrm{BD}})_s = L^+ \mathcal{G}_{\mathbf{f}} \qquad\qquad\qquad\qquad\quad (\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{\mathrm{BD}})_s = \mathrm{Fl}_{\mathbf{f}}.$$

Restriction from the disc $\hat{D}_R$ to the disc defined by the ideal $(t - x_1)$ (resp. $(t - x_2)$) defines a map

$$L^+ \mathcal{G}_{\mathbf{f}}^{\mathrm{BD}} \to L^+ \mathcal{G}_{\mathbf{f}}, \quad (\text{resp. } L^+ \mathcal{G}_{\mathbf{f}}^{\mathrm{BD}} \to L^+ \mathcal{G}_{\mathbf{f}} \times C).$$

4.2.3. *The Beilinson–Drinfeld convolution Grassmannians.* We now recall the moduli interpretation of $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$. By [HR20, Lemma 3.4], $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(R)$ is the set of isomorphism classes of pairs $(\mathcal{E}, \alpha)$ where $\mathcal{E}$ is a $\mathcal{G}_{\mathbf{f}}$-torsor on $\hat{D}_R$ and $\alpha \colon \mathcal{E}\big|_{\hat{D}_R^\circ} \xrightarrow{\sim} \mathcal{E}^0\big|_{\hat{D}_R^\circ}$ is a trivialization. Here $\mathcal{E}^0$ is a fixed trivial $\mathcal{G}_{\mathbf{f}}$-torsor. Similarly, $L_I \mathcal{G}_{\mathbf{f}}(R) = (\mathcal{E}, \alpha, \beta)$ where $(\mathcal{E}, \alpha)$ is as before and $\beta \colon \mathcal{E}^0\big|_{\hat{D}_R} \xrightarrow{\sim} \mathcal{E}\big|_{\hat{D}_R}$.

The moduli interpretation of $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$ allows us to construct a convolution Grassmannian as follows, generalizing [AR, §3.5.2] (where $|I| = 2$). If $|I| > 1$, choose a distinguished element $i_0 \in I$. For $R \in \mathrm{AffSch}_{C^I}^{\mathrm{op}}$, let $\hat{D}_{R,I-\{i_0\}}^\circ$ be the complement in $\hat{D}_R$ of the divisor defined by the ideal generated by $\prod_{i \in I \setminus \{i_0\}} (t - x_i)$. Thus, $\hat{D}_R^\circ$ is obtained from $\hat{D}_{R,I-\{i_0\}}^\circ$ by further removing the divisor defined by the ideal $(t - x_{i_0})$. Similarly, let $\hat{D}_{R,\{i_0\}}^\circ$ be the complement in $\hat{D}_R$ of the divisor defined by the ideal $(t - x_{i_0})$.

27

Definition 4.8. The *Beilinson–Drinfeld convolution Grassmannian relative to $i_0 \in I$ and $\mathbf{f}$* is the functor $\mathrm{AffSch}_{C^I}^{\mathrm{op}} \to$ Set defined by $\widetilde{\mathrm{Gr}}_{\mathcal{G}_{\mathbf{f}},I}(R) = \{(\mathcal{E}^1, \mathcal{E}^2, \alpha_1, \alpha_2)\}$ where the $\mathcal{E}^i$ are $\mathcal{G}_{\mathbf{f}}$-torsors on $\hat{D}_R$ and

$$\alpha_1 \colon \mathcal{E}^1\big|_{\hat{D}^\circ_{R,I-\{i_0\}}} \xrightarrow{\sim} \mathcal{E}^0\big|_{\hat{D}^\circ_{R,I-\{i_0\}}}, \quad \alpha_2 \colon \mathcal{E}^2\big|_{\hat{D}^\circ_{R,\{i_0\}}} \xrightarrow{\sim} \mathcal{E}^1\big|_{\hat{D}^\circ_{R,\{i_0\}}}.$$

The *convolution map* $m_{i_0} \colon \widetilde{\mathrm{Gr}}_{\mathcal{G}_{\mathbf{f}},I} \to \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$ is defined on points by

$$\{(\mathcal{E}^1, \mathcal{E}^2, \alpha_1, \alpha_2)\} \mapsto \{(\mathcal{E}^2, \alpha_1\big|_{\hat{D}^\circ_R} \circ \alpha_2\big|_{\hat{D}^\circ_R})\}.$$

The convolution Grassmannian $\widetilde{\mathrm{Gr}}_{\mathcal{G}_{\mathbf{f}},I}$ will be used to prove the ind-projectivity of $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$. Toward this end, we note that as a functor on $\mathrm{AffSch}_{C^I}^{\mathrm{op}}$ we have $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I-\{i_0\}} \times C)(R) = \{(\mathcal{E}, \alpha)\}$ where $\mathcal{E}$ is a $\mathcal{G}_{\mathbf{f}}$-torsor on $\hat{D}_R$ and $\alpha \colon \mathcal{E}\big|_{\hat{D}^\circ_{R,I-\{i_0\}}} \xrightarrow{\sim} \mathcal{E}^0\big|_{\hat{D}^\circ_{R,I-\{i_0\}}}$ is a trivialization. Similarly, $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},\{i_0\}} \times C^{I-\{i_0\}})(R) = \{(\mathcal{E}, \alpha)\}$ where now $\alpha \colon \mathcal{E}\big|_{\hat{D}^\circ_{R,\{i_0\}}} \xrightarrow{\sim} \mathcal{E}^0\big|_{\hat{D}^\circ_{R,\{i_0\}}}$. There is a right $L_I^+ \mathcal{G}_{\mathbf{f}}$-torsor $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I-\{i_0\}} \times C)^{(\infty)} \to \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I-\{i_0\}} \times C$ whose R-points are given by $\{(\mathcal{E}, \alpha, \beta)\}$ where $\{(\mathcal{E}, \alpha)\} \in (\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I-\{i_0\}} \times C)(R)$ and $\beta \colon \mathcal{E}\big|_{\hat{D}_R} \xrightarrow{\sim} \mathcal{E}^0\big|_{\hat{D}_R}$. The group $L_I^+ \mathcal{G}_{\mathbf{f}}$ also acts on $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},\{i_0\}} \times C^{I-\{i_0\}}$ on the left by changing the trivialization $\alpha \colon \mathcal{E}\big|_{\hat{D}^\circ_{R,\{i_0\}}} \xrightarrow{\sim} \mathcal{E}^0\big|_{\hat{D}^\circ_{R,\{i_0\}}}$. These observations lead to the following alternative description of $\widetilde{\mathrm{Gr}}_{\mathcal{G}_{\mathbf{f}},I}$.

Lemma 4.9. *The convolution Grassmannian $\widetilde{\mathrm{Gr}}_{\mathcal{G}_{\mathbf{f}},I}$ is isomorphic to the following étale quotient with respect to the diagonal action of $L_I^+ \mathcal{G}_{\mathbf{f}}$:*

$$\widetilde{\mathrm{Gr}}_{\mathcal{G}_{\mathbf{f}},I} \cong (\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I-\{i_0\}} \times C)^{(\infty)} \times_{C^I}^{L_I^+ \mathcal{G}_{\mathbf{f}}} (\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},\{i_0\}} \times C^{I-\{i_0\}}).$$

Proof. This follows from [HR20, Lemma 3.4] which says that every $\mathcal{G}_{\mathbf{f}}$-torsor on $\hat{D}_R$ is trivializable after some étale cover $R \to R'$, cf. also the proof of [AR, Proposition 2.3.11]. $\square$

4.2.4. *Ind-projectivity.* We are now ready to prove that $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$ is ind-projective.

Proposition 4.10. *Let H be a C-group scheme arising from the base change along $C = \mathbf{A}_S^1 \to \mathbf{A}_\mathbf{Z}^1$ of a smooth affine group scheme over $\mathbf{A}_\mathbf{Z}^1$ with geometrically connected fibers. Then the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{H,I}$ is represented by a separated ind-finite type C-scheme. For any facet $\mathbf{f}$, $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$ is furthermore ind-projective over C^I.*

Proof. Since sheafification commutes with base change we may assume $S = \mathrm{Spec}\,\mathbf{Z}$. If $G = \mathrm{GL}_n$ and $\mathbf{f} = \mathbf{f}_0$ then $\mathcal{G}_{\mathbf{f}} = \mathrm{GL}_{n,C}$, the general linear group over C. Ind-projectivity of $\mathrm{Gr}_{\mathrm{GL}_{n,C},I}$ is proved using Quot schemes in [HR20, Lemma 3.8]. For general H, since $\mathbf{A}_\mathbf{Z}^1$ has dimension 2 then by [PZ13, Corollary 11.7] we may choose a closed subgroup embedding $H \to \mathrm{GL}_{n,C}$ for some n such that $(\mathrm{GL}_{n,C}/H)_{\mathrm{fppf}}$ is quasi-affine. The induced map $\mathrm{Gr}_H \to \mathrm{Gr}_{\mathrm{GL}_{n,C}}$ is a quasi-compact immersion by [HR20, Proposition 3.10], so $\mathrm{Gr}_{H,I}$ is represented by a separated ind-finite-type C-scheme [HR20, Corollary 3.11].

For ind-projectivity of $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$ it then suffices to show ind-properness. We start with the case $I = \{*\}$. The argument for ind-properness in [PZ13, Proposition 6.5] (where the base is a complete discrete valuation ring with perfect residue field) applies over C as well; we sketch the argument for the convenience of the reader. First, the geometric fibers of $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}} \to C$ are ind-proper because Gr and $\mathrm{Fl}_{\mathbf{f}}$ are ind-proper [Fal03, PR08]. For $\mu \in X_*(T)^+$ let $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{\leq \mu}$ be the scheme-theoretic image of the map $\mathrm{Gr}^{\leq \mu} \times_S C_\eta \to \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}$. Using that the fibers of $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}$ are ind-proper and that $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}} \to \mathrm{Gr}_{\mathrm{GL}_{n,C}}$ is an immersion with ind-proper target, it can be shown that $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{\leq \mu} \to C$ is proper, cf. the proof of [PZ13, Proposition 6.5] or [Ric16, Theorem 2.19]. To conclude that $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}$ is ind-proper it remains to show that the fibers $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{\leq \mu})_s$ cover $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_s$. This can be checked on reduced loci of the geometric fibers. Then the claim follows from the fact that when $S = \mathrm{Spec}\,k$ is the spectrum of an algebraically closed field, the reduced locus of $(\mathrm{Gr}_{\mathcal{G}}^{\leq \mu})_s$ contains the μ-admissible locus $\mathcal{A}^{\mathbf{f}}(\mu)$ [Ric16, Lemma 3.12] as in Definition 4.5. We then conclude by noting that $\mathrm{Fl}_{\mathbf{f}}$ is covered by the closed Schubert schemes $\mathrm{Fl}_{t(\nu)}(\mathbf{f}, \mathbf{f})$ associated to translation elements $\nu \in X_*(T) \subset W$.

For $|I| > 1$ we proceed by induction on $|I|$, using the technique in [AR, §3.5.2] (where $|I| = 2$) to write $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$ as the image of an ind-proper scheme. First, as in [AR, Lemma 2.3.10] the ind-projective scheme $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},\{*\}}$ admits an $L_{\{*\}}^+ \mathcal{G}_{\mathbf{f}}$-stable presentation by projective C-schemes each admitting an $L_{\{*\}}^+ \mathcal{G}_{\mathbf{f}}$-equivariant relatively ample line bundle. (The line bundle is pulled back from $\mathrm{Gr}_{\mathrm{GL}_{n,C}}$, using that the latter is constructed from Quot schemes which embed into ordinary Grassmannians.) By [AR, Remark 2.1.10], Lemma 4.9 shows that $\widetilde{\mathrm{Gr}}_{\mathcal{G}_{\mathbf{f}},I}$ is represented by an ind-proper C^I-scheme, cf. also [AR, Proposition 2.3.10]. Since the convolution map m_{i_0} in Definition 4.8 is surjective, this implies that $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$ is ind-proper. $\square$

4.2.5. *Iterated Grassmannians.* To define the iterated Grassmannians we follow the setup in [AR, §2.3.4]. In this subsection we fix $I = \{1, 2\}$ when discussing divisors. Let $R \in \mathrm{AffSch}_C^{\mathrm{op}}$. We may view R as an affine scheme over C^2 by composing with the inclusion $C \cong C \times \{0\} \subset C^2$, and under this identification we have the disc $\hat{D}_R$ associated to the ideal generated by $(t - x_1)t$. Let $V((t - x_1)t)$ be the divisor associated to this ideal. For P equal to one of the four symbols $\emptyset, \underline{0}, \underline{x}, \underline{x} \cup \underline{0}$, we define

$$\hat{D}_P^\circ = \begin{cases} \hat{D}_R & P = \emptyset \\ \hat{D}_R \setminus V(t) & P = \underline{0} \\ \hat{D}_R \setminus V(t - x_1) & P = \underline{x} \\ \hat{D}_R \setminus V((t - x_1)t) = \hat{D}_R^\circ & P = \underline{x} \cup \underline{0}. \end{cases}$$

Definition 4.11. Let $\underline{P} = (P_1, \ldots, P_n)$ where $P_i \in \{\emptyset, \underline{0}, \underline{x}, \underline{x} \cup \underline{0}\}$ for all i. The iterated Grassmannian associated to $\underline{P}$ is the functor $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P}) \colon \mathrm{AffSch}_C^{\mathrm{op}} \to \mathrm{Set}$ such that

$$\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})(R) = \{(\mathcal{E}^i, \alpha_i)\}_{1 \le i \le n}$$

where the $\mathcal{E}^i$ are $\mathcal{G}_{\mathbf{f}}$-bundles on $\hat{D}_R$ and the α_i are isomorphisms $\alpha_i \colon \mathcal{E}^i\big|_{\hat{D}_{P_i}^\circ} \xrightarrow{\sim} \mathcal{E}^{i-1}\big|_{\hat{D}_{P_i}^\circ}$ (so α_1 is a trivialization).

The fibers of the iterated Grassmannians are as follows.

Lemma 4.12. *Let* $\underline{P} = (P_1, \ldots, P_n)$. *We have a canonical isomorphism*

$$\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})_\eta = \underbrace{LG \times^{L^+ G} \cdots \times^{L^+ G} \mathrm{Gr}}_{i \text{ factors}} \times \underbrace{LG \times^{L^+ G_{\mathbf{f}}} \cdots \times^{L^+ G_{\mathbf{f}}} \mathrm{Fl}_{\mathbf{f}}}_{j \text{ factors}} \times C_\eta$$

where i is the number of symbols belonging to $\{\underline{x}, \underline{x} \cup \underline{0}\}$ and j is number of symbols belonging to $\{\underline{0}, \underline{x} \cup \underline{0}\}$. We also have a canonical isomorphism

$$\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})_s = \underbrace{LG \times^{L^+ G_{\mathbf{f}}} \cdots \times^{L^+ G_{\mathbf{f}}} \mathrm{Fl}_{\mathbf{f}}}_{k \text{ factors}}$$

where k is the number of symbols $P_i \ne \emptyset$. Furthermore, if $\underline{P}$ is a singleton we have canonical isomorphisms

$$\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\emptyset) = C, \quad \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{0}) = \mathrm{Fl}_{\mathbf{f}} \times C, \quad \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{x}) = \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}, \quad \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{x} \cup \underline{0}) = \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{\mathrm{BD}}.$$

Proof. This is immediate from the moduli description of these functors; see [AR, Lemma 2.3.6, Table 2.3.1 ff.]. $\square$

Lemma 4.12 implies that $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})$ is representable if $\underline{P}$ is a singleton. For general $\underline{P}$ we consider the functor $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{(\infty)}(\underline{P}) \colon \mathrm{AffSch}_C^{\mathrm{op}} \to \mathrm{Set}$ such that $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{(\infty)}(\underline{P})(R)$ is the set of isomorphism classes $\{(\mathcal{E}^i, \alpha_i), \beta\}_{1 \le i \le n}$ where $\{(\mathcal{E}^i, \alpha_i)\}_{1 \le i \le n}$ is as in Definition 4.11 and $\beta \colon \mathcal{E}^n \xrightarrow{\sim} \mathcal{E}^0$ is a trivialization of $\mathcal{G}$-bundles on $\hat{D}_R$. There is a natural right action of $L^+ \mathcal{G}_{\mathbf{f}}^{\mathrm{BD}}$ on $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{(\infty)}(\underline{P})$ which changes β, cf. [AR, Eqn. (2.3.9)]. The map $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{(\infty)}(\underline{P}) \to \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})$ which forgets β is an étale-locally trivial $L^+ \mathcal{G}_{\mathbf{f}}^{\mathrm{BD}}$-torsor by [AR, Lemma 2.3.9]. There is also a left action of $L^+ \mathcal{G}_{\mathbf{f}}^{\mathrm{BD}}$ on $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{(\infty)}(\underline{P})$ and $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})$ which changes α_1, cf. [AR, Eqn. (2.3.5)].

Proposition 4.13. *The iterated Grassmannian $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})$ is represented by an ind-proper C-scheme. Moreover, it admits a presentation as a colimit of proper $L^+ \mathcal{G}_{\mathbf{f}}^{\mathrm{BD}}$-stable subschemes such that the action of $L^+ \mathcal{G}_{\mathbf{f}}^{\mathrm{BD}}$ on each subscheme factors through some $L^{+,(m)} \mathcal{G}_{\mathbf{f}}^{\mathrm{BD}}$.*

Proof. This is proved in [AR, Proposition 2.3.11] when $S = \mathrm{Spec}\,\mathbf{C}$ but the proof applies to the general case since [HR20, §3] works in this generality. $\square$

Remark 4.14. If $Z \subset \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})$ is a finite-dimensional closed subscheme such that the action of $L^+ \mathcal{G}_{\mathbf{f}}^{\mathrm{BD}}$ factors through $L^{+,(m)} \mathcal{G}_{\mathbf{f}}^{\mathrm{BD}}$, then there is a canonical equivalence $\mathrm{DM}(L^+ \mathcal{G}_{\mathbf{f}}^{\mathrm{BD}} \backslash Z) = \mathrm{DM}(L^{+,(m)} \mathcal{G}_{\mathbf{f}}^{\mathrm{BD}} \backslash Z)$. This follows from the argument in [RS20, Proposition 2.2.11], using that the kernel of $L^+ \mathcal{G}_{\mathbf{f}}^{\mathrm{BD}} \to L^{+,(0)} \mathcal{G}_{\mathbf{f}}^{\mathrm{BD}}$ is split pro-unipotent by [RS20, Proposition A.9]. When $\underline{P} = P$ is a singleton, this equivalence also restricts to $\mathrm{DTM}^{(\mathrm{anti})}$ by the arguments in [RS20, Proposition 3.1.27] once we show that the latter category exists in Theorem 4.37(1).

4.2.6. *Nearby cycles on the Hecke stack.*

Definition 4.15. Let $\underline{P}$ be as in Definition 4.11 and let $\mathbf{f}$ be a facet. The *Hecke stack* associated to $\underline{P}$ and $\mathbf{f}$ is the prestack quotient $\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(\underline{P}) = \mathcal{L}^+ \mathcal{G}_{\mathbf{f}}^{\mathrm{BD}} \backslash \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})$.

We define the functor

$$\Upsilon_{\underline{P}} \colon \mathrm{DM}(\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})_\eta) \to \mathrm{Nilp}_{\mathbf{Q}} \mathrm{DM}(\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})_s)$$

to be the unipotent nearby cycles functor associated to the structure map of $\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})$. We also define the functor

$$\mathsf{Z}_{\mathbf{f}} \colon \mathrm{DM}(L^+ G \backslash \mathrm{Gr}) \to \mathrm{Nilp}_{\mathbf{Q}} \mathrm{DM}(L^+ G_{\mathbf{f}} \backslash \mathrm{Fl}_{\mathbf{f}}), \quad \mathsf{Z}_{\mathbf{f}}(\mathcal{F}) = \Upsilon_{\underline{x}}(\mathcal{F} \boxtimes_S \mathbf{Z}_{C_\eta}),$$

where we use the identification $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_\eta = \mathrm{Gr} \times_S C_\eta$.

Remark 4.16. In the expression $\mathcal{F} \boxtimes_S \mathbf{Z}_{C_\eta}$, implicitly we apply a restriction functor, i.e., !-pullback along the map of prestack quotients $(L^+\mathcal{G}_{\mathbf{f}}^{\mathrm{BD}} \backslash \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_\eta \to (L^+\mathcal{G}_{\mathbf{f}} \backslash \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_\eta$.

The functor Υ on $\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}$ exists for formal reasons, cf. Section 3.11: such a functor exists for the motives on the ind-scheme $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})$, and then on the prestack quotient by the action of the pro-smooth $\mathbf{A}^1$-group scheme $L^+\mathcal{G}_{\mathbf{f}}$.

For later use, e.g. in Section 4.2.7, also recall this: let $X = \mathrm{colim}\, X_i$ be an ind-scheme the with an action of a pro-smooth group scheme $G = \lim G_j$ such that the action of G on each X_i factors through a smooth quotient G_j. If the kernel of $G \to G_0$ is split pro-unipotent, by [RS20, Proposition 2.4.4] we have $\mathrm{DM}(X_i/G) = \mathrm{DM}(X_i/G_j)$. We furthermore have an external product $\boxtimes$ on DM of such prestacks if they are placid in the sense of [RS21, §A]. By Proposition 4.13 and the discussion following Definition 4.6, all of these assumptions are satisfied for $\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})$.

4.2.7. *Convolution.* In order to refine the considerations in [CvdHS22, §4.2] to a fully ∞-categorical treatment, we apply the constructions in Section 2.2.8 to $L^+G_{\mathbf{f}} \subset LG$. Indeed, LG is a placid ind-scheme [Gai20, §C.3] and $L^+G_{\mathbf{f}}$ is a pro-smooth group scheme, and the Zariski (and thus Nisnevich) sheafification of the presheaf quotient $LG/L^+G_{\mathbf{f}}$ is representable by the ind-proper ind-scheme $\mathrm{Fl}_{\mathbf{f}}$. By construction, the multiplication map of the (non-symmetric) monoidal structure on $\mathrm{DM}(L^+G_{\mathbf{f}} \backslash LG/L^+G_{\mathbf{f}})$ is the usual convolution functor

$$\star : \mathrm{DM}(L^+G_{\mathbf{f}} \backslash LG/L^+G_{\mathbf{f}}) \times \mathrm{DM}(L^+G_{\mathbf{f}} \backslash LG/L^+G_{\mathbf{f}}) \to \mathrm{DM}(L^+G_{\mathbf{f}} \backslash LG/L^+G_{\mathbf{f}})$$

$$\mathcal{F}_1 \star \mathcal{F}_2 := m_! \pi^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2).$$

Here π and m are the natural quotient and multiplication maps of prestacks:

$$L^+G_{\mathbf{f}} \backslash LG/L^+G_{\mathbf{f}} \times L^+G_{\mathbf{f}} \backslash LG/L^+G_{\mathbf{f}} \xleftarrow{\;\pi\;} L^+G_{\mathbf{f}} \backslash LG \times^{L^+G_{\mathbf{f}}} LG/L^+G_{\mathbf{f}} \xrightarrow{\;m\;} L^+G_{\mathbf{f}} \backslash LG/L^+G_{\mathbf{f}}.$$

The underlying motive of $\pi^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ on the Zariski quotient $(LG \times^{L^+G_{\mathbf{f}}} \mathrm{Fl}_{\mathbf{f}})_{\mathrm{Zar}}$ is sometimes also denoted $\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2$. Under ℓ-adic realization one recovers the usual notion of convolution, cf. [RS21, Proposition 3.14].

Proposition 4.17. *The convolution product turns* $\mathrm{DTM}_{L^+G_{\mathbf{f}}}(LG/L^+G_{\mathbf{f}})^{(\mathrm{anti})}$ *into a monoidal ∞-category.*

Proof. By [CvdHS22, 4.11] the functor $\star$ preserves anti-effective (resp. all) stratified Tate motives, so the monoidal structure on $\mathrm{DM}(L^+G_{\mathbf{f}} \backslash LG/L^+G_{\mathbf{f}})$ established above restricts to this full subcategory. $\square$

The notion of local compactness (as defined in Section 2.2.6) is also preserved by the convolution product.

Lemma 4.18. *The convolution product preserves the subcategory* $\mathrm{DTM}_{L^+G_{\mathbf{f}}}(\mathrm{Fl}_{\mathbf{f}})^{\mathrm{lc}} \subseteq \mathrm{DTM}_{L^+G_{\mathbf{f}}}(\mathrm{Fl}_{\mathbf{f}})$ *and therefore turns it into a monoidal ∞-category.*

Proof. Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{DTM}_{L^+G_{\mathbf{f}}}(\mathrm{Fl}_{\mathbf{f}})^{\mathrm{lc}}$. Then $u^!(\mathcal{F}_1 \star \mathcal{F}_2)$ agrees with the pushforward of the twisted exterior product $\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2$ (cf. [CvdHS22, Lemma 4.16]) along the multiplication map

$$\mathrm{Fl}_{\mathbf{f}} \widetilde{\times} \mathrm{Fl}_{\mathbf{f}} := LG \overset{L^+G_{\mathbf{f}}}{\times} \mathrm{Fl}_{\mathbf{f}} \to \mathrm{Fl}_{\mathbf{f}}.$$

Since this multiplication map is ind-proper, pushforward along it preserves compactness, so it suffices to show $\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2$ is compact.

This twisted exterior product is defined such that its pullback along the $L^+G_{\mathbf{f}}$-torsor $LG \times \mathrm{Fl}_{\mathbf{f}} \to \mathrm{Fl}_{\mathbf{f}} \widetilde{\times} \mathrm{Fl}_{\mathbf{f}}$ agrees with the pullback of $u^!(\mathcal{F}_1) \boxtimes u^!(\mathcal{F}_2)$ along the $L^+G_{\mathbf{f}}$-torsor $LG \times \mathrm{Fl}_{\mathbf{f}} \to \mathrm{Fl}_{\mathbf{f}} \times \mathrm{Fl}_{\mathbf{f}}$, or similarly for suitable truncations of $L^+G_{\mathbf{f}}$. The lemma then follows from the fact that pullback along such (Zariski-locally trivial) torsors preserves and reflects compact objects. $\square$

We have the following global variant of convolution as in [AR, Eqn. (2.3.8)]. There is a natural union operation $\cup$ on the symbols $\emptyset, \underline{0}, \underline{x}, \underline{x} \cup \underline{0}$. For $1 \le i \le j \le n$ let

$$m_{i,j} : \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_1, \dots, P_n) \to \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_1, \dots, P_{i-1}, P_i \cup \dots \cup P_j, P_{j+1}, \dots, P_n)$$

be the map which sends $\{(\mathcal{E}^i, \alpha_i)\}_{1 \le i \le n}$ to

$$\{(\mathcal{E}^1, \dots, \mathcal{E}^{i-1}, \mathcal{E}^j, \mathcal{E}^{j+1}, \dots, \mathcal{E}^n, \alpha_1, \dots, \alpha_{i-1}, \alpha_j', \alpha_{j+1}, \dots, \alpha_n)\}.$$

Here $\alpha_j' = \alpha_j|_{\hat{D}_{P'}^\circ} \circ \dots \circ \alpha_i|_{\hat{D}_{P'}^\circ}$ and $P' = P_i \cup \dots \cup P_j$. The convolution maps $m_{i,j}$ are $\mathcal{L}^+\mathcal{G}_{\mathbf{f}}^{\mathrm{BD}}$-equivariant. Over C_η and C_s the $m_{i,j}$ restrict to local convolution maps for Gr and $\mathrm{Fl}_{\mathbf{f}}$. These convolution maps are used throughout the proofs in Section 4.5.

We construct a convolution product over C_η as follows. Consider the diagram

$$\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_1) \times_C \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_2) \xleftarrow{\;p\;} \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{(\infty)}(P_1) \times_C \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_2) \xrightarrow{\;q\;} \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_1, P_2). \qquad (4.1)$$

The map p is the base change of the étale-locally trivial $\mathcal{L}^+\mathcal{G}_{\mathbf{f}}^{\mathrm{BD}}$-torsor $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{(\infty)}(P_1) \to \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_1)$ from Section 4.2.5. The map q is an étale torsor for the diagonal action of $\mathcal{L}^+\mathcal{G}_{\mathbf{f}}^{\mathrm{BD}}$ as in [AR, Corollary 2.3.14]. We also have the $\mathcal{L}^+\mathcal{G}_{\mathbf{f}}^{\mathrm{BD}}$-equivariant convolution map $m_{1,2} : \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_1, P_2) \to \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_1 \cup P_2)$, so we get a map $\overline{m}_{1,2} : \mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P_1, P_2) \to$

$\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P_1 \cup P_2)$. The map p is equivariant for the action of $(\mathcal{L}^+\mathcal{G}_{\mathbf{f}}^{\mathrm{BD}})^2$ by left translation on both factors. After taking fibers over η and the Zariski-quotient by this action on the target, p_η descends to a projection map

$$\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P_1)_{\mathrm{Zar},\eta} \times_C \mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P_2)_{\mathrm{Zar},\eta} \xleftarrow{\overline{p}_\eta} \mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P_1, P_2)_\eta. \tag{4.2}$$

Zariski-sheafification is necessary, but étale sheafification is not, since p_η and q_η are Zariski-locally trivial. A similar projection exists over C_s.

Remark 4.19. A priori the maps p_η and q_η are only étale quotients, but since $LG \to \mathrm{Fl}_{\mathbf{f}}$ admits sections Zariski-locally by Definition 4.2 ff., then the Zariski-local triviality of p_η and q_η follows from Lemma 4.12, cf. also [CvdHS22, Lemma 4.16 ff.]. One can avoid taking Zariski-sheafifications in (4.2) at the cost of working with products of pre-stack quotients of local loop groups in both the source and target, e.g. $LG/L^+G_{\mathbf{f}}$ instead of $\mathrm{Fl}_{\mathbf{f}}$. This loop group description does not extend globally to $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_1, P_2)$, and we do not assert that q is Zariski-locally trivial.

We define

$$\star_{C_\eta} : \mathrm{DM}(\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P_1)_\eta) \times \mathrm{DM}(\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P_2)_\eta) \to \mathrm{DM}(\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P_1 \cup P_2)_\eta)$$

$$\mathcal{F}_1 \star_{C_\eta} \mathcal{F}_2 := (\overline{m}_{1,2})_{\eta!} \circ \overline{p}_\eta^!(\mathcal{F}_1 \boxtimes_{C_\eta} \mathcal{F}_2).$$

The construction of this functor is analogous to that in [CvdHS22, §4.2.4]. We are implicitly using the fact that DM satisfies Nisnevich descent so that we can replace Zariski quotients with pre-stack quotients, e.g. $\mathrm{DM}(\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P)_{\mathrm{Zar},\eta}) = \mathrm{DM}(\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P)_\eta)$. Working over C_s gives another convolution product, which we denote by $\star$ since it recovers the convolution product on $\mathrm{DM}(L^+G_{\mathbf{f}} \backslash LG / L^+G_{\mathbf{f}})$. There are natural associativity constraints on $\star_{C_\eta}$ and $\star$ at the level of homotopy categories constructed as in [AR, Eqn. (2.4.5)].

Lemma 4.20. *Let $\underline{P} = (P_1, P_2)$. For $\mathcal{F}_1 \in \mathrm{DM}(\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P_1)_\eta)$ and $\mathcal{F}_2 \in \mathrm{DM}(\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P_2)_\eta)$, there are canonical morphisms in $\mathrm{Nilp}_{\mathbf{Q}} \mathrm{DM}(\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P_1, P_2)_s)$ and $\mathrm{Nilp}_{\mathbf{Q}} \mathrm{DM}(\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P_1 \cup P_2)_s)$, functorial in $\mathcal{F}_1$ and $\mathcal{F}_2$:*

$$\overline{p}_s^!(\Upsilon_{P_1}(\mathcal{F}_1) \boxtimes_{C_s} \Upsilon_{P_2}(\mathcal{F}_2)) \to \Upsilon_{\underline{P}}(\overline{p}_\eta^!(\mathcal{F}_1 \boxtimes_{C_\eta} \mathcal{F}_2)) \tag{4.3}$$

$$\Upsilon_{P_1}(\mathcal{F}_1) \star \Upsilon_{P_2}(\mathcal{F}_2) \to \Upsilon_{P_1 \cup P_2}(\mathcal{F}_1 \star_{C_\eta} \mathcal{F}_2) \tag{4.4}$$

Proof. By continuity we may assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are bounded. Applying $(\overline{m}_{1,2})_{s!}$ to both sides of (4.3) and using compatibility with proper pushforward along $\overline{m}_{1,2}$ gives (4.4). To construct, (4.3) we first replace the maps p and q in (4.1) by $\mathcal{L}^{+,(n)}\mathcal{G}_{\mathbf{f}}^{\mathrm{BD}}$-torsors for some integer n. The functor $\Upsilon_{\underline{P}}$ is an equivariant nearby cycles functor along $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_1, P_2)$, so we may view the motives in (4.3) as $L^+G_{\mathbf{f}}$-equivariant motives on $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_1, P_2)_s$.

Since q_s admits sections Zariski-locally, then to give a map as in (4.3), it is equivalent to apply $q_s^!$ and then give an $(L^+G_{\mathbf{f}})^2$-equivariant map of motives on the source of q_s. By compatibility of Υ with smooth pullback, we need an $(L^+G_{\mathbf{f}})^2$-equivariant map

$$p_s^!(\Upsilon_{P_1}(\mathcal{F}_1) \boxtimes_{C_s} \Upsilon_{P_2}(\mathcal{F}_2)) \to \Upsilon_{\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{(\infty)}(P_1) \times_C \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_2)}(p_\eta^!(\mathcal{F}_1 \boxtimes_{C_\eta} \mathcal{F}_2)). \tag{4.5}$$

The underlying map of motives in (4.5) is obtained from applying $p_s^!$ to a Künneth map for nearby cycles along $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_1) \times_S \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_2) \to C^2$. Künneth maps commute with smooth pullback, so this is equivalently a Künneth map for nearby cycles along $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{(\infty)}(P_1) \times_C \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_2)$. The particular Künneth map we need is obtained from Proposition 3.27 followed by Corollary 3.26. By construction of these natural transformations, the resulting map (4.5) is Σ-linear (where Σ acts on the domain by restriction along the comultiplication (2.4), cf. (3.17)).

It remains to see that (4.5) is $(L^+G_{\mathbf{f}})^2$-equivariant. In fact, $p_\eta^!(\mathcal{F}_1 \boxtimes_{C_\eta} \mathcal{F}_2)$ is $(\mathcal{L}^+\mathcal{G}_{\mathbf{f}}^{\mathrm{BD}})_\eta^3$-equivariant, so (4.5) can be refined to a Künneth map for $(\mathcal{L}^+\mathcal{G}_{\mathbf{f}}^{\mathrm{BD}})^3$-equivariant nearby cycles along $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{(\infty)}(P_1) \times_C \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_2)$. The desired $(L^+G_{\mathbf{f}})^2$-equivariance comes from restriction along the map $(L^+G_{\mathbf{f}})^2 \to (L^+G_{\mathbf{f}})^3$, $(g_1, g_2) \mapsto (g_1, g_2, g_2)$. $\square$

Remark 4.21. There are diagrams similar (4.2) and (4.1) for $n > 1$ factors, in particular

$$\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P_1)_{\mathrm{Zar},\eta} \times_C \cdots \times_C \mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P_n)_{\mathrm{Zar},\eta} \xleftarrow{\overline{p}} \mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P_1, \ldots P_n)_\eta$$

where $\underline{P} = (P_1, \ldots, P_{i-1}, P_i \cup \cdots \cup P_j, P_{j+1}, \ldots, P_n)$. Considerations similar to those in Lemma 4.20 lead to a natural map in $\mathrm{Nilp}_{\mathbf{Q}} \mathrm{DM}$:

$$\overline{p}_s^!(\Upsilon_{P_1}(\mathcal{F}_1) \boxtimes_{C_s} \cdots \boxtimes_{C_s} \Upsilon_{P_n}(\mathcal{F}_n)) \to \Upsilon_{(P_1, \ldots, P_n)}(\overline{p}_\eta^!(\mathcal{F}_1 \boxtimes_{C_\eta} \cdots \boxtimes_{C_\eta} \mathcal{F}_n)) \tag{4.6}$$

We will show in Theorem 4.37(3) that (4.6) is an isomorphism for Tate motives. Applying pushforward along $\overline{m}_{i,j} : \mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P_1, \ldots P_n)_\eta \to \mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})$ then gives an isomorphism of convolution products which will be used throughout Section 4.5.

4.3. **Multiplicative group actions.** In order to construct Whitney–Tate stratifications of Beilinson–Drinfeld Grassmannians we must first take a detour and consider certain multiplicative group actions. The general setup is as follows. Fix a cocharacter $\lambda \in X_*(T)$. There is an inclusion $\mathbf{G}_{\mathrm{m},C^I} \to L_I^+(\mathbf{G}_{\mathrm{m},C})$ coming from the identifications $\mathbf{G}_{\mathrm{m},C^I}(R) = R^\times \subset R[\![D]\!]^\times = L_I^+(\mathbf{G}_{\mathrm{m},C})(R)$. Let $\lambda_C \colon \mathbf{G}_{\mathrm{m},C} \to T_C$ be the base change of λ by C. By composing the previous inclusion with $L_I^+(\lambda_C) \colon L_I^+(\mathbf{G}_{\mathrm{m},C}) \to L_I^+(T_C)$ and the natural map $L_I^+(T_C) \to L_I^+ \mathcal{G}_{\mathbf{f}}$, we get a $\mathbf{G}_{\mathrm{m},C^I}$-action on $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$. This also induces a $\mathbf{G}_{\mathrm{m},C}$-action on $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{\mathrm{BD}}$ and on the $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})$. The following lemma allows us to apply the hyperbolic localization isomorphism [Ric19, Theorem B].

Lemma 4.22. *The $\mathbf{G}_{\mathrm{m},C^I}$-action on $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$ associated to a cocharacter $\lambda \in X_*(T)$ is Zariski-locally linearizable, as is the $\mathbf{G}_{\mathrm{m},C}$-action on $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})$.*

Proof. By [PZ13, Corollary 11.7] and [HR20, Proposition 3.10] there is a closed subgroup embedding $\mathcal{G}_{\mathbf{f}} \to \mathrm{GL}_{n,C}$ for some integer n which induces a $\mathbf{G}_{\mathrm{m},C^I}$-equivariant immersion $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I} \to \mathrm{Gr}_{\mathrm{GL}_{n,C},I}$. This immersion is in fact closed because $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$ is ind-proper. Hence the $\mathbf{G}_{\mathrm{m},C^I}$-action on $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$ is Zariski-locally linearizable since the proof of [HR20, Lemma 3.16] shows that the same is true for the $\mathbf{G}_{\mathrm{m},C^I}$-action on $\mathrm{Gr}_{\mathrm{GL}_{n,C},I}$. By Lemma 4.12 this also treats the case where $\underline{P}$ is a singleton.

The general case can be deduced as follows. Let $m_i \colon \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P}) \to \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_1 \cup \cdots \cup P_i)$ be the partial convolution map when sends $\{(\mathcal{E}^i, \alpha_i)\}_{1 \le i \le n}$ to $\{(\mathcal{E}^i, \alpha_1 \circ \ldots \alpha_i|_{\hat{D}^\circ_{P_1 \cup \cdots \cup P_i}})\}$ for $1 \le i \le n$. The product of the m_i over C induces a map

$$f \colon \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P}) \to \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_1) \times_C \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_1 \cup P_2) \times_C \cdots \times_C \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_1 \cup \cdots \cup P_n),$$

which is $\mathbf{G}_{\mathrm{m},C}$-equivariant for the diagonal action of $\mathbf{G}_{\mathrm{m},C}$ on the target. Hence, it suffices to show that f is a closed embedding. Since both ind-schemes are ind-proper, it suffices to show that f is a monomorphism. Thus, suppose that f sends two R-points, $\{(\mathcal{E}^i, \alpha_i)\}_{1 \le i \le n}$ and $\{(\mathcal{F}^i, \beta_i)\}_{1 \le i \le n}$, to the same element. These elements are equal in $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})$ if and only if there exist isomorphisms $\mathcal{E}^i \xrightarrow{\sim} \mathcal{F}^i$ over $\hat{D}_R$ for $1 \le i \le n$ making an appropriate diagram commute. Equality may be checked after passing to an étale cover of R, so by [HR20, Lemma 3.4] we may assume that the $\mathcal{E}^i$ and $\mathcal{F}^i$ are trivializable. We can then choose equivalent representatives in $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})$ such that $\mathcal{E}^i = \mathcal{F}^i = \mathcal{E}^0$ for all i. Note that for all $P \in \{\emptyset, \underline{0}, \underline{x}, \underline{x} \cup \underline{0}\}$, $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P)(\hat{D}^\circ_P)$ is a subgroup of $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\hat{D}^\circ_R) := \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(R(\!(D)\!))$. Assuming that $\mathcal{E}^i = \mathcal{F}^i = \mathcal{E}^0$ for all i, it is straightforward to check that $\{(\mathcal{E}^i, \alpha_i)\}_{1 \le i \le n} = \{(\mathcal{F}^i, \beta_i)\}_{1 \le i \le n}$ in $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})$ if and only if

$$(\beta_1 \circ \cdots \circ \beta_i)^{-1}(\alpha_1 \circ \cdots \circ \alpha_i) \in \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\hat{D}_R), \quad \text{for all } 1 \le i \le n.$$

On the other hand, $\{(\mathcal{E}^0, \alpha)\} = \{(\mathcal{E}^0, \beta)\}$ in $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_1 \cup \cdots \cup P_i)$ if and only if $\beta^{-1}\alpha \in \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\hat{D}_R)$. Hence $\{(\mathcal{E}^i, \alpha_i)\}_{1 \le i \le n} = \{(\mathcal{F}^i, \beta_i)\}_{1 \le i \le n}$, and f is a closed embedding. $\qquad\square$

We use the notation $(-)^0$, $(-)^+$, and $(-)^-$ to denote the fixed points, attractors, and repellers for an ind-scheme with $\mathbf{G}_{\mathrm{m}}$-action in the sense of [Ric19, HR20]. The cocharacter λ induces a $\mathbf{G}_{\mathrm{m}}$-action on G by conjugation, $t \cdot g = \lambda(t)g\lambda(t)^{-1}$. Then $P^+ = G^+$ and $P^- = G^-$ are opposite parabolic subgroups of G with Levi factor $M = P^+ \cap P^- = G^0$.

Lemma 4.23. *Consider the $\mathbf{G}_{\mathrm{m},C}$-action on $\mathcal{G}_{\mathbf{f}}$ induced by $\lambda \in X_*(T)$.*

(1) The functors $\mathcal{P}_{\mathbf{f}}^+ := \mathcal{G}_{\mathbf{f}}^+$, $\mathcal{P}_{\mathbf{f}}^- := \mathcal{G}_{\mathbf{f}}^-$ and $\mathcal{M}_{\mathbf{f}} := \mathcal{G}_{\mathbf{f}}^0$ are smooth, affine C-groups with geometrically connected fibers.

(2) The Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{P}_{\mathbf{f}}^\pm,I}$ is represented by separated ind-finite type C-scheme for all I.

(3) Furthermore, $\mathcal{M}_{\mathbf{f}}$ is the Bruhat–Tits group scheme for M associated to the facet $\mathbf{f}$ as in Section 4.2.1.

Proof. We may assume that $S = \mathrm{Spec}\,\mathbf{Z}$ by [Ric19, Corollary 1.16]. The geometric properties in (1) follow from [CGP10, Proposition 2.1.8]. By Proposition 4.10 this also implies (2). For (3), we note that the apartments $\mathscr{A}(G,T)$ and $\mathscr{A}(M,T)$ are naturally identified. The facet $\mathbf{f}$ lies in a unique facet for M, which we denote in the same way for simplicity. Then for each prime p the fiber $\mathcal{M}_{\mathbf{f}} \times_C \mathbf{Z}_p[t]$ is a parahoric $\mathbf{Z}_p[t]$-group scheme for M by [HR21, Lemma 5.16]. This uses the unique characterization of these parahoric group schemes in [PZ13, Theorem 4.1, Corollary 4.2]. There is a similar fiberwise characterization of parahoric $\mathbf{Z}[t]$-group schemes [Lou23, Theorem 1.3], so this implies that $\mathcal{M}_{\mathbf{f}}$ is a parahoric $\mathbf{Z}[t]$-group scheme. $\qquad\square$

We will often abbreviate $P^+ = P$ and $\mathcal{P}_{\mathbf{f}}^+ = \mathcal{P}_{\mathbf{f}}$. The follow lemma describes the fibers of $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})^0$ and $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})^\pm$.

Lemma 4.24. *Consider the following maps associated to the $\mathbf{G}_{\mathrm{m}}$-action on $\mathrm{Fl}_{\mathbf{f}}$.*

$$(\mathrm{Fl}_{\mathbf{f}})^0 \xleftarrow{\ q^\pm\ } (\mathrm{Fl}_{\mathbf{f}})^\pm \xrightarrow[\ p^\pm\ ]{\ \iota\ } \mathrm{Fl}_{\mathbf{f}}$$

(1) The map $q^{\pm}$ is ind-affine, has geometrically connected fibers, and induces a bijection on connected components $\pi_0((\mathrm{Fl}_{\mathbf{f}})^{\pm}) \cong \pi_0((\mathrm{Fl}_{\mathbf{f}})^0)$.

(2) The map $p^{\pm}$ is schematic and restricts to an immersion on each connected component of $(\mathrm{Fl}_{\mathbf{f}})^{\pm}$. The map $p^{\pm}$ is also bijective on k-points for every field k.

(3) The map ι is a closed immersion.

(4) If S is an algebraically closed field then there is a canonical isomorphism $\pi_0((\mathrm{Fl}_{\mathbf{f}})^0) = W_{M,\mathrm{aff}} \backslash W / W_{\mathbf{f}}$ where $W_{M,\mathrm{aff}}$ is the affine Weyl group of M with respect to $\mathbf{a}_0$.

(5) Suppose that $\mathbf{f}$ is in the closure of $\mathbf{a}_0$, and that λ is a regular cocharacter. Then $\pi_0((\mathrm{Fl}_{\mathbf{f}})^0) = W / W_{\mathbf{f}}$ and there is a canonical isomorphism of reduced loci

$$(\mathrm{Fl}_{\mathbf{f}})^0_{\mathrm{red}} = \bigsqcup_{W/W_{\mathbf{f}}} S.$$

Proof. Parts (1) and (3) are generalities about étale-locally linearizable $\mathbf{G}_{\mathrm{m}}$-actions [HR21, Theorem 2.1]. For part (2), the map $p^{\pm}$ is schematic again by [HR21, Theorem 2.1]. To see that $p^{\pm}$ restricts to an immersion on each connected component of $(\mathrm{Fl}_{\mathbf{f}})^{\pm}$ we use the fact that the $\mathbf{G}_{\mathrm{m}}$-action is Zariski-locally linearizable, cf. also the proof of [CvdHS22, Lemma 3.5]. The claim then follows since on an affine $\mathbf{G}_{\mathrm{m},C}$-scheme Y the map $Y^+ \to Y$ is an closed immersion [Ric19, Lemma 1.9], and the map $q^{\pm}$ commutes with taking $\mathbf{G}_{\mathrm{m}}$-equivariant affine Zariski covers [Ric19, Lemma 1.11]. Finally, $p^{\pm}$ is a monomorphism by [Ric19, Remark 1.19] and it is surjective on k-points by [Sta24, Tag 0BXZ].

For part (4), let M_{sc} be the simply connected cover of M_{der} and let $P_{\mathrm{sc}}^{\pm} = M_{\mathrm{sc}} \ltimes U^{\pm}$ where $P^{\pm} = M \ltimes U^{\pm}$. By the argument in [AGLR22, Theorem 5.2] (see also [HR21, §4]), when S is an algebraically closed field the reduced locus of $(\mathrm{Fl}_{\mathbf{f}})^{\pm}$ agrees with the reduced locus of the disjoint union of the connected semi-infinite orbits $LP_{\mathrm{sc}} \cdot \dot{w}$ where $w \in W_{M,\mathrm{aff}} \backslash W / W_{\mathbf{f}}$ and $\dot{w}$ is any lift to $LG(S)$ (and the reduced locus of $(\mathrm{Fl}_{\mathbf{f}})^0$ agrees with the $LM_{\mathrm{sc}} \cdot \dot{w}$).

For part (5), we have $M = T$ so that $W_{M,\mathrm{aff}}$ is trivial in part (4). By [Ric19, Corollary 1.16] we may assume that $S = \mathrm{Spec}\,\mathbf{Z}$. Then there is a map $f : \sqcup_{W/W_{\mathbf{f}}} \mathrm{Spec}\,\mathbf{Z} \to (\mathrm{Fl}_{\mathbf{f}})^0_{\mathrm{red}}$ obtained by taking representatives of elements of W in $LG(\mathbf{Z})$. The map f is a closed immersion by part (3), so by the proof of part (4) it is an isomorphism on reduced loci over fields. Next, we claim that $(\mathrm{Fl}_{\mathbf{f}})^0_{\mathrm{red}}$ is ind-smooth over $\mathbf{Z}$. If $\mathbf{f} = \mathbf{f}_0$ then ind-smoothness follows since $(\mathrm{Gr}_G)^0 = \mathrm{Gr}_T$ by [HR20, Theorem 3.17]. If $\mathbf{c}$ is any facet in the closure of another facet $\mathbf{c}'$ then the argument in [HR21, Lemma 4.9] shows that the natural map $(\mathrm{Fl}_{\mathbf{c}'})^0 \to (\mathrm{Fl}_{\mathbf{c}})^0$ is smooth and surjective. This uses that $\mathrm{Fl}_{\mathbf{c}'} \to \mathrm{Fl}_{\mathbf{c}}$ is smooth (it is a torsor for a homogeneous space for G) and a generality about $\mathbf{G}_{\mathrm{m}}$-actions over a general base [HR21, Lemma 2.2]. Taking $\mathbf{c}' = \mathbf{a}_0$ and $\mathbf{c} = \mathbf{f}_0$ shows that $(\mathrm{Fl}_{\mathbf{a}_0})^0_{\mathrm{red}}$ is ind-smooth, and then taking $\mathbf{c} = \mathbf{f}$ shows that $(\mathrm{Fl}_{\mathbf{f}})^0_{\mathrm{red}}$ is ind-smooth by [Sta24, Tag 02K5]. Hence the $\mathbf{Q}$-fiber of $(\mathrm{Fl}_{\mathbf{f}})^0_{\mathrm{red}}$ is dense. Moreover, after restricting to any closed Schubert cell in $\mathrm{Fl}_{\mathbf{f}}$, the reduced fixed point locus is a disjoint union of finitely smooth (in particular, normal) integral subschemes, each admitting a birational map from $\mathrm{Spec}\,\mathbf{Z}$ that is bijective on points. Thus, f is an isomorphism by Zariski's main theorem [Sta24, Tag 05K0]. $\qquad\square$

The following theorem due to Haines–Richarz describes the geometry of the $\mathbf{G}_{\mathrm{m},C^I}$-action on $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$.

Theorem 4.25. *We have a commutative diagram with the following properties.*

$$
\begin{array}{ccccc}
\mathrm{Gr}_{\mathcal{M}_{\mathbf{f}},I} & \xleftarrow{\;q_{\mathbf{f}}^{\pm}\;} & \mathrm{Gr}_{\mathcal{P}_{\mathbf{f}}^{\pm},I} & \xrightarrow{\;p_{\mathbf{f}}^{\pm}\;} & \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I} \\
\downarrow{\scriptstyle\iota^0} & & \downarrow{\scriptstyle\iota^{\pm}} & & \downarrow{\scriptstyle\mathrm{id}} \\
(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I})^0 & \xleftarrow{\;q^{\pm}\;} & (\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I})^{\pm} & \xrightarrow{\;p^{\pm}\;} & \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}
\end{array}
$$

(1) *The maps $q_{\mathbf{f}}^{\pm}$ and $p_{\mathbf{f}}^{\pm}$ are induced by the natural maps $\mathcal{M}_{\mathbf{f}} \leftarrow \mathcal{P}_{\mathbf{f}}^{\pm} \to \mathcal{G}_{\mathbf{f}}$ associated to the $\mathbf{G}_{\mathrm{m},C}$-action on $\mathcal{G}_{\mathbf{f}}$.*

(2) *The maps $q^{\pm}$ and $p^{\pm}$ are the natural maps associated to the $\mathbf{G}_{\mathrm{m},C^I}$-action on $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$.*

(3) *The map $q^{\pm}$ is ind-affine, has geometrically connected fibers, and induces a bijection on connected components $\pi_0((\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I})^{\pm}) \cong \pi_0((\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I})^0)$.*

(4) *The map $p^{\pm}$ is schematic and restricts to an immersion on each connected component of $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I})^{\pm}$.*

(5) *The restrictions of ι^0 and $\iota^{\pm}$ to $(C_{\eta})^I$ are isomorphisms.*

(6) *Furthermore, the natural map $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I})^0 \to \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$ is a closed immersion.*

Proof. Since the action of $\mathbf{G}_{\mathrm{m},C}$ on $\mathrm{Gr}_{\mathcal{M}_{\mathbf{f}}}$ is trivial the natural map $\mathrm{Gr}_{\mathcal{M}_{\mathbf{f}}} \to \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}$ factors through $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})^0$. This defines ι^0. The Rees construction explained in [HR20, §3.3.1], which uses the description of $\mathrm{Gr}_{\mathcal{P}_{\mathbf{f}}^{\pm}}$ as a moduli space of $\mathcal{P}_{\mathbf{f}}^{\pm}$-torsors, gives a canonical isomorphism $\mathrm{Gr}_{\mathcal{P}_{\mathbf{f}}^{\pm}} \cong (\mathrm{Gr}_{\mathcal{P}_{\mathbf{f}}^{\pm}})^{\pm}$ (it is assumed in loc. cit. that $\mathcal{G}_{\mathbf{f}}$ is reductive but this assumption is only necessary for showing that ι^0 and $\iota^{\pm}$ are isomorphisms). Then $\iota^{\pm}$ is defined by composing with the natural map $(\mathrm{Gr}_{\mathcal{P}_{\mathbf{f}}^{\pm}})^{\pm} \to (\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}^{\pm}})^{\pm}$. This constructs the commutative diagram with the maps in (1) and

33

(2). Parts (3), (4), and (6) are generalities about $\mathbf{G}_\mathrm{m}$-actions and can be proved as in Lemma 4.24. Part (5) is a special case of [HR20, Theorem 3.17]. □

4.4. Whitney–Tateness of Beilinson–Drinfeld Grassmannians.

4.4.1. *Stratifications.* From now on we assume that all facets are contained in the closure of $\mathbf{a}_0$. Then we may consider the stratification of Gr by the Gr^μ for $\mu \in X_*(T)^+$, and the stratification of $\mathrm{Fl}_\mathbf{f}$ by the $\mathrm{Fl}_w^\circ(\mathbf{f}, \mathbf{f})$ for $w \in W_\mathbf{f} \backslash W / W_\mathbf{f}$. We build a stratification of $\mathrm{Gr}_{\mathcal{G}_\mathbf{f}}(P)$ using these strata as follows.

Definition and Lemma 4.26. Let $P \in \{\emptyset, \underline{0}, \underline{x}, \underline{x} \cup \underline{0}\}$. We define a stratification of $\mathrm{Gr}_{\mathcal{G}_\mathbf{f}}(P)$ by taking the union of the following strata over C_η and C_s with respect to the isomorphisms in Lemma 4.12, where $\mu \in X_*(T)^+$ and $w \in W_\mathbf{f} \backslash W / W_\mathbf{f}$.

P	$\mathrm{Gr}_{\mathcal{G}_\mathbf{f}}(P)_\eta$	$\mathrm{Gr}_{\mathcal{G}_\mathbf{f}}(P)_s$
$\emptyset$	C_η	C_s
$\underline{0}$	$\mathrm{Fl}_w^\circ(\mathbf{f}, \mathbf{f}) \times C_\eta$	$\mathrm{Fl}_w^\circ(\mathbf{f}, \mathbf{f}) \times C_s$
$\underline{x}$	$\mathrm{Gr}^\mu \times C_\eta$	$\mathrm{Fl}_w^\circ(\mathbf{f}, \mathbf{f}) \times C_s$
$\underline{x} \cup \underline{0}$	$\mathrm{Fl}_w^\circ(\mathbf{f}, \mathbf{f}) \times \mathrm{Gr}^\mu \times C_\eta$	$\mathrm{Fl}_w^\circ(\mathbf{f}, \mathbf{f}) \times C_s$

Proof. We must show that the closure of a stratum is a union of strata. In general, the strata are the orbits for the actions of $(L^+\mathcal{G}_\mathbf{f}^{\mathrm{BD}})_\eta$ and $(L^+\mathcal{G}_\mathbf{f}^{\mathrm{BD}})_s$. The closures of the strata over C_η are thus stable under the action of $L^+\mathcal{G}_\mathbf{f}^{\mathrm{BD}}$, since this group is pro-smooth. Hence the fibers over C_s of these closures are also stable under the action of $(L^+\mathcal{G}_\mathbf{f}^{\mathrm{BD}})_s$. □

Remark 4.27. By Lemma 4.12 and [CvdHS22, Proposition 3.7] the stratifications of $\mathrm{Gr}_{\mathcal{G}_\mathbf{f}}(P)_\eta$ and $\mathrm{Gr}_{\mathcal{G}_\mathbf{f}}(P)_s$ in Definition and Lemma 4.26 are anti-effective universally Whitney–Tate. In particular, we have the categories $\mathrm{DTM}(\mathrm{Gr}_{\mathcal{G}_\mathbf{f}}(P)_\eta)^{\mathrm{c,anti}}$ and $\mathrm{DTM}(\mathrm{Gr}_{\mathcal{G}_\mathbf{f}}(P)_s)^{\mathrm{c,anti}}$ of compact, anti-effective stratified Tate motives.

To show that categories of Tate motives glue over C, we will use hyperbolic localization in the next section. Example 4.29 below is illustrative of the general case; first we need the following result.

Lemma 4.28. *Suppose the action of $L^+ G_{\mathbf{f}'}$ on $\mathrm{Fl}_w^\circ(\mathbf{f}', \mathbf{f})$ factors through some jet group $H := L^{+,(m)} G_{\mathbf{f}'}$. Let $H_w \subset H$ be the stabilizer of the basepoint of $\mathrm{Fl}_w^\circ(\mathbf{f}', \mathbf{f})$ and let $e \colon S/H_w \to H \backslash \mathrm{Fl}_w^\circ(\mathbf{f}', \mathbf{f})$ be the induced map of pre-stacks.*
Then there are canonical equivalences

$$\mathrm{DM}(L^+ G_{\mathbf{f}'} \backslash \mathrm{Fl}_w^\circ(\mathbf{f}', \mathbf{f})) = \mathrm{DM}(H \backslash \mathrm{Fl}_w^\circ(\mathbf{f}', \mathbf{f})) \xrightarrow[\sim]{e^!} \mathrm{DM}(S/H_w).$$

These equivalences restrict to DTM. Furthermore, some H-equivariant Tate motive in $\mathrm{DTM}(\mathrm{Fl}_w^\circ(\mathbf{f}', \mathbf{f}))$ is anti-effective iff its *-pullback to the basepoint $S \subset \mathrm{Fl}_w^\circ(\mathbf{f}', \mathbf{f})$ is anti-effective. Finally, if S satisfies Beilinson–Soulé vanishing condition then forgetting the equivariance induces an equivalence $\mathrm{MTM}(S/H_w) = \mathrm{MTM}(S)$.

Proof. The first equivalence follows as in Remark 4.14. For the second we recall that $\mathrm{Fl}_w^\circ(\mathbf{f}', \mathbf{f}) = (H/H_w)_{\mathrm{Zar}}$ by [CvdHS22, Proposition 3.2], so that e is an equivalence after Zariski-sheafification. Since DM is invariant under Zariski-sheafification we conclude that $e^!$ is an equivalence, cf. [RS20, Lemma 2.2.21]. The same arguments as in [RS20, Propositions 3.1.23, 3.1.27], [RS] show that these equivalences respect $\mathrm{DTM}^{(\mathrm{anti})}$ (the only difference is that we use Zariski instead of étale quotients here). Finally, the equivalence $\mathrm{MTM}(S/H_w) = \mathrm{MTM}(S)$ follows as in [RS20, Proposition 3.2.23] using that H_w is cellular and fiberwise connected by [RS20, Lemma 4.3.7], cf. also [CvdHS22, Proposition 3.4]. □

Example 4.29. Let us recall the simplest non-trivial example of such degenerations, which already appeared in [Gai01, §1.2.3]. Namely, let $G = \mathrm{PGL}_2$ and let μ be the unique minuscule dominant cocharacter. Then $\mathrm{Gr}^\mu \cong \mathbf{P}^1$, and the reduced special fiber of $\overline{\mathrm{Gr}^\mu \times \mathbf{G}_\mathrm{m}} \subset \mathrm{Gr}_{\mathcal{G}_{\mathbf{a}_0}}$ is isomorphic to two copies of $\mathbf{P}^1$ which intersect transversally in a single point 0, which we identify with zero in both copies of $\mathbf{P}^1$. Let us denote this special fiber by X, and the points at infinity by ∞_1 and ∞_2.

Let $j \colon \mathrm{Gr}^\mu \times \mathbf{G}_\mathrm{m} \to \overline{\mathrm{Gr}^\mu \times \mathbf{G}_\mathrm{m}} \leftarrow X \colon i$ be the inclusions. To show that $i^* j_* \mathbf{Z}$ is Tate, we consider the semi-infinite orbits on Fl with respect to a regular dominant cocharacter. The semi-infinite orbits divide X into three subschemes: a copy of $\mathbf{A}^1$ with origin 0, a copy of $\mathbf{A}^1$ with origin identified with ∞_1 (up to relabelling), and a copy of S identified with ∞_2. Switching to an anti-dominant regular cocharacter swaps ∞_1 and ∞_2.

Note that hyperbolic localization preserves Tate motives on both Gr and Fl, and it commutes with $i^* j_*$. Additionally, the hyperbolic localization of $i^* j_* \mathbf{Z}$ along the semi-infinite orbit identified with ∞_2 is just the stalk $(i^* j_* \mathbf{Z})_{\infty_2}$, which is therefore Tate. By using an anti-dominant regular cocharacter, $(i^* j_* \mathbf{Z})_{\infty_1}$ is also Tate. Now by Iwahori-equivariance and Lemma 4.28, $(i^* j_* \mathbf{Z})_{X \backslash \{0\}}$ is Tate. It remains to show that $(i^* j_* \mathbf{Z})_0$ is Tate. Viewing $(i^* j_* \mathbf{Z})_0$ as a motive supported on X, this stalk is trivially identified with the hyperbolic localization of $(i^* j_* \mathbf{Z})_0$

along the copy of $\mathbf{A}^1$ with origin 0. The advantage is that $(i^*j_*\mathbf{Z})_0$ is the cofiber of $(i^*j_*\mathbf{Z})_{X\setminus\{0\}} \to i^*j_*\mathbf{Z}$, and both of these terms have Tate hyperbolic localizations. Thus, $(i^*j_*\mathbf{Z})_0$ is Tate.

4.4.2. *Hyperbolic localization on affine flag varieties.* Let us fix a regular cocharacter $\lambda \in X_*(T)$. As in Section 4.3 this induces an action of $\mathbf{G}_{\mathrm{m},C}$ on $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}$ and $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{\mathrm{BD}}$.

Lemma 4.30. *The reduced loci of the fixed points $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})^0$ and $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{\mathrm{BD}})^0$ can be described as follows.*

(1) We have canonical isomorphisms
$$(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})^0_{\eta,\mathrm{red}} \cong \coprod_{X_*(T)} C_\eta, \quad (\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{\mathrm{BD}})^0_{\eta,\mathrm{red}} \cong \coprod_{X_*(T)\times W/W_{\mathbf{f}}} C_\eta.$$

(2) The reduced loci of both $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})^0_s$ and $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{\mathrm{BD}})^0_s$ are canonically isomorphic to $\coprod_{W/W_{\mathbf{f}}} S$.

(3) The reduced closure of any connected component of $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})^0_\eta$ (resp. $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{\mathrm{BD}})^0_\eta$) is canonically isomorphic to C.

Proof. Parts (1) and (2) follow from Lemma 4.24(5). For part (3), note that $(\mathrm{Gr}_{T_C})_{\mathrm{red}} = \coprod_{X_*(T)} C$. Hence the natural map $(\mathrm{Gr}_{T_C})_{\mathrm{red}} \to \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}$ admits a section over each connected component of the source, and thus it is a closed immersion by ind-properness. Then by Theorem 4.25 we have a closed embedding $(\mathrm{Gr}_{T_C})_{\mathrm{red}} \to (\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})^0$. This implies that (3) holds for $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}$. To prove that (3) holds for $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{\mathrm{BD}}$, we consider the convolution map $m\colon \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{x},\underline{0}) \to \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{\mathrm{BD}}$, which is an isomorphism over C_η. The reduced closure of a connected component of $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{\mathrm{BD}})^0_\eta$ inside $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{x},\underline{0})$ is isomorphic to C, as can be seen using the isomorphism $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{x},\underline{0}) \cong \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}^{(\infty)}}(\underline{x}) \times^{\mathcal{L}^+\mathcal{G}_{\mathbf{f}}^{\mathrm{BD}}} \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{0})$ and the case of $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{x}) = \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}$. Hence the reduced closure inside $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{\mathrm{BD}}$ is an integral C-scheme C' whose structure map $C' \to C$ is birational. The map $C' \to C$ is also bijective on points by considering the case where S is an algebraically closed field. Since C is normal we have $C' = C$ by Zariski's main theorem. $\qquad\square$

Remark 4.31. By Lemma 4.24(5) and Lemma 4.30, when $\underline{P}$ is a singleton there is a canonical stratification of $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})^0$ by copies of C_η and S (the latter strata live over C_s). This stratification is universally anti-effective Whitney–Tate, since the same is true for the stratification $C = \mathbf{A}^1_S = \mathbf{G}_{\mathrm{m},S} \sqcup S$. Indeed, for the inclusions $j\colon \mathbf{G}_{\mathrm{m},S} \to \mathbf{A}^1_S \leftarrow S\colon i$ we have $i^*j_*\mathbf{Z} = \mathbf{Z} \oplus \mathbf{Z}(-1)[-1]$ by relative purity. Thus, when $\underline{P}$ is a singleton we may consider the category $\mathrm{DTM}(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})^0)^{\mathrm{anti}}$.

We now prove that hyperbolic localization preserves Tate motives on $\mathrm{Fl}_{\mathbf{f}}$. In the proof we will use the Demazure resolutions, which we now recall. Let $w \in W_{\mathrm{aff}}$ and let $\dot{w} = s_1 \cdots s_n$ be a reduced decomposition by simple reflections in the walls of $\mathbf{a}_0$. Let $\mathcal{P}_i \subset LG$ be the minimal parahoric subgroup containing $\mathcal{I}$ and a representative for s_i (see [PR08, §8.a] for more details). The Demazure scheme associated to $\dot{w}$ is $D(\dot{w}) = \mathcal{P}_1 \times^{\mathcal{I}} \cdots \times^{\mathcal{I}} \mathcal{P}_n/\mathcal{I}$. Multiplication in LG induces an $\mathcal{I}$-equivariant map $D(\dot{w}) \to \mathrm{Fl}_w(\mathbf{a}_0,\mathbf{a}_0)$ which is an isomorphism over $\mathrm{Fl}^\circ_w(\mathbf{a}_0,\mathbf{a}_0)$, cf. [Fal03, Definition 5 ff.]. The $\mathbf{G}_{\mathrm{m}}$-action on $D(\dot{w})$ induced by the regular cocharacter $\lambda \in X_*(T)$ has the following properties.

Lemma 4.32. *Let $w \in W_{\mathrm{aff}}$ and let $\dot{w} = s_1 \cdots s_n$ be a reduced decomposition.*

(1) The reduced locus of the fixed points $D(\dot{w})^0 \subset D(\dot{w})$ is a disjoint union of 2^n copies of S.

(2) The natural map $\tilde{p}^+\colon D(\dot{w})^+ \to D(\dot{w})$ is bijective on points and restricts to an immersion on connected components.

(3) The restriction of the natural map $\tilde{q}^+\colon D(\dot{w})^+ \to D(\dot{w})^0$ to reduced loci is a disjoint union of affine spaces.

(4) There exists a filtrable decomposition of $D(\dot{w})$ by the connected components of $D(\dot{w})^+$.

Proof. The product of the partial convolution maps $(p_1,\ldots,p_n) \mapsto p_1 \cdots p_i$ induces a $\mathbf{G}_{\mathrm{m}}$-equivariant closed embedding $D(\dot{w}) \to \mathrm{Fl}^n$. Hence the fact that $\tilde{p}^+$ restricts to an immersion on connected components follows from the case of Fl as in Lemma 4.24(2). The map $\tilde{p}^+$ is a bijection on points because $D(\dot{w})$ is projective, so in particular all maps $\mathbf{G}_{\mathrm{m}} \to D(\dot{w})$ defined over a field extend to $\mathbf{A}^1$ (and it is a monomorphism because $D(\dot{w})$ is separated). This proves (2). When $S = k$ is an algebraically closed field the remaining parts (except for the precise number of fixed points) are true more generally for smooth projective $\mathbf{G}_{\mathrm{m}}$-schemes with isolated fixed points by work of Białynicki-Birula [BB73, BB76]. Rather than generalizing these results to an arbitrary base we will give a direct proof for $D(\dot{w})$ following [CX22, Theorem 2.31].

First, we note that $D(\dot{s}_i) \cong \mathbf{P}^1$. Since λ is regular then $D(\dot{s}_i)^0_{\mathrm{red}} \cong S \sqcup S$, corresponding to the two fixed points $\{e\}, \{s_i\}$ where e is the basepoint. Furthermore, $D(\dot{s}_i)^+_{\mathrm{red}} \cong \mathbf{A}^1_S \sqcup S$. The connected component of $D(\dot{s}_i)^+_{\mathrm{red}}$ which is isomorphic to $\mathbf{A}^1_S$ attracts toward $\{s_i\}$ (resp. $\{e\}$) if λ attracts (resp. repels) the affine root group corresponding to s_i. This proves the remaining parts in the case $l(w) = 1$.

For the general case we induct on $l(w)$ using the isomorphism $D(\dot{s}) = D(\dot{t}) \times^{\mathcal{I}} D(\dot{s}_n)$, where $\dot{t} = s_1 \cdots s_{n-1}$. The projection $\pi\colon D(\dot{s}) \to D(\dot{t})$ is a $\mathbf{G}_{\mathrm{m}}$-equivariant, Zariski-locally trivial $\mathbf{P}^1$-bundle. By applying the case $l(w) = 1$ to $D(\dot{s}_n)$, it follows that for each connected component $Y_i \subset D(\dot{t})^+_{\mathrm{red}}$, the fiber $\pi^{-1}(Y_i)$ is set-theoretically the union of

two connected components X_i^1, X_i^2 of $D(\dot{s})_{\text{red}}^+$. One of these connected components, say X_i^1, maps isomorphically on reduced loci onto Y_i by the map π. The other connected component X_i^2 has the structure of a line bundle over Y_i on reduced loci. Now (1) follows by induction. It also follows by induction that $\tilde{q}^+$ is a disjoint union of vector bundles. To see that these bundles are trivial we can reduce by base change to the case $S = \mathbf{Z}$, where it follows from the Quillen–Suslin theorem. This proves (3). Finally, to prove (4) we may suppose by induction that we have a filtrable decomposition by closed subschemes $Z_1 \subset \cdots \subset Z_{2^{n-1}} = D(\dot{t})$ with relative complements $Z_i \setminus Z_{i-1} = Y_i$. Then it is easy to see that we have a filtrable decomposition of $D(\dot{s})$ with relative complements $X_1^1, X_1^2, \ldots, X_{2^{n-1}}^1, X_{2^{n-1}}^2$. $\square$

Proposition 4.33. *In the notation of Theorem* 4.25, *hyperbolic localization induces a well-defined functor*

$$q_!^+(p^+)^* \colon \mathrm{DTM}((\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_\eta)^{\mathrm{c,anti}} \to \mathrm{DTM}((\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_\eta^0)^{\mathrm{c,anti}}.$$

The same is true more generally for $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P})_\eta$ *when* $\underline{P}$ *is a singleton, and also for the respective fibers over* C_s.

Proof. Since $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{\mathrm{BD}})_\eta = \mathrm{Gr} \times \mathrm{Fl}_{\mathbf{f}} \times C_\eta$, we may deduce the case of $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^{\mathrm{BD}}$ from the case of $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}$ using the Künneth formula [JY21a, Theorem 2.4.6] for !-pushforward. By the description of the fibers over C_η and C_s in Lemma 4.12, we are reduced to proving a similar statement statement about hyperbolic localization on $\mathrm{Fl}_{\mathbf{f}}$. Recall that $(\mathrm{Fl}_{\mathbf{f}})_{\mathrm{red}}^0$ is a disjoint union of copies of S by Lemma 4.24(5). As the stratification of $\mathrm{Fl}_{\mathbf{f}}$ by the $\mathrm{Fl}_w^\circ(\mathbf{a}_0, \mathbf{f})$ is finer than the stratification by the $\mathrm{Fl}_w^\circ(\mathbf{f}, \mathbf{f})$, and taking fixed points and attractors is compatible with passing to equivariant closed subschemes [Dri18, Lemma 1.4.9], we can restrict ourselves to motives supported on $\mathrm{Fl}_w(\mathbf{a}_0, \mathbf{f})$, for some $w \in W/W_{\mathbf{f}}$.

Let $\tilde{w} \in W$ be the unique lift of w such that $\mathrm{Fl}_{\tilde{w}}^\circ(\mathbf{a}_0, \mathbf{a}_0) \cong \mathrm{Fl}_w^\circ(\mathbf{a}_0, \mathbf{f})$ via the canonical projection. After applying a $\mathbf{G}_{\mathrm{m}}$-equivariant right translation by an element of Ω (which we may do since Ω normalizes $\mathcal{I}$) we may assume that $\tilde{w} \in W_{\mathrm{aff}}$. To ease the notation we simply write w instead of $\tilde{w}$ for the rest of the proof. Chose a reduced decomposition $\dot{w} = s_1 \cdots s_n$. Then we have a $\mathbf{G}_{\mathrm{m}}$-equivariant Demazure resolution $m \colon D(\dot{w}) \to \mathrm{Fl}_w(\mathbf{a}_0, \mathbf{f})$, which is an isomorphism over $\mathrm{Fl}_w^\circ(\mathbf{a}_0, \mathbf{f})$.

We claim that $m_! \mathbf{Z} \in \mathrm{DTM}(\mathrm{Fl}_w(\mathbf{a}_0, \mathbf{f}))^{\mathrm{c,anti}}$. To prove this, first suppose that $\mathbf{f} = \mathbf{a}_0$. Then as in the proof of [Fal03, Lemma 9], m factors as a composition of $\mathcal{I}$-equivariant maps with fibers trivial or isomorphic to $\mathbf{P}^1$. The structure map $f \colon \mathbf{P}_S^1 \to S$ satisfies $f_!(\mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}(-1)[-2]$, which is compact and anti-effective Tate. By $\mathcal{I}$-equivariance and Lemma 4.28, the properties of Tateness, compactness, and anti-effectiveness may be checked fiberwise. Thus, these three properties follow from the cohomology of $\mathbf{P}^1$. For general $\mathbf{f}$ it remains to show that pushforward along $\mathrm{Fl}_w(\mathbf{a}_0, \mathbf{a}_0) \to \mathrm{Fl}_w(\mathbf{a}_0, \mathbf{f})$ preserves compact anti-effective stratified Tate motives. This follows by similar reasoning using that the reduced fibers of this map are unions of Schubert varieties in classical flag varieties, which in particular are stratified by affine spaces (the Bruhat decomposition), cf. the proof of [CX22, Lemma 2.13].

We now proceed similarly to [CvdHS22, Proposition 3.11], using an induction on $l(w)$. Let $T \subseteq \mathrm{Fl}_w(\mathbf{a}_0, \mathbf{f})^+$ be a connected component, and denote by $f \colon \tilde{T} := m^{-1}(T) \to T$ the restriction of m. Then $\tilde{T}$ is set-theoretically a union of connected components of $D(\dot{w})^+$. Consider the open immersion $j \colon T \cap \mathrm{Fl}_w^\circ(\mathbf{a}_0, \mathbf{f}) \subseteq T$ with closed complement $i \colon Z \subseteq T$. By induction on $l(w)$ we need only show that $q_!^+(j_! \mathbf{Z}) \in \mathrm{DTM}(\mathrm{Fl}_w(\mathbf{a}_0, \mathbf{f})^0)^{\mathrm{c,anti}}$. As m is an isomorphism over $\mathrm{Fl}_w^\circ(\mathbf{a}_0, \mathbf{f})$, applying $q_!^+$ to the localization sequence associated to $f_!(\mathbf{Z}_{\tilde{T}})$ gives an exact triangle $q_!^+(j_! \mathbf{Z}) \to q_!^+ f_!(\mathbf{Z}_{\tilde{T}}) \to q_!^+ i_! i^* f_!(\mathbf{Z}_{\tilde{T}})$. Lemma 4.32 shows that $\tilde{T}$ has a filtrable decomposition with cells isomorphic to affine spaces. Since the structure map $g \colon \mathbf{A}_S^1 \to S$ satisfies $g_! \mathbf{Z} = \mathbf{Z}(-1)[-2]$, then $q_!^+ f_!(\mathbf{Z}_{\tilde{T}}) \in \mathrm{DTM}(\mathrm{Fl}_w(\mathbf{a}_0, \mathbf{f})^0)^{\mathrm{c,anti}}$. It remains to show that $q_!^+ i_! i^* f_!(\mathbf{Z}_{\tilde{T}}) \in \mathrm{DTM}(\mathrm{Fl}_w(\mathbf{a}_0, \mathbf{f})^0)^{\mathrm{c,anti}}$. By base change, $i_! i^* f_!(\mathbf{Z}_{\tilde{T}})$ is a $*$-restriction of $m_! \mathbf{Z}$. Since $m_! \mathbf{Z} \in \mathrm{DTM}(\mathrm{Fl}_w(\mathbf{a}_0, \mathbf{f}))^{\mathrm{c,anti}}$ then we conclude by induction, as Z is contained in a union of smaller affine Schubert cells. $\square$

Remark 4.34. Since hyperbolic localization commutes with base change, Proposition 4.33 is also true for the category $\mathrm{DTM}(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})^{\mathrm{c,anti}}$ (once we show that this category is well-defined).

4.4.3. *Whitney–Tateness.* By Lemma 4.24, $\pi_0((\mathrm{Fl}_{\mathbf{f}})^+) = W/W_{\mathbf{f}}$. Let $\mathcal{S}_w$ be the reduced locus of the corresponding connected component, which we view as a locally closed sub ind-scheme $\mathcal{S}_w \subset \mathrm{Fl}_{\mathbf{f}}$.

The following lemma is the equal characteristic analogue of [AGLR22, Lemma 5.3]. It can be proven in a similar way, but we sketch the proof for convenience of the reader.

Lemma 4.35. *For any* $w \in W/W_{\mathbf{f}}$, *there exists a regular cocharacter* $\mathbf{G}_{\mathrm{m}} \to T \subseteq G$ *such that for the induced* $\mathbf{G}_{\mathrm{m}}$-*action,* $\mathcal{S}_w \cap \mathrm{Fl}_w(\mathbf{a}_0, \mathbf{f}) \cong S$.

Proof. As in the proof of Proposition 4.33 we may assume that $\tilde{w} \in W_{\mathrm{aff}}$. Since the $\mathcal{I}$-orbit $\mathrm{Fl}_w^\circ(\mathbf{a}_0, \mathbf{f}) \subseteq \mathrm{Fl}_w(\mathbf{a}_0, \mathbf{f})$ is a $\mathbf{G}_{\mathrm{m}}$-stable open subscheme containing the fixed point of $\mathcal{S}_w$, we have $\mathcal{S}_w \cap \mathrm{Fl}_w(\mathbf{a}_0, \mathbf{f}) = \mathcal{S}_w \cap \mathrm{Fl}_w^\circ(\mathbf{a}_0, \mathbf{f})$. Fix a reduced decomposition $\dot{w} = s_1 \ldots s_n$, and for each $1 \leq i \leq n$ consider the unique positive affine root α_i corresponding to the wall separating the alcoves $s_1 \ldots s_{i-1}(\mathbf{a}_0)$ and $s_1 \ldots s_i(\mathbf{a}_0)$. By considering the Demazure resolution $D(\dot{w}) \to \mathrm{Fl}_w(\mathbf{a}_0, \mathbf{f})$, which is an isomorphism over $\mathrm{Fl}_w^\circ(\mathbf{a}_0, \mathbf{f})$, one can show that $\mathrm{Fl}_w^\circ(\mathbf{a}_0, \mathbf{f}) = L^+ \mathcal{U}_{\alpha_1} \cdots L^+ \mathcal{U}_{\alpha_i} \cdot w$ where the $\mathcal{U}_{\alpha_i}$ are the corresponding affine root groups of $G_{\mathbf{f}}$, cf. [AGLR22, (5.11)]. Now, [HN02, Corollary 5.6] shows that

there exists a regular cocharacter repelling all the $L^+\mathcal{U}_{\alpha_j}$. By the above, this cocharacter then also repels $\mathrm{Fl}_w^\circ(\mathbf{a}_0, \mathbf{f})$, and we conclude that $\mathcal{S}_w \cap \mathrm{Fl}_w(\mathbf{a}_0, \mathbf{f}) \cong S$. $\qquad\square$

For every regular cocharacter $\lambda \in X_*(T)$, we have the hyperbolic localization functor $(q_s^+)_!(p_s^+)^*u^!\colon \mathrm{DM}(\mathcal{I}\backslash\mathrm{Fl}_{\mathbf{f}}) \to \mathrm{DM}(\coprod_{W/W_{\mathbf{f}}} S)$ (the dependence on λ is not reflected in the notation). The following conservativity result, used in the proof of Theorem 4.37 below, is similar to [AGLR22, Propositions 6.4, 6.6].

Lemma 4.36. *Let $u^!\colon \mathrm{DM}(\mathcal{I}\backslash\mathrm{Fl}_{\mathbf{f}}) \to \mathrm{DM}(\mathrm{Fl}_{\mathbf{f}})$ be the forgetful functor.*

(1) Let $M \in \mathrm{DM}(\mathcal{I}\backslash\mathrm{Fl}_{\mathbf{f}})$ be a bounded motive. Then $u^!M$ is Tate (resp. compact, resp. anti-effective) if and only if $(q_s^+)_!(p_s^+)^*u^!M \in \mathrm{DM}(\coprod_{W/W_{\mathbf{f}}} S)$ is Tate (resp. compact, resp. anti-effective) for every regular cocharacter λ.

(2) Let $\phi\colon M_1 \to M_2$ be a map between bounded motives in $\mathrm{DM}(\mathcal{I}\backslash\mathrm{Fl}_{\mathbf{f}})$. Then ϕ is an isomorphism if and only if $(q_s^+)_!(p_s^+)^*u^!\phi$ is an isomorphism for every regular cocharacter λ.

Proof. We prove both parts by noetherian induction on the support of the motives in $\mathrm{DM}(\mathcal{I}\backslash\mathrm{Fl}_{\mathbf{f}})$, starting with part (1). One direction is proved in Proposition 4.33. For the other direction, let $\mathrm{Fl}_w^\circ(\mathbf{a}_0, \mathbf{f})$ be a maximal Schubert cell in the support of M. By Lemma 4.35 we may choose $\lambda \in X_*(T)$ such that $\mathcal{S}_w \cap \mathrm{Fl}_w(\mathbf{a}_0, \mathbf{f}) \cong S$. Then the hypothesis implies that the $*$-restriction of $i_w^*u^!M$ to this copy of S is Tate (resp. compact, resp. anti-effective), where $i_w\colon \mathrm{Fl}_w^\circ(\mathbf{a}_0, \mathbf{f}) \to \mathrm{Fl}_{\mathbf{f}}$ is the inclusion. Then we conclude by Lemma 4.28 that $i_w^*u^!M$ is Tate (resp. compact, resp. anti-effective). Now the motive M' defined by the fiber sequence $M' \to u^!M \to i_{w*}i_w^*u^!M$ is $\mathcal{I}$-equivariant, and $(q_s^+)_!(p_s^+)^*M'$ is Tate (resp. compact, resp. anti-effective) for all regular cocharacters λ. Thus, we may conclude the proof of (1) by induction. The proof of part (2) is similar, using conservativity of $u^!$ and the fact that an isomorphism over $\mathcal{S}_w \cap \mathrm{Fl}_w(\mathbf{a}_0, \mathbf{f}) \cong S$ spreads out to an isomorphism over $\mathrm{Fl}_w^\circ(\mathbf{a}_0, \mathbf{f})$ by $\mathcal{I}$-equivariance and Lemma 4.28. $\qquad\square$

We can now prove the Whitney–Tateness of $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P)$.

Theorem 4.37. *Let $\mathbf{f}$ be a facet contained in the closure of $\mathbf{a}_0$ and let $P \in \{\emptyset, \underline{0}, \underline{x}, \underline{x}\cup\underline{0}\}$.*

(1) The stratification of $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P)$ in Definition and Lemma 4.26 is anti-effective universally Whitney–Tate.

(2) The nearby cycles functor Υ_P in Definition 4.15 preserves locally compact anti-effective Tate motives.

(3) For $P_i \in \{\emptyset, \underline{0}, \underline{x}, \underline{x}\cup\underline{0}\}$ and $\mathcal{F}_i \in \mathrm{DTM}(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P_i))$, the Künneth map (4.6) is an isomorphism.

Proof. First, we note that if $P = \emptyset$ then $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\emptyset) = \mathbf{G}_{\mathrm{m}} \sqcup \mathbf{A}^0$ is universally anti-effective Whitney–Tate by a computation involving relative purity. Moreover, (2) follows in this case since $\Upsilon_\emptyset$ is trivial on the constant family C by Lemma 3.12. Here we use that $\mathrm{DTM}(C_\eta)^{\mathrm{c},(\mathrm{anti})}$ is generated (under finite colimits, retracts and shifts) by $\mathbf{Z}_{C_\eta}$ and its twists.

The proofs of (1) and (2) for $P \in \{\underline{0}, \underline{x}, \underline{x}\cup\underline{0}\}$ are essentially the same; for simplicity we will only write down the case where $P = \underline{x}$, so $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{P}) = \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}$. By [CvdHS22, Proposition 3.7] (and [CvdHS22, Corollary 3.10] when $P = \underline{x}\cup\underline{0}$), $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_\eta$ and $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_s$ are anti-effective universally Whitney–Tate. In particular, we only need to understand what happens when we pass from $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_\eta$ to $(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_s$. Thus, fix some $\mu \in X_*(T)^+$ and consider the anti-effective Tate motive $\mathcal{F} := j_{\mu,*}(\mathbf{Z})$, where $j_\mu\colon \mathrm{Gr}^\mu \times C_\eta \to \mathrm{Gr} \times C_\eta \cong (\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_\eta$ is the associated immersion. We must show that $i^*j_*\mathcal{F} \in \mathrm{DTM}((\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_s)^{\mathrm{anti}}$, where j and i are defined below.

Note that the motive $\mathcal{F}$ is $(L^+G \times C_\eta)$-equivariant, and in particular $\mathbf{G}_{m,C_\eta}$-equivariant for the action induced by any regular cocharacter $\lambda \in X_*(T)$. We then have the following commutative diagram

$$
\begin{array}{ccccc}
(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_s^0 & \xleftarrow{q_s^+} & (\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_s^+ & \xrightarrow{p_s^+} & (\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_s \\
\downarrow{\scriptstyle i^0} & & \downarrow{\scriptstyle i^+} & & \downarrow{\scriptstyle i} \\
(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})^0 & \xleftarrow{q^+} & (\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})^+ & \xrightarrow{p^+} & \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}} \\
\uparrow{\scriptstyle j^0} & & \uparrow{\scriptstyle j^+} & & \uparrow{\scriptstyle j} \\
(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_\eta^0 & \xleftarrow{q_\eta^+} & (\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_\eta^+ & \xrightarrow{p_\eta^+} & (\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_\eta,
\end{array}
$$

where the four small squares are cartesian by [Ric19, Corollary 1.16]. Using base change and hyperbolic localization for the $\mathbf{G}_{\mathrm{m},C}$-equivariant motive $j_*\mathcal{F}$, we compute

$$
q_{s!}^+ p_s^{+*}(i^*j_*\mathcal{F}) \cong q_{s!}^+ i^{+*}(p^{+*}j_*\mathcal{F}) \cong i^{0*}q_!^+(p^{+*}j_*\mathcal{F}) \cong i^{0*}q_*^-(p^{-!}j_*\mathcal{F})
$$
$$
\cong i^{0*}q_*^-(j_*^-p_\eta^{-!}\mathcal{F}) \cong i^{0*}j_*^0(q_{\eta*}^-p_\eta^{-!}\mathcal{F}) \cong i^{0*}j_*^0(q_{\eta!}^+p_\eta^{+*}\mathcal{F}).
$$

By Proposition 4.33, the hyperbolic localization functor $q_{\eta!}^+p_\eta^{+*}$ preserves anti-effective stratified Tate motives. Also, $i^{0*}j_*^0$ preserves anti-effective stratified Tate motives by Lemma 4.30(3) (or Lemma 4.24(5) when $P = \underline{0}$) and the corresponding result for $\mathbf{A}^1 = \mathbf{G}_{\mathrm{m}} \sqcup \mathbf{A}^0$. Thus, we see that $q_{s!}^+p_s^{+*}(i^*j_*\mathcal{F}) \in \mathrm{DTM}((\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_s^0)^{\mathrm{anti}}$. As $i^*j_*\mathcal{F}$ is

$\mathcal{I}$-equivariant by smooth base change, we conclude by Lemma 4.36(1) that $i^* j_* \mathcal{F} \in \mathrm{DTM}((\mathrm{Gr}_{\mathcal{G}_\mathbf{f}})_s)^{\mathrm{anti}}$. Thus, $\mathrm{Gr}_{\mathcal{G}_\mathbf{f}}$ is anti-effective Whitney–Tate stratified.

The condition that this stratification is universal means that the base change map [CvdHS22, Eqn. (2.4)] is an isomorphism. Again, we only need to understand what happens when passing from $(\mathrm{Gr}_{\mathcal{G}_\mathbf{f}})_\eta$ to $(\mathrm{Gr}_{\mathcal{G}_\mathbf{f}})_s$. Since hyperbolic localization commutes with base change, then by Lemma 4.36(2) we again reduce to the case of $\mathbf{A}^1 = \mathbf{G}_\mathrm{m} \sqcup \mathbf{A}^0$.

For (2) a similar argument applies, using Lemma 4.36(1) and that Ψ_P commutes with hyperbolic localization by Proposition 3.17, thus reducing us to the case of $\mathbf{A}^1$ where the result follows from Lemma 3.12. Except for preservation of compactness, (2) also follows from (1) by Proposition 3.16.

For (3), let $X = \mathrm{Gr}_{\mathcal{G}_\mathbf{f}}(P_1) \times_C \cdots \times_C \mathrm{Gr}_{\mathcal{G}_\mathbf{f}}(P_n)$ and let $\mathcal{F}_i \in \mathrm{DTM}(\mathrm{Gr}_{\mathcal{G}_\mathbf{f}}(P_i))$. Using the componentwise left actions of $L^+ \mathcal{G}_\mathbf{f}^{\mathrm{BD}}$ on X, any n-tuple of regular cocharacters $(\lambda_1, \ldots, \lambda_n)$ induces a Zariski-locally linearizable $\mathbf{G}_{\mathrm{m},C}$-action on X by Lemma 4.22. For such a $\mathbf{G}_{\mathrm{m},C}$-action, the reduced closure of any connect component of X_η^0 is isomorphic to C by Lemma 4.24(5) and Lemma 4.30(3). Also, the $\mathcal{F}_i$ and their box product are each $\mathbf{G}_\mathrm{m}$-monodromic since strata in affine flag varieties are $\mathbf{G}_\mathrm{m}$-stable.

The construction of the Künneth map (4.6) involves nearby cycles over $\mathbf{A}^n$, but this is only done in order to keep track of the monodromy. By inspecting the proof of Proposition 3.27 and Corollary 3.26 one sees that the underlying map of motives in (4.6) can be constructed using a base change map over $\mathbf{A}^1$ (followed by compatibility of Υ with smooth pullback). In particular, to check that is an isomorphism we may also apply hyperbolic localization, since the latter commutes with Υ by Proposition 3.17 and with the formation of base change maps. This reduces us to the Künneth map for the trivial family over $\mathbf{A}^1$. Since the formation of Künneth maps commutes with colimits, we reduce to the case where $\mathcal{F}_i = \mathbf{Z}_{\mathbf{G}_\mathrm{m}}$ for all i, in which case we have an isomorphism by Lemma 3.12. $\square$

4.5. The central functor.

4.5.1. *A functor to the center.*

Theorem 4.38. *For $\mathcal{A} \in \mathrm{DTM}(L^+ G \backslash LG / L^+ G)$ and $\mathcal{F} \in \mathrm{DTM}(L^+ G_\mathbf{f} \backslash LG / L^+ G_\mathbf{f})$ there are canonical isomorphisms*

$$\mathsf{Z}_\mathbf{f}(\mathcal{A}) \star \mathcal{F} \cong \Upsilon_{\underline{x} \cup \underline{0}}(\mathcal{A} \boxtimes \mathcal{F} \boxtimes \mathbf{Z}_{C_\eta}) \cong \mathcal{F} \star \mathsf{Z}_\mathbf{f}(\mathcal{A}) \tag{4.7}$$

in $\mathrm{Nilp}_\mathbf{Q} \mathrm{DTM}(L^+ G_\mathbf{f} \backslash LG / L^+ G_\mathbf{f})$. The composites of these isomorphisms are compatible with the associativity constraint on $\mathrm{Ho}(\mathrm{DTM}(L^+ G_\mathbf{f} \backslash LG / L^+ G_\mathbf{f}))$ in the sense that the following diagram commutes for $\mathcal{A} \in \mathrm{DTM}(L^+ G \backslash LG / L^+ G)$ and $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{DTM}(L^+ G_\mathbf{f} \backslash LG / L^+ G_\mathbf{f})$.

$$
\begin{array}{ccc}
 & \mathsf{Z}_\mathbf{f}(\mathcal{A}) \star (\mathcal{F}_1 \star \mathcal{F}_2) \xrightarrow{(4.7)} (\mathcal{F}_1 \star \mathcal{F}_2) \star \mathsf{Z}_\mathbf{f}(\mathcal{A}) & \\
\nearrow & & \searrow \\
(\mathsf{Z}_\mathbf{f}(\mathcal{A}) \star \mathcal{F}_1) \star \mathcal{F}_2 & & \mathcal{F}_1 \star (\mathcal{F}_2 \star \mathsf{Z}_\mathbf{f}(\mathcal{A})) \\
\downarrow {\scriptstyle (4.7)} & & \uparrow {\scriptstyle (4.7)} \\
(\mathcal{F}_1 \star \mathsf{Z}_\mathbf{f}(\mathcal{A})) \star \mathcal{F}_2 \longrightarrow \mathcal{F}_1 \star (\mathsf{Z}_\mathbf{f}(\mathcal{A}) \star \mathcal{F}_2) &
\end{array}
$$

Proof. The isomorphisms are constructed as in [AR, Proposition 3.2.1]. Briefly, the first isomorphism comes from applying Theorem 4.37(3) and Lemma 4.20 in the case $P_1 = \underline{x}$ and $P_2 = \underline{0}$. For the second we take $P_1 = \underline{0}$ and $P_2 = \underline{x}$, noting that we have a canonical isomorphism $(\mathrm{Gr}_{\mathcal{G}_\mathbf{f}}^{\mathrm{BD}})_\eta = \mathrm{Gr} \times \mathrm{Fl}_\mathbf{f} \times C_\eta \cong \mathrm{Fl}_\mathbf{f} \times \mathrm{Gr} \times C_\eta$ which swaps the factors. We also use the fact that $\Psi_{\underline{0}}$ is trivial on the sheaves in Corollary 3.14, where [CvdHS22, Corollary 3.10] ensures that the hypothesis on the Künneth map is satisfied. The commutativity of the diagram is proved exactly as in [AR, Theorem 3.2.3], using Theorem 4.37(3) for $n > 2$ factors along with Remark 4.21. $\square$

Remark 4.39. If $\mathbf{f} = \mathbf{f}_0$ then $\mathsf{Z}_{\mathbf{f}_0} = \mathrm{id}$ by Corollary 3.14 and [CvdHS22, Corollary 3.10]. Thus, Theorem 4.38 endows $\mathrm{Ho}(\mathrm{DTM}(L^+ G \backslash LG / L^+ G))$ with the structure of a symmetric monoidal category. In Proposition 4.47 we compare this to the commutativity constraint on $\mathrm{MTM}(L^+ G \backslash LG / L^+ G)$ constructed in [CvdHS22, Proposition 5.48] via the fusion interpretation.

Remark 4.40. For general $\mathbf{f}$, the isomorphisms in Theorem 4.38 endow $\mathsf{Z}_\mathbf{f}$ with a factorization through the Drinfeld center $\mathcal{Z}(\mathrm{Ho}(\mathrm{DTM}(L^+ G_\mathbf{f} \backslash LG / L^+ G_\mathbf{f})))$, cf. [AR, §3.1]:

$$\mathsf{Z}_\mathbf{f} : \mathrm{Ho}(\mathrm{DTM}(L^+ G \backslash LG / L^+ G)) \to \mathcal{Z}(\mathrm{Ho}(\mathrm{DTM}(L^+ G_\mathbf{f} \backslash LG / L^+ G_\mathbf{f}))).$$

4.5.2. *Monoidality.*

Theorem 4.41. *For $\mathcal{A}_1$, $\mathcal{A}_2 \in \mathrm{DTM}(L^+G\backslash LG/L^+G)$ there is a canonical isomorphism*

$$\mathsf{Z}_\mathbf{f}(\mathcal{A}_1) \star \mathsf{Z}_\mathbf{f}(\mathcal{A}_2) \cong \mathsf{Z}_\mathbf{f}(\mathcal{A}_1 \star \mathcal{A}_2) \tag{4.8}$$

in $\mathrm{Nilp}_\mathbf{Q}\,\mathrm{DTM}(L^+G_\mathbf{f}\backslash LG/L^+G_\mathbf{f})$. These isomorphisms are compatible with the centrality constraints (4.7) in the sense that the following diagram commutes for $\mathcal{A}_1$, $\mathcal{A}_2 \in \mathrm{DTM}(L^+G\backslash LG/L^+G)$ and $\mathcal{F} \in \mathrm{DTM}(L^+G_\mathbf{f}\backslash LG/L^+G_\mathbf{f})$ (we have omitted the associativity isomorphisms, but see [AR, Equation (3.4.1)] for the full diagram).

$$
\begin{array}{ccc}
\mathsf{Z}_\mathbf{f}(\mathcal{A}_1 \star \mathcal{A}_2) \star \mathcal{F} & \xrightarrow{\ (4.7)\ } & \mathcal{F} \star \mathsf{Z}_\mathbf{f}(\mathcal{A}_1 \star \mathcal{A}_2) \\
\downarrow{\scriptstyle(4.8)} & & \uparrow{\scriptstyle(4.8)} \\
\mathsf{Z}_\mathbf{f}(\mathcal{A}_1) \star \mathsf{Z}_\mathbf{f}(\mathcal{A}_2) \star \mathcal{F} \xrightarrow{(4.7)} \mathsf{Z}_\mathbf{f}(\mathcal{A}_1) \star \mathcal{F} \star \mathsf{Z}_\mathbf{f}(\mathcal{A}_2) & \xrightarrow{(4.7)} & \mathcal{F} \star \mathsf{Z}_\mathbf{f}(\mathcal{A}_1) \star \mathsf{Z}_\mathbf{f}(\mathcal{A}_2)
\end{array}
$$

Proof. The isomorphism (4.8) comes from applying Theorem 4.37(3) and Lemma 4.20 in the case $P_1 = P_2 = \underline{x}$. The commutativity of the diagram is proved exactly as in [AR, Theorem 3.4.1]. □

Remark 4.42. Theorem 4.41 endows $\mathsf{Z}_\mathbf{f}$ with the structure of a *monoidal* functor to the Drinfeld center of $\mathrm{Ho}(\mathrm{DTM}(L^+G_\mathbf{f}\backslash LG/L^+G_\mathbf{f}))$.

4.5.3. *Change of facet.*

Theorem 4.43. *Let $\mathbf{c}$ be a facet contained in the closure of $\mathbf{f}$ and let $\pi\colon \mathrm{Fl}_\mathbf{f} \to \mathrm{Fl}_\mathbf{c}$ be the projection.*

(1) For $\mathcal{A} \in \mathrm{DTM}(L^+G\backslash LG/L^+G)$ there is a canonical isomorphism

$$\pi_* \mathsf{Z}_\mathbf{f}(\mathcal{A}) \cong \mathsf{Z}_\mathbf{c}(\mathcal{A})$$

of the underlying non-equivariant motives in $\mathrm{Nilp}_\mathbf{Q}\,\mathrm{DTM}(\mathrm{Fl}_\mathbf{c})$.

(2) For $\mathcal{A}_1$, $\mathcal{A}_2 \in \mathrm{DTM}(L^+G\backslash LG/L^+G)$ there is a canonical isomorphism

$$\pi_*(\mathsf{Z}_\mathbf{f}(\mathcal{A}_1) \star \mathsf{Z}_\mathbf{f}(\mathcal{A}_2)) \cong \mathsf{Z}_\mathbf{c}(\mathcal{A}_1) \star \mathsf{Z}_\mathbf{c}(\mathcal{A}_2) \tag{4.9}$$

of the underlying non-equivariant motives in $\mathrm{Nilp}_\mathbf{Q}\,\mathrm{DTM}(\mathrm{Fl}_\mathbf{c})$.

(3) There is a commutative diagram of isomorphisms in $\mathrm{Ho}(\mathrm{DTM}(L^+G_\mathbf{c}\backslash LG/L^+G_\mathbf{c}))$ as follows:

$$
\begin{array}{ccc}
\pi_*(\mathsf{Z}_\mathbf{f}(\mathcal{A}_1) \star \mathsf{Z}_\mathbf{f}(\mathcal{A}_2)) & \xrightarrow{\pi_*(4.7)} & \pi_*(\mathsf{Z}_\mathbf{f}(\mathcal{A}_2) \star \mathsf{Z}_\mathbf{f}(\mathcal{A}_1)) \\
\downarrow{\scriptstyle(4.9)} & & \downarrow{\scriptstyle(4.9)} \\
\mathsf{Z}_\mathbf{c}(\mathcal{A}_1) \star \mathsf{Z}_\mathbf{c}(\mathcal{A}_2) & \xrightarrow{(4.7)} & \mathsf{Z}_\mathbf{c}(\mathcal{A}_2) \star \mathsf{Z}_\mathbf{c}(\mathcal{A}_1).
\end{array}
$$

Remark 4.44. If $\mathbf{f} = \mathbf{a}_0$ and $\mathbf{c} = \mathbf{f}_0$ then part (1) provides a canonical identification of $\pi_* \mathsf{Z}_{\mathbf{a}_0}$ with the forgetful functor $\mathrm{DTM}(L^+G\backslash LG/L^+G) \to \mathrm{DTM}(\mathrm{Gr}_G)$.

Proof. Part (1) follows from the fact that Υ commutes with pushforward along the proper map $\mathrm{Gr}_{\mathcal{G}_\mathbf{f}} \to \mathrm{Gr}_{\mathcal{G}_\mathbf{c}}$. The isomorphism (4.9) in part (2) is constructed by combining part (1) and (4.8). Alternatively, (4.9) may be constructed as follows. By combining part (1) with base change and the Künneth formula for the proper map $\pi \widetilde{\times} \pi\colon LG \times^{L^+G_\mathbf{f}} \mathrm{Fl}_\mathbf{f} \to LG \times^{L^+G_\mathbf{c}} \mathrm{Fl}_\mathbf{c}$ we get an isomorphism of non-equivariant motives,

$$(\pi \widetilde{\times} \pi)_!(\mathsf{Z}_\mathbf{f}(\mathcal{A}_1)\widetilde{\boxtimes}\mathsf{Z}_\mathbf{f}(\mathcal{A}_2)) \cong (\mathsf{Z}_\mathbf{c}(\mathcal{A}_1)\widetilde{\boxtimes}\mathsf{Z}_\mathbf{c}(\mathcal{A}_2)).$$

Applying $m_!$ also gives an isomorphism (4.9), which agrees with the previous construction by the same arguments as in [AR, Lemma 3.4.3]. Finally, part (3) follows as in [AR, Lemma 3.4.4] by compatibility of Υ with proper pushforward (e.g, along $\mathrm{Gr}_{\mathcal{G}_\mathbf{f}} \to \mathrm{Gr}_{\mathcal{G}_\mathbf{c}}$). □

4.5.4. *Mixed Tate motives and the motivic Satake equivalence.*
In this subsection we assume that S satisfies the Beilinson–Soulé conjecture (this is unnecessary when working with reduced motives). Recall from Section 2.2.4 that $\mathrm{DTM}(S)$ carries a t-structure with heart $\mathrm{MTM}(S)$. By [CvdHS22, Lemma 2.13] the $\mathbf{Z}(k)$ for $k \in \mathbf{Z}$ compactly generate $\mathrm{MTM}(S)$. By [CvdHS22, Lemma 2.15] the strata $\mathrm{Fl}^\circ_w(\mathbf{f}, \mathbf{f})$ also satisfy the Beilinson–Soulé conjecture. We normalize such that $\mathbf{Z}[\dim \mathrm{Fl}^\circ_w(\mathbf{f}, \mathbf{f})] \in \mathrm{MTM}(\mathrm{Fl}^\circ_w(\mathbf{f}, \mathbf{f}))$. By [CvdHS22, Lemma 2.15], the t-structures on these strata glue to a t-structure on $\mathrm{DTM}(\mathrm{Fl}_\mathbf{f})$.

Definition and Lemma 4.45. Let $\mathbf{f}$ be a facet and let $P \in \{\emptyset, \underline{x}, \underline{0}, \underline{x} \cup \underline{0}\}$.

(1) The abelian subcategory $\mathrm{MTM}(\mathrm{Gr}_{\mathcal{G}_\mathbf{f}}(P)) \subset \mathrm{DTM}(\mathrm{Gr}_{\mathcal{G}_\mathbf{f}}(P))$ of mixed Tate motives is well-defined. The same is true for the Hecke stack $\mathrm{Hck}_{\mathcal{G}_\mathbf{f}}(P)$.

(2) The natural forgetful functor $\mathrm{MTM}(\mathrm{Hck}_{\mathcal{G}_\mathbf{f}}(P)) \to \mathrm{MTM}(\mathrm{Gr}_{\mathcal{G}_\mathbf{f}}(P))$ is fully faithful, and the image is stable under subquotients.

(3) The perverse truncation functors preserve anti-effective motives, so that the full abelian subcategory $\mathrm{MTM}(\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P))^{\mathrm{anti}} \subset \mathrm{MTM}(\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P))$ is well-defined.

Proof. We first observe that the stratification $\mathbf{A}^1 = \mathbf{G}_{\mathrm{m}} \sqcup \mathbf{A}^0$ is Whitney–Tate and cellular relative to S in the sense of [CvdHS22, Remark 2.10], so that $\mathrm{MTM}(C)$ is well-defined by [CvdHS22, Lemma 2.15]. Next, we note that the Schubert cells $\mathrm{Fl}^\circ_w(\mathbf{f}, \mathbf{f})$ are smooth, and the finer stratifications by Iwahori-orbits are cellular and Whitney–Tate by [CvdHS22, Proposition 3.7]. By [CvdHS22, Remark 2.10], this implies that the stratification of $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P)$ in Definition and Lemma 4.26 is admissible relative to C in the sense of [CvdHS22, Definition 2.9]. The t-structures in part (1) are then constructed as in [CvdHS22, Lemma 2.15] (for $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P)$) and in [CvdHS22, Lemma 2.24] (for $\mathrm{Hck}_{\mathcal{G}_{\mathbf{f}}}(P)$). In the latter case we also need [RS20, Proposition A.4.9], which implies that the kernel of $L^+\mathcal{G}_{\mathbf{f}}^{\mathrm{BD}} \to L^{+,(0)}\mathcal{G}_{\mathbf{f}}^{\mathrm{BD}}$ is split pro-unipotent. Part (2) is proved exactly as in [CvdHS22, Lemma 4.30], except we replace the stratification of $\mathbf{A}^I$ by partial diagonals by the stratification $\mathbf{A}^1 = \mathbf{G}_{\mathrm{m}} \sqcup \mathbf{A}^0$. Finally, the construction in (1) carries over to anti-effective motives since the stratification of $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(P)$ is anti-effective, giving (3). $\qquad\square$

Proposition 4.46. *Let* $\mathbf{f} = \mathbf{f}_0$, *so that* $\mathcal{G}_{\mathbf{f}_0} = G_C$. *Let* $P = \underline{x} \cup \underline{0}$, *and let* $j\colon \mathrm{Hck}_{G_C}(P)_\eta \to \mathrm{Hck}_{G_C}(P)$ *and* $i\colon \mathrm{Hck}_{G_C}(P)_s \to \mathrm{Hck}_{G_C}(P)$ *be the inclusions.*

(1) *The following nearby cycles functor is t-exact*
$$\Upsilon_P((-) \boxtimes \mathbf{Z}_{C_\eta})\colon \mathrm{DTM}_{L^+G \times L^+G}(\mathrm{Gr} \times \mathrm{Gr}) \to \mathrm{DTM}_{L^+G}(\mathrm{Gr}).$$

(2) *The following functors are also t-exact*
$$j_!((-) \boxtimes \mathbf{Z}_{C_\eta}[1]), \ j_*((-) \boxtimes \mathbf{Z}_{C_\eta}[1])\colon \mathrm{DTM}_{L^+G \times L^+G}(\mathrm{Gr} \times \mathrm{Gr}) \to \mathrm{DTM}(\mathrm{Hck}_{G_C}(P)).$$

(3) *For* $\mathcal{A} \in \mathrm{MTM}_{L^+G \times L^+G}(\mathrm{Gr} \times \mathrm{Gr})$ *we have a canonical isomorphism*
$$\Upsilon_P(\mathcal{A} \boxtimes \mathbf{Z}_{C_\eta}) \cong {}^{\mathrm{p}}\mathrm{H}^{-1}(i^* j_{!*}(\mathcal{A} \boxtimes \mathbf{Z}_{C_\eta}[1])).$$

Proof. By continuity we may restrict to bounded motives. To prove (1), we may compose with the t-exact and conservative constant term functor $\mathrm{CT}_B[\deg]$ by [CvdHS22, Proposition 5.9]. This commutes with Υ_P by Proposition 3.17, where we apply $\mathrm{CT}_{B \times B}[\deg_{B \times B}] \boxtimes \mathrm{id}_{\mathrm{DTM}(C_\eta)}$ over C_η. (In contrast to [CvdHS22], we will not include the degree shift in the definition of CT_B.) This crucially uses that $\mathrm{Gr}_{G_C}(P)_s = \mathrm{Gr}$. By Lemma 4.30(3), the reduced locus of the closure of each connected component of $(\mathrm{Gr}_{G_C}(P))^0_\eta$ is isomorphic to C. This reduces (1) to the case where G is trivial, in which case the result follows from Corollary 3.14 (where we set $Y = S$).

To prove (2), we note that $i^* j_*$ and $i^! j_!$ commute with constant terms (both assertions follow from base change since $\mathrm{CT}_B = q_!^+ p^{+*} = q_*^- p^{-!}$). The same is true for $i^! j_*$ and $i^* j_!$ since both of these are zero. Thus, we may similarly apply constant terms to reduce to the case where G is trivial, in which case (2) follows from relative purity on $\mathbf{A}^1 = \mathbf{G}_{\mathrm{m}} \sqcup \mathbf{A}^0$.

To prove (3), we argue as in [AR, Lemma 9.1.9]. Let $\mathcal{F} = \mathcal{A} \boxtimes \mathbf{Z}_{C_\eta}[1]$. The canonical map $i^* j_* \mathcal{F} \to \Upsilon_P(\mathcal{F})$ (Remark 3.11) is a map between objects in $\mathrm{DTM}(\mathrm{Gr})$, where the latter lies in cohomological degree -1 by part (1). Applying ${}^{\mathrm{p}}\mathrm{H}^{-1}$ then gives a map ${}^{\mathrm{p}}\mathrm{H}^{-1} i^* j_* \mathcal{F} \to \Upsilon_P(\mathcal{F})$ which we claim is an isomorphism. Indeed, applying the t-exact and conservative constant term functor CT_B commutes with everything in sight. Thus, to check this is an isomorphism, we may assume $G = T$, in which case the claim holds by Corollary 3.14 and Lemma 4.30(3). By (2) and the exact triangle $j_! \mathcal{F} \to j_* \mathcal{F} \to i_* i^* j_* \mathcal{F}$, we may identify ${}^{\mathrm{p}}\mathrm{H}^{-1}(i^* j_* \mathcal{F})$ with the kernel of $j_! \mathcal{F} \to j_* \mathcal{F}$. This is the same as the kernel of $j_! \mathcal{F} \to j_{!*} \mathcal{F}$, and we conclude by using the exact triangle $j_! \mathcal{F} \to j_{!*} \mathcal{F} \to i_* i^* j_{!*} \mathcal{F}$. $\qquad\square$

Proposition 4.47. *Let* $\mathcal{A}_1, \mathcal{A}_2 \in \mathrm{MTM}(L^+G \backslash LG / L^+G)$. *Then the commutativity isomorphism* ${}^{\mathrm{p}}\mathrm{H}^0(\mathcal{A}_1 \star \mathcal{A}_2) \cong {}^{\mathrm{p}}\mathrm{H}^0(\mathcal{A}_2 \star \mathcal{A}_1)$ *constructed via the fusion product in* [CvdHS22, Proposition 5.48], *omitting the sign modifications in* [CvdHS22, Remark 5.45], *agrees with the perverse truncation of (4.7) for* $\mathbf{f} = \mathbf{f}_0$.

Proof. The unmodified commutativity isomorphism in [CvdHS22, Proposition 5.48] is constructed as follows. Let $I = \{1, 2\}$ and consider the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{G_C, I} \to C^I$. Let $\Delta C \subset C^I$ be the diagonal and let $C^\circ \subset C^I$ be the complement. Then the fusion isomorphism implies that
$$\mathrm{Gr}_{G_C, I}\big|_{\Delta C} = \mathrm{Gr} \times C, \quad \mathrm{Gr}_{G_C, I}\big|_{C^\circ} = \mathrm{Gr} \times \mathrm{Gr} \times C^\circ.$$

Let $i_I\colon \mathrm{Gr}_{G_C, I}\big|_{\Delta C} \to \mathrm{Gr}_{G_C, I}$ and $j_I\colon \mathrm{Gr}_{G_C, I}\big|_{C^\circ} \to \mathrm{Gr}_{G_C, I}$ be the inclusions. There is a canonical isomorphism
$$i_I^* \circ j_{I, !*}({}^{\mathrm{p}}\mathrm{H}^0(\mathcal{A}_1 \boxtimes \mathcal{A}_2) \boxtimes \mathbf{Z}_{C^\circ}[2]) \cong {}^{\mathrm{p}}\mathrm{H}^0(\mathcal{A}_1 \star \mathcal{A}_2) \boxtimes \mathbf{Z}_{\Delta C}[2].$$

The commutativity isomorphism is constructed by observing that $i_I^* \circ j_{I, !*}$ is invariant under the isomorphism of $\mathrm{Gr}_{G_C, I}$ which swaps the coordinates of C^I.

To relate this to nearby cycles, note that there is a canonical isomorphism
$$\mathrm{Gr}_{G_C, I} \cong C \times \mathrm{Gr}_{G_C}^{\mathrm{BD}}$$

which identifies $\mathrm{Gr}_{G_C,I}\big|_{\Delta C}$ with $\{0\}\times\mathrm{Gr}_{G_C}^{\mathrm{BD}}$. This is constructed using the additive structure on C, cf. [AR, Lemma 3.3.6]. Since the extra factor of C may be absorbed into the base S, the unmodified commutativity isomorphism may alternatively be constructed using the canonical isomorphism

$$i^*j_{!*}(^{\mathrm{p}}\mathrm{H}^0(\mathcal{A}_1\boxtimes\mathcal{A}_2)\boxtimes\mathbf{Z}_{C_\eta}[1]))\cong{}^{\mathrm{p}}\mathrm{H}^0(\mathcal{A}_1*\mathcal{A}_2)[1].$$

By Proposition 4.46(3), this agrees with the perverse truncation of the commutativity isomorphism constructed using nearby cycles. $\qquad\square$

Remark 4.48. The computation of constant terms $\mathrm{CT}_B[\deg]\colon \mathrm{MTM}(L^+G\backslash LG/L^+G)\to\mathrm{MTM}(L^+T\backslash LT/L^+T)$ using nearby cycles in [CvdHS22, Proposition 5.33] assembles to a monoidal structure on $\mathrm{CT}_B[\deg]$. By arguments similar to the proof of Proposition 4.47, this monoidal structure coincides with the one constructed in [CvdHS22, Proposition 5.48], again omitting the sign modifications in [CvdHS22, Remark 5.45]. Similar remarks apply to the fiber functor to $\mathrm{MTM}(C)$ obtained by pushforward, cf. [AR, Proposition 3.3.8].

4.5.5. *Centrality.*

Theorem 4.49. *Let $\mathcal{A}, \mathcal{B}\in\mathrm{MTM}(L^+G\backslash LG/L^+G)$ be such that $\mathcal{A}\boxtimes\mathcal{B}$ is mixed Tate. Then the following diagram commutes in $\mathrm{Ho}(\mathrm{DTM}(L^+G_{\mathbf{f}}\backslash LG/L^+G_{\mathbf{f}}))$, where the subscripts indicate the facet to which an isomorphism is applied.*

$$
\begin{array}{ccc}
\mathsf{Z}_{\mathbf{f}}(\mathcal{A})\star\mathsf{Z}_{\mathbf{f}}(\mathcal{B}) & \xrightarrow{\;(4.8)_{\mathbf{f}}\;} & \mathsf{Z}_{\mathbf{f}}(\mathcal{A}\star\mathcal{B}) \\
\big\downarrow{\scriptstyle(4.7)_{\mathbf{f}}} & & \big\downarrow{\scriptstyle\mathsf{Z}_{\mathbf{f}}\circ(4.7)_{\mathbf{f}_0}} \\
\mathsf{Z}_{\mathbf{f}}(\mathcal{B})\star\mathsf{Z}_{\mathbf{f}}(\mathcal{A}) & \xrightarrow{\;(4.8)_{\mathbf{f}}\;} & \mathsf{Z}_{\mathbf{f}}(\mathcal{B}\star\mathcal{A})
\end{array}
\tag{4.10}
$$

Remark 4.50. To construct the isomorphisms $(4.7)_{\mathbf{f}}$ and $\mathsf{Z}_{\mathbf{f}}\circ(4.7)_{\mathbf{f}_0}$ we must choose which factor should be considered central for the purpose of commuting it with the other factor. By Proposition 4.47 the isomorphism $\mathsf{Z}_{\mathbf{f}}\circ(4.7)_{\mathbf{f}_0}$ is independent of this choice, and the proof will show that $(4.7)_{\mathbf{f}}$ is also independent of this choice, which amounts to choosing an order of the coordinates on C^2 for which to perform iterated nearby cycles.

Remark 4.51. Note that $\mathcal{A}\boxtimes\mathcal{B}$ is always mixed Tate if one restricts to coefficients in a field. Even if $\mathcal{A}\boxtimes\mathcal{B}$ is not mixed Tate, it is reasonable to expect commutativity of the diagram analogous to (4.10) where one takes the perverse truncation of all convolution products. The proof of this commutativity in [AR, Theorem 3.5.1] uses t-exactness of $\Upsilon_{\underline{x}\cup 0}[-1]$, which is currently unavailable in the motivic setting.

Proof. The proof is analogous to [AR, Theorem 3.5.1]. We will construct all of the necessary isomorphisms; commutativity may then be checked as in *loc. cit.* We fix $I=\{1,2\}$ throughout the proof. Recall the global convolution Grassmannian $m_{i_0}\colon \widetilde{\mathrm{Gr}}_{\mathcal{G}_{\mathbf{f}},I}\to\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$ from Definition 4.8, where $i_0=2$. In Lemma 4.9 we also constructed $L_I^+\mathcal{G}_{\mathbf{f}}$-torsors

$$\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}\times\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}\xleftarrow{\;p\;}(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}\times C)^{(\infty)}\times_{C^2}(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}\times C)\xrightarrow{\;q\;}\widetilde{\mathrm{Gr}}_{\mathcal{G}_{\mathbf{f}},I}.$$

Let us denote the nearby cycles functor on $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$ (resp. $\widetilde{\mathrm{Gr}}_{\mathcal{G}_{\mathbf{f}},I}$) by Υ_I (resp. $\Upsilon_{\tilde{I}}$).

Let $\mathcal{A}'=\mathcal{A}\boxtimes\mathbf{Z}_{C_\eta}\in\mathrm{DTM}((\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_\eta)$, and define $\mathcal{B}'$ likewise. While [CvdHS22, §4] works over the base C^2, it applies equally well over $C_\eta^2=\mathbf{G}_{\mathrm{m}}\times\mathbf{G}_{\mathrm{m}}$. Thus, by [CvdHS22, Lemma 4.16] we may form the twisted product $\mathcal{A}'\widetilde{\boxtimes}\mathcal{B}'\in\mathrm{DM}((\widetilde{\mathrm{Gr}}_{\mathcal{G}_{\mathbf{f}},I})_\eta)$. Associated to the usual order on I have the iterated nearby cycles functor $\Upsilon_{\tilde{I},1}\circ\Upsilon_{\tilde{I},2}$. If Δ is the inclusion of diagonal over C_η^2 then we also have the nearby cycles functor $\Upsilon_{(\underline{x},\underline{x})}\circ\Delta^*$, since the fiber of $\widetilde{\mathrm{Gr}}_{\mathcal{G}_{\mathbf{f}},I}$ over Δ is $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}(\underline{x},\underline{x})$.

By Lemma 3.29 and Corollary 3.26 there are maps

$$\Upsilon_{\tilde{I},1}\circ\Upsilon_{\tilde{I},2}(\mathcal{A}'\widetilde{\boxtimes}\mathcal{B}')\leftarrow\Upsilon_{\tilde{I}}(\mathcal{A}'\widetilde{\boxtimes}\mathcal{B}')\rightarrow\Upsilon_{(\underline{x},\underline{x})}\Delta^*(\mathcal{A}'\widetilde{\boxtimes}\mathcal{B}').\tag{4.11}$$

We claim that these are isomorphisms. Both maps are compatible with smooth base change, and p and q admit sections Zariski-locally over C_η^2 and C_s^2 (over C_η^2 this is [CvdHS22, Lemma 4.8]). Thus, it suffices to check that the analogous maps are isomorphisms for $\mathcal{A}'\boxtimes\mathcal{B}'\in\mathrm{DM}(\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^2)_\eta$. For this we proceed as in Theorem 4.37, using Lemma 4.36 and hyperbolic localization to reduce to the case of Tate motives supported on $\mathbf{G}_{\mathrm{m}}^2\subset\mathbf{A}^2$. By Proposition 3.27 we have $\Upsilon_{\mathbf{A}^2/\mathbf{A}^2}(\mathbf{Z}_{\mathbf{G}_{\mathrm{m}}^2})=\mathbf{Z}_S$, so that we indeed have isomorphisms by tracing through the construction of these maps.

We also claim that there are canonical isomorphisms

$$\Upsilon_{\tilde{I}}(\mathcal{A}'\widetilde{\boxtimes}\mathcal{B}')\cong\Upsilon_{\underline{x}}(\mathcal{A}')\widetilde{\boxtimes}\Upsilon_{\underline{x}}(\mathcal{B}')\cong\mathsf{Z}_{\mathbf{f}}(\mathcal{A})\widetilde{\boxtimes}\mathsf{Z}_{\mathbf{f}}(\mathcal{B}).\tag{4.12}$$

The second is a definition, and the first is proved similarly to Theorem 4.37(3), using p and q to reduce to showing that the Künneth map $\Upsilon_{\underline{x}}(\mathcal{A}')\boxtimes\Upsilon_{\underline{x}}(\mathcal{B}')\to\Upsilon_{\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}}^2}(\mathcal{A}'\boxtimes\mathcal{B}')$ in Proposition 3.27 is an isomorphism. This latter fact can also be proved using hyperbolic localization, so this constructs (4.12).

Since the maps in (4.11) are compatible with proper pushforward, we have a commutative diagram of isomorphisms

$$m_{s!}(\Upsilon_{\tilde{I},1} \circ \Upsilon_{\tilde{I},2}(\mathcal{A}' \widetilde{\boxtimes} \mathcal{B}')) \xleftarrow{\ \sim\ } m_{s!}\Upsilon_{\tilde{I}}(\mathcal{A}' \widetilde{\boxtimes} \mathcal{B}') \xrightarrow{\ \sim\ } m_{s!}\Upsilon_{(\underline{x},\underline{x})}\Delta^*(\mathcal{A}' \widetilde{\boxtimes} \mathcal{B}') \tag{4.13}$$

$$\downarrow{\scriptstyle\sim} \qquad\qquad\qquad \downarrow{\scriptstyle\sim} \qquad\qquad\qquad \downarrow{\scriptstyle\sim}$$

$$\Upsilon_{I,1} \circ \Upsilon_{I,2}(m_{\eta!}(\mathcal{A}' \widetilde{\boxtimes} \mathcal{B}')) \xleftarrow{\ \sim\ } \Upsilon_I m_{\eta!}(\mathcal{A}' \widetilde{\boxtimes} \mathcal{B}') \xrightarrow{\ \sim\ } \Upsilon_{\underline{x}}\Delta^* m_{\eta!}(\mathcal{A}' \widetilde{\boxtimes} \mathcal{B}').$$

We now make several observations about this diagram, similar to those in the proof of [AR, Theorem 3.5.1]. First, we have a canonical isomorphism $\Delta^*(\mathcal{A}' \widetilde{\boxtimes} \mathcal{B}') \cong A \widetilde{\boxtimes} \mathcal{B} \boxtimes \mathbf{Z}_{\mathbf{C}_\eta}$, and the resulting automorphism of $\mathsf{Z}_{\mathbf{f}}(\mathcal{A}) \widetilde{\boxtimes} \mathsf{Z}_{\mathbf{f}}(\mathcal{B})$ obtained by combining (4.11) and (4.12) is the identity.

Second, if we take $S = \mathbf{G}_{\mathrm{m}}$, $Y = (\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}})_\eta$, $X = \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}}} \times S$, $N = \mathcal{A}'$, and $M = \mathcal{B}' \boxtimes \mathbf{Z}_S$, then the map (3.20) is an isomorphism. Indeed, this can be checked after applying hyperbolic localization, which reduces us to the constant family $Y \times \mathbf{A}^1$, where the result follows from Corollary 3.14 and [CvdHS22, Corollary 3.10]. Combining (3.20) with smooth pullback along p and q and pushforward along m gives a canonical isomorphism

$$\Upsilon_{I,1} \circ \Upsilon_{I,2}(m_{\eta!}(\mathcal{A}' \widetilde{\boxtimes} \mathcal{B}')) \cong \Upsilon_{\underline{x} \cup 0}(\mathcal{A} \boxtimes \mathsf{Z}_{\mathbf{f}}(\mathcal{B}) \boxtimes \mathbf{Z}_{C_\eta}). \tag{4.14}$$

Recall that the right side of (4.14) is the key motive in the construction of (4.7)$_{\mathbf{f}}$.

Third, by base change we have a canonical isomorphism

$$\Upsilon_{\underline{x}}\Delta^* m_{\eta!}(\mathcal{A}' \widetilde{\boxtimes} \mathcal{B}') \cong \mathsf{Z}_{\mathbf{f}}(\mathcal{A} \star \mathcal{B}). \tag{4.15}$$

Via the identification $m_{s!}\Upsilon_{(\underline{x},\underline{x})}\Delta^*(\mathcal{A}' \widetilde{\boxtimes} \mathcal{B}') \cong \mathsf{Z}_{\mathbf{f}}(\mathcal{A}) \star \mathsf{Z}_{\mathbf{f}}(\mathcal{B})$ coming from (4.12), the right vertical map of (4.13) is identified with (4.8)$_{\mathbf{f}}$.

Let $j\colon \mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}\big|_{C_\eta^2 \setminus \Delta_\eta} \to (\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I})_\eta$ be the inclusion over the locus with pairwise distinct nonzero coordinates. Note that m is an isomorphism over $C_\eta^2 \setminus \Delta_\eta$. By [CvdHS22, Theorem 5.43] and its proof (which shows that a global convolution product is mixed Tate as soon as the box product is mixed Tate), the motive $m_{\eta!}(\mathcal{A}' \widetilde{\boxtimes} \mathcal{B}')[2]$ lies in the Satake category $\mathrm{Sat}^{G,I}$. Then by [CvdHS22, Proposition 5.18],

$$m_{\eta!}(\mathcal{A}' \widetilde{\boxtimes} \mathcal{B}')[2] \cong j_{!*}(\mathcal{A} \boxtimes \mathcal{B} \boxtimes \mathbf{Z}_{C_\eta^2 \setminus \Delta_\eta}[2]).$$

There is an isomorphism of $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}$ which swaps the two coordinates of C^2. Over $C_\eta^2 \setminus \Delta_\eta$ this isomorphism swaps the two factors of Gr in $\mathrm{Gr}_{\mathcal{G}_{\mathbf{f}},I}\big|_{C_\eta^2 \setminus \Delta_\eta} = \mathrm{Gr} \times \mathrm{Gr} \times C_\eta^2 \setminus \Delta_\eta$, and over C_s^2 it is the identity map. Thus, we get a canonical isomorphism

$$\Upsilon_I m_{\eta!}(\mathcal{A}' \widetilde{\boxtimes} \mathcal{B}') \cong \Upsilon_I m_{\eta!}(\mathcal{B}' \widetilde{\boxtimes} \mathcal{A}').$$

Similarly we get an isomorphism

$$\Upsilon_{\underline{x}}\Delta^* m_{\eta!}(\mathcal{A}' \widetilde{\boxtimes} \mathcal{B}') \cong \Upsilon_{\underline{x}}\Delta^* m_{\eta!}(\mathcal{B}' \widetilde{\boxtimes} \mathcal{A}')$$

which identifies with $\mathsf{Z}_{\mathbf{f}} \circ (4.7)_{\mathbf{f}_0}$ under (4.15). On the other hand, the swapping isomorphism sends $\Upsilon_{I,1} \circ \Upsilon_{I,2}$ to $\Upsilon_{I,2} \circ \Upsilon_{I,1}$. The resulting isomorphism

$$\Upsilon_{I,1} \circ \Upsilon_{I,2}(m_{\eta!}(\mathcal{A}' \widetilde{\boxtimes} \mathcal{B}')) \cong \Upsilon_{I,2} \circ \Upsilon_{I,1}(m_{\eta!}(\mathcal{B}' \widetilde{\boxtimes} \mathcal{A}'))$$

is identified under (4.14) (and its analogue for $\Upsilon_{I,2} \circ \Upsilon_{I,1}$) with (4.8)$_{\mathbf{f}}$. This finally brings all of the isomorphisms in (4.10) into the picture, the commutativity of which now follows from that of (4.13) and all of the identifications we have made. $\qquad\square$

5. Wakimoto motives

The classical Wakimoto sheaves, due to Mirković, are Iwahori-equivariant sheaves on the affine flag variety that categorify the Bernstein elements in the Iwahori–Hecke algebra, cf. [AR, §5]. They are used in [AB09] to construct the so-called Arkhipov–Bezrukavnikov equivalence, which relates Iwahori–Whittaker sheaves on Fl to $\widehat{G}$-equivariant coherent sheaves on the Springer resolution of $\widehat{G}$, where $\widehat{G}$ is the Langlands dual group of G. The treatment of these Wakimoto sheaves in [AB09] crucially uses t-exactness of nearby cycles, which is currently not available in the motivic setting. A similar problem arises when using diamonds, such as in [ALWY23]. To solve this problem, the authors of loc. cit. refine the study of Wakimoto sheaves, and use them to show t-exactness of the central functor from sheaves on the affine Grassmannian to sheaves on the full affine flag variety. In this section, we will define and study similar objects in the motivic setting, with a view towards a motivic Arkhipov–Bezrukavnikov equivalence, and use them as in [ALWY23] to deduce that $\mathsf{Z}_{\mathbf{a}_0}$ is t-exact. The general strategy will follow [AR, §4], with the necessary modifications to suit our purposes.

5.1. **Standard and costandard motives.** Before we define the motivic Wakimoto functors, let us recall the standard and costandard functors, along with some basic properties. Let $\iota \colon \mathrm{Fl}^{\dagger} \to \mathrm{Fl}$ be the stratification by the $\mathcal{I}$-orbits Fl^w, for $w \in W$. We also write $\iota_w \colon \mathcal{I} \backslash \mathrm{Fl}^w \to \mathcal{I} \backslash \mathrm{Fl}$ for the induced map after modding out $\mathcal{I}$. We denote by $h_w \colon \mathcal{I} \backslash \mathrm{Fl}^w \to \mathcal{I} \backslash S$ the quotient of the structure map.

Definition 5.1. For any $w \in W$, the standard functor is defined as

$$\Delta_w \colon \mathrm{DM}(\mathcal{I} \backslash S) \to \mathrm{DM}(\mathcal{I} \backslash \mathrm{Fl}) \colon \mathcal{F} \mapsto \iota_{w!} h_w^* \mathcal{F}[l(w)],$$

Similarly, the costandard functor for w is defined as

$$\nabla_w \colon \mathrm{DM}(\mathcal{I} \backslash S) \to \mathrm{DM}(\mathcal{I} \backslash \mathrm{Fl}) \colon \mathcal{F} \mapsto \iota_{w*} h_w^* \mathcal{F}[l(w)].$$

By [CvdHS22, Proposition 3.7], these functors send Tate motives on $\mathcal{I} \backslash S$ to stratified Tate motives on $\mathcal{I} \backslash \mathrm{Fl}$, and we will usually view Δ_w and ∇_w as functors $\mathrm{DTM}_{\mathcal{I}}(S) \to \mathrm{DTM}_{\mathcal{I}}(\mathrm{Fl})$. The following proposition is the affine analogue of [EK19, Lemma 4.21], and we follow its proof.

Proposition 5.2. *Both functors $\iota_*, \iota_! \colon \mathrm{DTM}(\mathrm{Fl}^{\dagger}) \to \mathrm{DTM}(\mathrm{Fl})$ are t-exact with respect to the t-structures in Section 2.2.4. The same is true when considering $\mathcal{I}$-equivariant motives on Fl. Therefore, for any $w \in W$, the standard and costandard functors Δ_w and ∇_w are t-exact.*

Proof. By homotopy invariance and $\mathrm{Fl}^w = \mathbf{A}_S^{l(w)}$, the t-exactness of $\iota_!$ is equivalent to the t-exactness of the Δ_w etc.

We will show the t-exactness of Δ_w, the case of ∇_w being similar. To this end, we consider the (co)standard functors as functors $\mathrm{DTM}(S) \to \mathrm{DTM}(\mathrm{Fl})$, i.e., we forget the equivariance.

Since Δ_w is defined via !-pushforward, and lying in $\mathrm{DTM}^{\leq 0}(\mathrm{Fl})$ is a condition on the *-pullback along the stratification, it is clear that Δ_w is right t-exact.

To show the left t-exactness, let $M \in \mathrm{DTM}^{\geq 0}(S)$. We will show $\Delta_w(M) \in \mathrm{DTM}^{\geq 0}(\mathrm{Fl})$ by induction on $l(w)$. If $l(w) = 0$, then ι_w is a closed immersion, pushforward along which is t-exact. If $l(w) > 0$, we can find $v \in W$ and a simple reflection s such that $w = vs$ and $l(w) = l(v) + 1$. Consider the codimension 1 facet $\mathbf{f}_s$ in the closure of $\mathbf{a}_0$ corresponding to s, and the projection $\pi \colon \mathrm{Fl} \to \mathrm{Fl}_{\mathbf{f}_s}$, which is an étale-locally trivial $\mathbf{P}^1$-fibration. As in [RS20, (5.1.2)], there is a distinguished triangle

$$\pi^* \pi_* \Delta_v(M) \to \Delta_v(M) \to \Delta_w(M).$$

Now for any $x \in W$, we have the cartesian diagram

$$
\begin{array}{ccc}
\mathrm{Fl}^{xs} \cup \mathrm{Fl}^x & \xrightarrow{\ k\ } & \mathrm{Fl} \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\
\mathrm{Fl}^{\circ}_{x \cdot W_s}(\mathbf{a}_0, \mathbf{f}_s) & \xrightarrow{\ i\ } & \mathrm{Fl}_{\mathbf{f}_s},
\end{array}
$$

where p is a $\mathbf{P}^1$-fibration, and we write $W_s := W_{\mathbf{f}_s}$ for simplicity. Moreover, $\mathrm{Fl}^{\circ}_{x \cdot W_s}(\mathbf{a}_0, \mathbf{f}_s)$ is an affine space, so that we have an equivalence $\mathrm{DTM}(\mathrm{Fl}^{\circ}_{x \cdot W_s}(\mathbf{a}_0, \mathbf{f}_s)) \cong \mathrm{DTM}(S)$, which is t-exact up to a shift. Applying $k^!$ to the distinguished triangle above and using base change, we get an exact triangle

$$p^* p_* k^! \Delta_v(M) \to k^! \Delta_v(M) \to k^! \Delta_w(M).$$

This reduces us to the claim that $\mathrm{cofib}(p^* p_* \to \mathrm{id})$ is left t-exact as an endofunctor on $\mathrm{DTM}(\mathbf{P}^1)$ stratified as $\mathbf{P}^1 = \mathbf{A}^1 \sqcup S$, which is an easy computation. $\qquad\square$

Proposition 5.3. *Let $v, w \in W$ and $M, N \in \mathrm{DTM}_{\mathcal{I}}(S)$.*

(1) If $l(vw) = l(v) + l(w)$, there are natural isomorphisms $\Delta_v(M) \star \Delta_w(N) \cong \Delta_{vw}(M \otimes N)$ and $\nabla_v(M) \star \nabla_w(N) \cong \nabla_{vw}(M \otimes N)$. These are compatible with associativity as in [AR, Lemma 4.1.4].

(2) There exist isomorphisms $\Delta_v(\mathbf{Z}) \star \nabla_{v^{-1}}(\mathbf{Z}) \cong \Delta_e(\mathbf{Z})(-l(v)) \cong \nabla_{v^{-1}}(\mathbf{Z}) \star \Delta_v(\mathbf{Z})$.

Proof. We follow the proof of [AR, Lemma 4.1.4]. The isomorphism for standards in (1) can be proven verbatim as in loc. cit. For costandards we additionally use [CvdHS22, Corollary 3.10] to make sure the *-pushforward along an open embedding commutes with the (twisted) exterior product; note that, while the proof in loc. cit. only states a Künneth map for *-pushforward with $\mathbf{Z}$, it immediately follows for general Tate motives.

We review the argument for (2) in order to explain the appearance of the extra Tate twist. Recall that forgetting the equivariance gives a fully faithful functor $\mathrm{MTM}_{\mathcal{I}}(\mathrm{Fl}) \to \mathrm{MTM}(\mathrm{Fl})$ [RS20, Proposition 3.2.20]. Since $\Delta_e(\mathbf{Z})(-l(v)) \in \mathrm{MTM}_{\mathcal{I}}(\mathrm{Fl})$ and lying in a certain degree can be checked after forgetting the equivariance, it is thus enough to prove (2) after forgetting the $\mathcal{I}$-action. Hence, we will again abuse notation and view Δ_v and $\Delta_{v^{-1}}$ as functors $\mathrm{DTM}(S) \to \mathrm{DTM}(\mathrm{Fl})$. By (1), we can assume that $w \in \Omega$ or $w = s$ is a simple reflection. The case $w \in \Omega$ is easy, as then $l(w) = 0$, so that $\mathrm{Fl}^{\leq w} \cong S \cong \mathrm{Fl}^{\leq w^{-1}}$ are closed in Fl.

43

On the other hand, for a simple reflection s, the convolution product $\Delta_s(\mathbf{Z}) \star \nabla_s(\mathbf{Z})$ is supported on $\mathrm{Fl}^{\leq s} = \mathrm{Fl}^s \cup \mathrm{Fl}^e$, as $s^2 = e$. Letting $k \colon \mathrm{Fl}^{\leq s} \to \mathrm{Fl}$ denote the closed immersion, we get a localization triangle $k_*\mathbf{Z} \to \iota_{e*}\mathbf{Z} \to \Delta_s(\mathbf{Z})$ in $\mathrm{DTM}(\mathrm{Fl})$. Convolving then gives an exact triangle

$$k_*\mathbf{Z} \star \nabla_s(\mathbf{Z}) \to \nabla_s(\mathbf{Z}) \to \Delta_s(\mathbf{Z}) \star \nabla_s(\mathbf{Z}).$$

The leftmost object in this triangle can be identified with $k_*\mathbf{Z}[1]$: by [RS21, Lemma 3.12 (i)], it is the *-pushforward of $\mathbf{Z}[1]$ along the convolution map $m' \colon \mathrm{Fl}^{\leq s} \overset{\mathcal{I}}{\widetilde{\times}} \mathrm{Fl}^s \to \mathrm{Fl}^{\leq s}$. As in the proof of [RS21, Proposition 3.19], this map can be identified with $\mathbf{P}^1 \times \mathbf{P}^1 \setminus \Delta(\mathbf{P}^1) \to \mathbf{P}^1 \times \mathbf{P}^1 \overset{\mathrm{pr}_2}{\longrightarrow} \mathbf{P}^1$, which is a Zariski-locally trivial $\mathbf{A}^1$-bundle. Hence the natural map $\mathbf{Z}[1] \to m'_* m'^* \mathbf{Z}[1]$ is an isomorphism, which implies that $k_*\mathbf{Z} \star \nabla_s(\mathbf{Z}) \cong k_*\mathbf{Z}[1]$. We claim moreover that $\Delta_s(\mathbf{Z}) \star \nabla_s(\mathbf{Z})$ is supported at the origin. Since

$$\mathrm{Hom}_{\mathrm{DM}(\mathrm{Fl}^{\leq s})}(\mathbf{Z}[1], \iota_{s,*}\mathbf{Z}[1]) \cong \mathrm{Hom}_{\mathrm{DM}(\mathbf{A}^1)}(\mathbf{Z}, \mathbf{Z}) \cong \mathbf{Z},$$

this will imply that the left arrow is isomorphic (up to an invertible scalar) to the natural inclusion $k_*\mathbf{Z}[1] \hookrightarrow \nabla_s(\mathbf{Z})$, the cone of which is $\Delta_e(\mathbf{Z})(-1)$. Thus we get $\Delta_s(\mathbf{Z}) \star \nabla_s(\mathbf{Z}) \cong \Delta_e(\mathbf{Z})(-1)$ (non-canonically), and similarly one can show $\nabla_s(\mathbf{Z}) \star \Delta_s(\mathbf{Z}) \cong \Delta_e(\mathbf{Z})(-1)$.

We are left to prove the claim that $\Delta_s(\mathbf{Z}) \star \nabla_s(\mathbf{Z})$ is supported at the origin. Let $q \colon LG \to \mathrm{Fl}$ be the quotient map as usual, and consider the diagram

$$
\begin{array}{ccccc}
U & \xrightarrow{\quad a \quad} & Y & \xrightarrow{\qquad\qquad b \qquad\qquad} & \mathrm{Fl}^s \\
\downarrow{\scriptstyle j_U} & & \downarrow{\scriptstyle j_Y} & & \downarrow{\scriptstyle j} \\
q^{-1}(\mathrm{Fl}^s) \overset{\mathcal{I}}{\times} \mathrm{Fl}^s & \xrightarrow{f} & q^{-1}(\mathrm{Fl}^s) \overset{\mathcal{I}}{\times} \mathrm{Fl}^{\leq s} & \xrightarrow{g} q^{-1}(\mathrm{Fl}^{\leq s}) \overset{\mathcal{I}}{\times} \mathrm{Fl}^{\leq s} & \xrightarrow{m} & \mathrm{Fl}^{\leq s}
\end{array}
$$

with cartesian squares, where f and g are the natural open immersions, and m is the (restriction of the) multiplication map. Denoting by $\bar{q} \colon q^{-1}(\mathrm{Fl}^s) \overset{\mathcal{I}}{\times} \mathrm{Fl}^{\leq s} \to \mathrm{Fl}^s$ the quotient map on the first factor, we see as in [AR, Lemma 4.1.4] that

$$q^{-1}(\mathrm{Fl}^s) \overset{\mathcal{I}}{\times} \mathrm{Fl}^{\leq s} \xrightarrow{(\bar{q}, m \circ g)} \mathrm{Fl}^s \times \mathrm{Fl}^{\leq s} \cong \mathbf{A}^1 \times \mathbf{P}^1$$

is an isomorphism, cf. also [RS21, (3.20)]. Now, in the diagram above, $U \subset Y$ is an open subset with complement isomorphic to $\mathbf{A}^1$. We denote the inclusion by $c \colon Y \setminus U \subset Y$. Note that $\Delta_s(\mathbf{Z}) \star \nabla_s(\mathbf{Z}) = m_! g_!(\mathbf{Z} \widetilde{\boxtimes} \nabla_s(\mathbf{Z})) = m_! g_! f_*(\mathbf{Z} \widetilde{\boxtimes} \mathbf{Z})$, where we again use [CvdHS22, Corollary 3.10] to ensure the commutation of f_* with the twisted exterior product. The restriction of this motive to $\mathrm{Fl}^s \subset \mathrm{Fl}^{\leq s}$ is given by $j^* m_! g_! f_*(\mathbf{Z} \widetilde{\boxtimes} \mathbf{Z}) \cong b_! a_* j_U^*(\mathbf{Z} \widetilde{\boxtimes} \mathbf{Z})$, which we want to show vanishes. Let $\mathcal{F} := j_Y^*(\mathbf{Z} \widetilde{\boxtimes} \mathbf{Z})$. By localization, we have an exact triangle

$$b_! c_! c^! \mathcal{F} \cong c^! \mathcal{F} \to b_! \mathcal{F} \to b_! a_* a^* \mathcal{F}.$$

Note that $Y \cong \mathbf{A}^2$, the map $b \colon \mathbf{A}^2 \to \mathrm{Fl}^s \cong \mathbf{A}^1$ identifies with projection onto the second factor, and $c \colon \mathbf{A}^1 \to \mathbf{A}^2$ identifies with the diagonal inclusion. Hence $b \circ c \cong \mathrm{id}$, and also $c^!$ and $b_!$ induce the same equivalence $M \mapsto M(-1)[-2]$ on Tate motives. From this it follows that $c^! \mathcal{F} \to b_! \mathcal{F}$ is an isomorphism. $\qquad\square$

In contrast to the affine Grassmannian the convolution product on Fl is not t-exact, even when working with coefficients in a field. However, we can still show t-exactness properties when convolving a standard and a costandard motive, which will be used to show the Wakimoto motives lie in MTM. Classically, this is shown by using t-exactness properties of affine pushforwards. Since this is not available in the motivic setting, we give a different argument.

Recall that $\mathcal{I} \backslash S$ is a placid prestack (in the sense of [RS21, §A.2]), so that $\mathrm{DM}(\mathcal{I} \backslash S)$ is monoidal by composing the exterior product from [RS21, Corollary A.15] with pullback along $\mathcal{I} \backslash S \to (\mathcal{I} \times \mathcal{I}) \backslash S$. After forgetting the equivariance, this monoidal structure is exactly the usual tensor product on $\mathrm{DM}(S)$, and in particular it preserves the subcategory of Tate motives. Recall also from [CvdHS22, Definition and Lemma 4.11] that convolution preserves Tate motives, so that we may view convolution with some $\mathcal{F} \in \mathrm{DTM}_{\mathcal{I}}(\mathrm{Fl})$ as an endofunctor of $\mathrm{DTM}_{\mathcal{I}}(\mathrm{Fl})$.

Lemma 5.4. *Let $v \in W$. Convolving with $\Delta_v(\mathbf{Z})$ on the left or right is left t-exact, while convolving with $\nabla_v(\mathbf{Z})$ on the left or right is right t-exact.*

Proof. We prove the lemma for right convolution with standard motives, the other cases being similar (see also [AR16, Proposition 4.6]). Choosing a reduced expression for v, we can moreover assume by Proposition 5.3 that $v = s$ is a simple reflection.

Let $\mathcal{F} \in \mathrm{DTM}_{\mathcal{I}}^{\geq 0}(\mathrm{Fl})$, which we may assume to be bounded. Then $\mathcal{F}$ is a finite extension of costandard objects $\nabla_w(\mathcal{F}_w)$, for $\mathcal{F}_w \in \mathrm{DTM}_{\mathcal{I}}^{\geq 0}(S)$, so we may assume $\mathcal{F} = \nabla_w(\mathcal{F}_w)$ for some $w \in W$. We now distinguish two cases. If $l(ws) < l(w)$, then by Proposition 5.3 we have

$$\nabla_w(\mathcal{F}_w) \star \Delta_s(\mathbf{Z}) \cong \nabla_{ws}(\mathcal{F}_w) \star \nabla_s(\mathbf{Z}) \star \Delta_s(\mathbf{Z}) \cong \nabla_{ws}(\mathcal{F}_w(-1)),$$

which lies in $\mathrm{DTM}_{\mathcal{I}}^{\geq 0}(\mathrm{Fl})$ by Proposition 5.2.

On the other hand, $\nabla_w(\mathcal{F}_w) \star \Delta_s(\mathbf{Z})$ can always be viewed as the $*$-pushforward of $h_w^*(\mathcal{F}_w[l(w)]) \widetilde{\boxtimes} \Delta_s(\mathbf{Z})$ along the restricted convolution map

$$\mathrm{Fl}^w \widetilde{\times} \mathrm{Fl}^{\leq s} \to \mathrm{Fl}.$$

But under the assumption that $l(w) < l(ws)$, the convolution map induces an isomorphism $\mathrm{Fl}^w \widetilde{\times} \mathrm{Fl}^{\leq s} \to \mathrm{Fl}^w \cup \mathrm{Fl}^{ws}$. As lying in degree ≥ 0 is a condition on the !-pullback, the desired left t-exactness can be checked by using base change, after which it follows from relative purity applied to the immersions of strata on the smooth scheme $\mathrm{Fl}^w \cup \mathrm{Fl}^{ws}$. $\qquad\square$

Combining this with Proposition 5.2, we get the following corollary.

Corollary 5.5. *Let $v, w \in W$. For any $M, N \in \mathrm{MTM}_{\mathcal{I}}(S)$ such that $M \otimes N \in \mathrm{MTM}_{\mathcal{I}}(S)$, the convolutions $\Delta_v(M) \star \nabla_w(N)$ and $\nabla_w(N) \star \Delta_v(M)$ lie in $\mathrm{MTM}_{\mathcal{I}}(\mathrm{Fl})$.*

Proof. The convolution $\Delta_v(M) \star \nabla_w(N)$ agrees with both $\Delta_v(M \otimes N) \star \nabla_w(\mathbf{Z})$ and $\Delta_v(\mathbf{Z}) \star \nabla_w(M \otimes N)$. The other convolution is similar. $\qquad\square$

5.2. **Wakimoto functors.** We now define the Wakimoto functors, the essential image of which will be called the Wakimoto motives. Recall that we fixed some split maximal torus and Borel $T \subseteq B \subseteq G$. In particular, we have the dominant cocharacters $X_*(T)^+ \subseteq X_*(T)$, and the dominance order $\trianglelefteq_B$ on $X_*(T)$ given by

$$\lambda \trianglelefteq_B \mu \iff \mu - \lambda \in X_*(T)^+.$$

We will define a Wakimoto functor for each cocharacter in $X_*(T)$, which agrees (up to a twist) with the costandard functors for dominant cocharacters. Moreover, this collection of functors should be compatible with the addition on $X_*(T)$ and the convolution product. In order to implement this idea on the level of ∞-categories we use the following lemma.

Lemma 5.6. *Let M be a discrete monoid, $\mathcal{C}$ a commutative algebra object in $\mathrm{Pr}^{\mathrm{L}}_{\omega, \mathrm{St}}$ (cf. Section 2.1.1 for notation), and $\mathcal{D}$ an algebra object in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}})$. Then there are equivalences*

$$\mathrm{Fun}_{\mathcal{C}}^{\otimes, \mathrm{L}}(\mathrm{Fun}(M, \mathcal{C}), \mathcal{D}) \cong \mathrm{Fun}^{\otimes, \mathrm{L}}(\mathrm{Fun}(M, \mathrm{Ani}), \mathcal{D}) \cong \mathrm{Fun}^{\otimes}(M, \mathcal{D}).$$

Here, $\mathrm{Fun}_{\mathcal{C}}^{\otimes, L}$ denotes the ∞-category of monoidal, colimit-preserving, $\mathcal{C}$-linear functors, and $\mathrm{Fun}(M, \mathcal{C})$ is equipped with the Day convolution product.

Proof. For the first equivalence, we note that $\mathrm{Fun}(M, \mathcal{C})$ is a product of copies of $\mathcal{C}$. Since $\mathcal{C}$ is compactly generated, it is dualizable in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$. Hence, tensoring with $\mathcal{C}$ (in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$) preserves limits, and in particular products.

The second equivalence is given by precomposing with the (symmetric monoidal) Yoneda embedding, as in [Rob14, Example 3.2.7]. $\qquad\square$

In order to apply this lemma, we want to extend

$$X_*(T)^+ \to \mathrm{DTM}_{\mathcal{I}}(\mathrm{Fl}) \colon \mu \mapsto \nabla_{t(\mu)}(\mathbf{Z})(\langle 2\rho, \mu\rangle) \tag{5.1}$$

to a monoidal functor $X_*(T) \to \mathrm{DTM}_{\mathcal{I}}(\mathrm{Fl})$, where 2ρ is the sum of the positive roots of G. (Here we use the ∞-categorical monoidal structure given by convolution, cf. Section 2.2.8). Since $\{\nabla_{t(\mu)}(\mathbf{Z})(\langle 2\rho, \mu\rangle) \mid \mu \in X_*(T)^+\} \subseteq \mathrm{MTM}_{\mathcal{I}}(\mathrm{Fl})$ by Proposition 5.2, which is the nerve of an ordinary category (as opposed to an ∞-category), Proposition 5.3 yields a monoidal structure on (5.1) (see also [AR, §4.2.3]). Now, as $X_*(T)$ is the Grothendieck completion of $X_*(T)^+$ and $\nabla_{t(\mu)}(\mathbf{Z})(\langle 2\rho, \mu\rangle)$ is invertible with inverse $\Delta_{t(-\mu)}(\mathbf{Z})$, we get a unique extension to a monoidal functor $X_*(T) \to \mathrm{DTM}_{\mathcal{I}}(\mathrm{Fl})$; the essential image of this functor still lies in $\mathrm{MTM}_{\mathcal{I}}(\mathrm{Fl})$ by Corollary 5.5.

Note that $\mathrm{DTM}_{\mathcal{I}}(S)$ is compactly generated by [Kra10, Theorem 7.2.1], as the full subcategory of $\mathrm{DM}(\mathcal{I}\backslash S)$ closed under colimits and containing the Tate twists, which is compactly generated by [RS20, Lemma 2.3.6]. Applying Lemma 5.6 to the situation where $M = X_*(T)$, $\mathcal{C} = \mathrm{DTM}_{\mathcal{I}}(S)$ and $\mathcal{D} = \mathrm{DTM}_{\mathcal{I}}(\mathrm{Fl})$ then permits the following definition.

Definition 5.7. The *full Wakimoto functor*

$$\mathbf{J}^B \colon \mathrm{Rep}_{\widehat{T}}(\mathrm{DTM}_{\mathcal{I}}(S)) \cong \mathrm{Fun}(X_*(T), \mathrm{DTM}_{\mathcal{I}}(S)) \to \mathrm{DTM}_{\mathcal{I}}(\mathrm{Fl})$$

is the monoidal functor which, under the equivalence from Lemma 5.6, corresponds to the functor $X_*(T) \to \mathrm{DTM}_{\mathcal{I}}(\mathrm{Fl})$ constructed above.

Moreover, for any $\mu \in X_*(T)$, we define $\mathbf{J}^B_\mu$ by precomposing $\mathbf{J}^B$ with the natural embedding $\mathrm{DTM}_{\mathcal{I}}(S) \to \mathrm{Rep}_{\widehat{T}}(\mathrm{DTM}_{\mathcal{I}}(S))$ corresponding to $\mu \in X_*(T) = X^*(\widehat{T})$. We call $\mathbf{J}^B_\mu$ the *Wakimoto functor associated to μ*. In particular, we have a decomposition $\mathbf{J}^B \cong \coprod_{\mu \in X_*(T)} \mathbf{J}^B_\mu$.

Remark 5.8. We record some basic properties of the $\mathbf{J}^B_\mu$, whose analogues in the classical setting are well-known.

(1) Since the definition above involves the dominant cocharacters specifically, the Wakimoto functors really depend on the choice of Borel B.

(2) For any $\lambda \in X_*(T)$ and $\mu \in X_*(T)^+$ such that $\mu \trianglerighteq_B \lambda$, there is an isomorphism of functors
$$\mathbf{J}_\lambda^B \cong \nabla_{t(\mu)}(\mathbf{Z})(\langle 2\rho, \mu \rangle) \star \Delta_{t(\lambda-\mu)} \cong \Delta_{t(\lambda-\mu)} \star \nabla_{t(\mu)}(\mathbf{Z})(\langle 2\rho, \mu \rangle).$$

(3) By Corollary 5.5, each $\mathbf{J}_\mu^B$ is t-exact. Hence $\mathbf{J}^B$ is t-exact as well.

(4) Monoidality of $\mathbf{J}^B$ implies that for $\lambda, \mu \in X_*(T)$ and $M, N \in \mathrm{DTM}_\mathcal{I}(S)$, we have an isomorphism
$$\mathbf{J}_\lambda^B(M) \star \mathbf{J}_\mu^B(N) \cong \mathbf{J}_{\lambda+\mu}^B(M \otimes N).$$

Remark 5.9. Our definition of the Wakimoto functors differs slightly from [AR], even in the mixed setting. By the twist appearing in Proposition 5.3(2), the functors $\mathbf{J}_\mu^B$ also have to involve some additional twist. But unlike [AR, §5.3.2], we cannot form half-twists, as we are working with motives. Our particular choice to twist the Wakimoto functors for dominant, rather than anti-dominant, cocharacters, is motivated by the Bernstein elements in the generic Iwahori–Hecke algebra, and we refer to Section 6 for more details.

Lemma 5.10. *Let $\lambda, \mu \in X_*(T)$, and $M, N \in \mathrm{DTM}_\mathcal{I}(S)$. If*
$$\mathrm{Maps}_{\mathrm{DTM}_\mathcal{I}(\mathrm{Fl})}(\mathbf{J}_\lambda^B(M), \mathbf{J}_\mu^B(N)) \neq 0,$$
then $\lambda - \mu$ is a sum of positive roots.

Proof. By convolving with some sufficiently dominant cocharacter, we may assume $\lambda, \mu \in X_*(T)^+$ are dominant. Then the Wakimoto functors agree with costandard functors up to a twist. Hence, by adjunction, the mapping spaces above vanish if $\iota_{t(\mu)}^* \iota_{t(\lambda)*}$ is trivial. By [AR, Lemma 4.1.2], this is the case unless $\lambda - \mu$ is a sum of positive roots (for $T \subseteq B$). $\square$

Recall that we introduced the Wakimoto functors in order to show t-exactness of $\mathbf{Z}_{\mathbf{a}_0}$. As the full subcategory of $\mathrm{DTM}_\mathcal{I}(\mathrm{Fl})$ spanned by the essential image of $\mathbf{J}^B$ is not stable and does not contain the essential image of $\mathbf{Z}_{\mathbf{a}_0}$, we will also consider the following categories.

Definition 5.11. We let
$$\mathrm{Wak}^+ \subseteq \mathrm{Wak} \subseteq \mathrm{DTM}_\mathcal{I}(\mathrm{Fl})$$
be the full stable cocomplete subcategory generated by the images of $\mathbf{J}_\mu^B$ for $\mu \in X_*(T)^+$, resp. $\mu \in X_*(T)$.

We also let $\mathrm{Wak}_{\mathrm{bd}} \subseteq \mathrm{DTM}_\mathcal{I}(\mathrm{Fl})$ be the full subcategory generated under extensions by the images of $\mathbf{J}_\mu^B$ for $\mu \in X_*(T)$. In particular, each object in Wak is a colimit of objects in $\mathrm{Wak}_{\mathrm{bd}}$.

By Remark 5.8 (4), these are monoidal subcategories (with respect to the convolution product). Note that any object in $\mathrm{Wak}_{\mathrm{bd}}$ is bounded, but it is not clear whether a bounded motive in Wak also lies in $\mathrm{Wak}_{\mathrm{bd}}$. . For a motive $\mathcal{F} \in \mathrm{DTM}_\mathcal{I}(\mathrm{Fl})$, we let $!\text{-}\mathrm{Supp}(\mathcal{F}) := \{w \in W \mid \iota_w^! \mathcal{F} \neq 0\}$, and we similarly define $*\text{-}\mathrm{Supp}(\mathcal{F})$.

Proposition 5.12. *An object $\mathcal{F} \in \mathrm{DTM}_\mathcal{I}(\mathrm{Fl})$ lies in Wak^+ if and only if $!\text{-}\mathrm{Supp}(\mathcal{F}) \subset \{t(\mu) \mid \mu \in X_*(T)^+\}$.*

Proof. Since the Schubert cells $\mathrm{Fl}^w \cong \mathbf{A}^{l(w)}$ are affine spaces, this follows from localization and homotopy invariance. $\square$

In order to show the central motives lie in Wak, we need another criterion, which in turn uses the following lemma.

Lemma 5.13. *Let $\mathcal{F} \in \mathrm{DTM}_\mathcal{I}(\mathrm{Fl})$ be bounded. Then there exists a finite subset $A_\mathcal{F} \subset W$ such that*
$$*\text{-}\mathrm{Supp}(\Delta_w(M) \star \mathcal{F}) \subset w \cdot A_\mathcal{F}, \qquad !\text{-}\mathrm{Supp}(\nabla_w(M) \star \mathcal{F}) \subset w \cdot A_\mathcal{F},$$
$$*\text{-}\mathrm{Supp}(\mathcal{F} \star \Delta_w(M)) \subset A_\mathcal{F} \cdot w, \qquad !\text{-}\mathrm{Supp}(\mathcal{F} \star \nabla_w(M)) \subset A_\mathcal{F} \cdot w,$$
for any $M \in \mathrm{DTM}_\mathcal{I}(S)$ and $w \in W$.

Proof. The proofs of [AR, Lemma 4.4.2, Proposition 4.4.4] go through without change, using only geometric arguments and basic properties of the 6-functor-formalism. $\square$

Proposition 5.14. *If a bounded object $\mathcal{F} \in \mathrm{DTM}_\mathcal{I}(\mathrm{Fl})$ satisfies $\Delta_{t(\nu)}(\mathbf{Z}) \star \mathcal{F} \cong \mathcal{F} \star \Delta_{t(\nu)}(\mathbf{Z})$ for all $\nu \in -X_*(T)^+$, then $\mathcal{F} \in \mathrm{Wak}_{\mathrm{bd}}$.*

Proof. Let $A_\mathcal{F}$ be a finite subset of W obtained via Lemma 5.13. Then we can find some $\nu \in -X_*(T)^+$ such that
$$t(\nu) \cdot A_\mathcal{F} \subseteq \{t(\mu) \cdot w \mid \mu \in -X_*(T)_{\mathrm{reg}}^+, w \in W_0\}, \quad A_\mathcal{F} \cdot t(\nu) \subseteq \{w \cdot t(\mu) \mid \mu \in -X_*(T)_{\mathrm{reg}}^+, w \in W_0\},$$
where $X_*(T)_{\mathrm{reg}}^+$ denotes the regular dominant cocharacters. As every W_0-orbit in $X_*(T)$ has a unique (anti-)dominant representative, we get
$$(t(\nu) \cdot A_\mathcal{F}) \cap (A_\mathcal{F} \cdot t(\nu)) \subseteq \{t(\mu) \mid \mu \in -X_*(T)_{\mathrm{reg}}^+\}.$$

Along with the assumption on $\mathcal{F}$ and Lemma 5.13, we deduce that

$$*\text{-}\operatorname{Supp}(\mathbf{J}_\nu^B(\mathbf{Z}) \star \mathcal{F}) \subseteq \{t(\mu) \mid \mu \in -X_*(T)_{\text{reg}}^+\}.$$

By localization and stratified Tateness of $\mathcal{F}$, we see that $\mathbf{J}_\nu^B(\mathbf{Z}) \star \mathcal{F}$ is an extension of standard objects associated to anti-dominant cocharacters. Since $\mathcal{F}$ is moreover bounded, only finitely many such standard objects are nontrivial. As these are Wakimoto motives, and Wak is closed under convolution, we see that $\mathcal{F} \cong \mathbf{J}_{-\nu}^B \star \mathbf{J}_\nu^B \star \mathcal{F} \in \operatorname{Wak}_{\text{bd}}$. $\qquad\square$

Using the centrality isomorphism from Theorem 4.38, the following corollary is immediate.

Corollary 5.15. *For any $\mathcal{F} \in \operatorname{DTM}_{L^+G}(\operatorname{Gr})$, the object $\mathbf{Z}_{\mathbf{a}_0}(\mathcal{F}) \in \operatorname{DTM}_{\mathcal{I}}(\operatorname{Fl})$ lies in* Wak.

Let us be more precise about what it means for a motive to lie in Wak. The next lemma says that any object in Wak admits a *Wakimoto filtration*. We consider the partial order $\preceq$ on $X_*(T)$, where $\lambda \preceq \mu$ if $\mu - \lambda$ is a sum of positive coroots.

Definition and Lemma 5.16. We view $X_*(T)$ as an ∞-category induced by the partially ordered set $(X_*(T), \preceq)$. Then there is a functor

$$\operatorname{Fil}\colon \operatorname{Wak} \to \operatorname{Fun}(X_*(T), \operatorname{Wak})\colon \mathcal{F} \mapsto (\lambda \mapsto \operatorname{Fil}_\lambda(\mathcal{F})),$$

along with a natural transformation $\operatorname{Fil}_\lambda \to \operatorname{id}$ for each λ, such that $\operatorname{Fil}_\lambda(\mathcal{F}) \to \mathcal{F}$ is the final morphism out of an object in the full subcategory of $\operatorname{DTM}_{\mathcal{I}}(\operatorname{Fl})$ generated under extensions and colimits by the images of $\mathbf{J}_\mu^B$ for $\mu \preceq \lambda$. Moreover, there is an isomorphism $\mathcal{F} \cong \operatorname{colim}_{\lambda \in X_*(T)} \operatorname{Fil}_\lambda(\mathcal{F})$ for each $\mathcal{F} \in \operatorname{Wak}$.

Proof. Let $\mathcal{F} \in \operatorname{Wak}_{\text{bd}}$. Then $\mathcal{F}$ lies in the subcategory of $\operatorname{DTM}_{\mathcal{I}}(\operatorname{Fl})$ generated by the essential images of finitely many Wakimoto functors, so that there exists $\mu \in X_*(T)^+$ sufficiently dominant for which $\mathcal{F} \star \mathbf{J}_\mu^B(\mathbf{Z}) \in \operatorname{Wak}^+$. The desired filtration for $\mathcal{F} \star \mathbf{J}_\mu^B(\mathbf{Z})$ is obtained by sending non-dominant cocharacters to 0, and $\lambda \in X_*(T)^+$ to $\iota_{\preceq\lambda,*}\iota_{\preceq\lambda}^!\mathcal{F}$, where $\iota_{\preceq\lambda}\colon \operatorname{Fl}^{\leq t(\lambda)} \to \operatorname{Fl}$ denotes the inclusion , and the transition maps for $\lambda' \preceq \lambda$ arise via localization. Convolving with the monoidal inverse of $\mathbf{J}_\mu^B(\mathbf{Z})$, as well as shifting the filtration by $-\mu$, then gives the desired filtration for $\mathcal{F}$. This is clearly functorial for uniformly bounded $\mathcal{F}$, and independent of the choice of μ. Passing to the colimit then defines $\operatorname{Fil}\colon \operatorname{Wak} \to \operatorname{Fun}(X_*(T), \operatorname{Wak})$ in general, and it clearly satisfies the required properties. $\qquad\square$

We note that the construction above is different, but equivalent to the construction of the Wakimoto filtration in [AR, ALWY23].

Corollary 5.17. *View $X_*(T)$ as a discrete ∞-category. Then the functor* Fil *from Definition and Lemma 5.16 induces a functor*

$$\operatorname{Grad}\colon \operatorname{Wak} \to \operatorname{Fun}(X_*(T), \operatorname{DTM}_{\mathcal{I}}(S)),$$

such that each $\mathcal{F}$ is a (possibly infinite) extension of all $\mathbf{J}_\lambda^B(\operatorname{Grad}_\lambda(\mathcal{F}))$'s.

We call $\operatorname{Grad}(\mathcal{F})$ the *associated graded of the Wakimoto filtration of $\mathcal{F}$*.

Proof. Fix $\lambda \in X_*(T)$, and consider the cofiber $\mathcal{F}'$ of the natural map $\operatorname{colim}_{\mu \prec \lambda} \operatorname{Fil}_\mu(\mathcal{F}) \to \operatorname{Fil}_\lambda(\mathcal{F})$, which is functorial. After convolving with $\mathbf{J}_\nu^B(\mathbf{Z})$ for sufficiently dominant $\nu \in X_*(T)^+$, this $\mathcal{F}'$ lies in the image of the fully faithful embedding $\iota_{t(\lambda)*}\colon \operatorname{DTM}_{\mathcal{I}}(\operatorname{Fl}^{t(\lambda)}) \to \operatorname{DTM}_{\mathcal{I}}(\operatorname{Fl})$. By homotopy invariance, we moreover have $\operatorname{DTM}_{\mathcal{I}}(\operatorname{Fl}^{t(\lambda)}) \cong \operatorname{DTM}_{\mathcal{I}}(S)$. It follows that $\mathcal{F}'$ identifies canonically with $\mathbf{J}_\lambda^B(\operatorname{Grad}_\lambda(\mathcal{F}))$ for some $\operatorname{Grad}_\lambda(\mathcal{F}) \in \operatorname{DTM}_{\mathcal{I}}(S)$, which concludes the proof. $\qquad\square$

The following property of this associated graded is immediate.

Corollary 5.18. *There is a natural isomorphism*

$$\operatorname{Grad} \circ \mathbf{J}^B \cong \operatorname{id}$$

of endofunctors on $\operatorname{Fun}(X_*(T), \operatorname{DTM}_{\mathcal{I}}(S))$. In other words, Grad *defines a splitting of $\mathbf{J}^B$.*

Finally, we mention some monoidality properties of Grad (although we do not construct a full ∞-categorical monoidal structure).

Proposition 5.19. *For $\lambda, \mu, \nu \in X_*(T)$, there exists a canonical morphism*

$$\alpha_{\lambda,\mu}\colon \operatorname{Grad}_\lambda(-) \otimes \operatorname{Grad}_\mu(-) \to \operatorname{Grad}_{\lambda+\mu}(- \star -)$$

of functors $\operatorname{Wak} \times \operatorname{Wak} \to \operatorname{DTM}(S)$, such that $\bigoplus_{\lambda+\mu=\nu} \alpha_{\lambda,\mu}$ is an isomorphism.

Proof. The proof is similar to [AR, Lemma 4.7.4 and Proposition 4.7.5]. However, it becomes simpler, as the failure of t-exactness of the convolution product is not an issue for us. For $\mathcal{F}, \mathcal{F}' \in \mathrm{Wak}$, there are natural morphisms

$$\mathrm{Fil}_\lambda(\mathcal{F}) \star \mathrm{Fil}_\mu(\mathcal{F}') \to \mathcal{F} \star \mathcal{F}' \to \mathrm{cofib}(\mathrm{Fil}_{\lambda+\mu}(\mathcal{F} \star \mathcal{F}') \to \mathcal{F} \star \mathcal{F}').$$

By Lemma 5.10, this composition must vanish, so that we get a canonical morphism

$$\mathrm{Fil}_\lambda(\mathcal{F}) \star \mathrm{Fil}_\mu(\mathcal{F}') \to \mathrm{Fil}_{\lambda+\mu}(\mathcal{F} \star \mathcal{F}').$$

By using Lemma 5.10 again, the morphisms

$$\operatorname*{colim}_{\lambda' \prec \lambda} \mathrm{Fil}_{\lambda'}(\mathcal{F}) \star \mathrm{Fil}_\mu(\mathcal{F}') \to \mathbf{J}^B_{\lambda+\mu}(\mathrm{Grad}_{\lambda+\mu}(\mathcal{F} \star \mathcal{F}'))$$

and

$$\mathrm{Fil}_\lambda(\mathcal{F}) \star \operatorname*{colim}_{\mu' \prec \mu} \mathrm{Fil}_\mu(\mathcal{F}') \to \mathbf{J}^B_{\lambda+\mu}(\mathrm{Grad}_{\lambda+\mu}(\mathcal{F} \star \mathcal{F}'))$$

vanish, which induces a canonical morphism

$$\alpha_{\lambda,\mu} \colon \mathrm{Grad}_\lambda(\mathcal{F}) \otimes \mathrm{Grad}_\mu(\mathcal{F}') \to \mathrm{Grad}_{\lambda+\mu}(\mathcal{F} \star \mathcal{F}').$$

To check that $\bigoplus_{\lambda+\mu=\nu} \alpha_{\lambda,\mu}$ is an isomorphism, we may restrict $\mathrm{Wak}_{\mathrm{bd}}$ by continuity. If $\mathcal{F} = \mathbf{J}^B_\lambda(M)$ for some $M \in \mathrm{DTM}(S)$, the claim follows from Remark 5.8 (4). For general bounded objects, this then follows by induction on the Wakimoto-filtration of $\mathcal{F}$. $\qquad\square$

5.3. t-exactness of the central functor. We are now ready to show $\mathsf{Z}_{\mathbf{a}_0}$ is t-exact. The idea is to use Corollary 5.15 to write a (bounded) central motive as an extension of Wakimoto motives. It then remains to see that for mixed Tate motives on Gr, the resulting Wakimoto motives appearing are also mixed Tate. This will be deduced using the geometric constant term functors on the affine flag variety as in [HR21, §6.1] and [AGLR22, §6.3]. In order to compare constant term functors for different affine flag varieties, we introduce a new notation, different from [CvdHS22, §5.1]. Namely, fix a cocharacter $\lambda \in X_*(T)$, and recall the notation from Section 4.3 and Theorem 4.25. In particular, the fixed points Fl^0 depend on the chosen λ, and we have the associated Levi subgroup $M \subseteq G$ and parabolic subgroup $P \subseteq G$, defined as the fixed points, resp. attractor, of the $\mathbf{G}_{\mathrm{m}}$-action on G induced by λ. In order to define the constant term functors with values in equivariant motives, we need to know the $L^+ M_{\mathbf{f}}$-action on Fl restricts to actions on $\mathrm{Fl}^\pm$ and Fl^0. This follows from the following general statement.

Lemma 5.20. *Let H/S be a group scheme acting on an ind-scheme X, and $\mathbf{G}_{\mathrm{m}} \to H$ a homomorphism. This yields a $\mathbf{G}_{\mathrm{m}}$-action on X by restriction, and a $\mathbf{G}_{\mathrm{m}}$-action on H by conjugation. Then the action of $H^0 \subseteq H$ on X restricts to an action on X^0, and similarly for attractors and repellers.*

Proof. Let S' be any S-scheme, and $h \in \mathrm{Hom}_S(S', H^0)$ and $x \in \mathrm{Hom}_S(S', X^0)$ two S'-valued points. Then for any $g \in \mathrm{Hom}_S(S', \mathbf{G}_{\mathrm{m}})$, we have $ghg^{-1} = h$ and $gx = x$. It follows that $ghx = ghg^{-1}gx = hx$, which proves the statement about the fixed points. The case of attractors and repellers can be handled similarly. $\qquad\square$

Now consider the maps

$$L^+ M_{\mathbf{a}_0} \backslash \mathrm{Fl}^0 \xleftarrow{\overline{q}^\pm_{\mathbf{a}_0}} L^+ M_{\mathbf{a}_0} \backslash \mathrm{Fl}^\pm \xrightarrow{\overline{p}^\pm_{\mathbf{a}_0}} L^+ M_{\mathbf{a}_0} \backslash \mathrm{Fl},$$

obtained by quotienting out the $L^+ M_{\mathbf{a}_0}$-action from the usual hyperbolic localization diagram.

Definition 5.21. The geometric constant term functor associated to P is

$$\mathrm{CT}^{\mathbf{a}_0}_P := (\overline{q}^+_{\mathbf{a}_0})_! (\overline{p}^+_{\mathbf{a}_0})^* \colon \mathrm{DM}(\mathcal{I} \backslash \mathrm{Fl}) \to \mathrm{DM}(L^+ M_{\mathbf{a}_0} \backslash \mathrm{Fl}^0).$$

Note that this really only depends on the parabolic P, rather than the cocharacter λ. By [CvdHS22, Proposition 2.5], this functor is canonically equivalent to $(\overline{q}^-_{\mathbf{a}_0})_* (\overline{p}^-_{\mathbf{a}_0})^!$. Moreover, these constant term functors preserve (anti-effective) stratified Tate motives by Proposition 4.33. Recall also the usual constant term functor on the affine Grassmannian from [CvdHS22, Definition 5.2]

$$\mathrm{CT}^{\mathbf{f}_0}_P := (\overline{q}^+_{\mathbf{f}_0})_! (\overline{p}^+_{\mathbf{f}_0})^* \colon \mathrm{DM}(L^+ G \backslash \mathrm{Gr}_G) \to \mathrm{DM}(L^+ M \backslash \mathrm{Gr}_M),$$

which similarly preserves stratified Tate motives. (In contrast to loc. cit., we have not included a shift in the definition of the constant term functor.)

Since in this section, we are mostly interested in the Iwahori-level, rather than general parahoric, we will denote the central functor $\mathsf{Z}_{\mathbf{a}_0}$ from Definition 4.15 by Z_G. This notation will make it easier to compare central functors for different groups. For example, by Proposition 3.17, there is a natural equivalence of functors

$$\mathrm{CT}^{\mathbf{a}_0}_P \circ \mathsf{Z}_G \cong \mathsf{Z}_M \circ \mathrm{CT}^{\mathbf{f}_0}_P, \tag{5.2}$$

where we implicitly use the inclusion $\mathrm{Gr}_M \subseteq \mathrm{Fl}^0$ on reduced loci from Theorem 4.25.

Now, assume that λ is regular anti-dominant (for the fixed Borel B), in which case $M = T$ is the maximal torus and $P = B^-$ is the opposite Borel. As before, cf. Lemma 4.24, we can write the repeller for the induced

$\mathbf{G}_{\mathrm{m}}$-action on Fl as $\coprod_{w\in W}\mathcal{S}_w$, a coproduct of semi-infinite orbits, keeping the notation from Section 4.4.3. These semi-infinite orbits are exactly the LU-orbits in Fl, where $U\subseteq B$ is the unipotent radical. Since the fixed point locus of this $\mathbf{G}_{\mathrm{m}}$-action consists of discrete points (up to taking reduced subschemes) indexed by W, we will also write $\mathrm{CT}_{B^-}^w\colon \mathrm{DTM}_{\mathcal{I}}(\mathrm{Fl})\to \mathrm{DTM}_{L^+T}(S)$ for the value of $\mathrm{CT}_{B^-}^{\mathbf{a}_0}$ at such a point $w\in W$.

To check which objects appear in the Wakimoto filtrations of central motives, we will apply the constant term functors, along with the following results, compare [AR, §4.5.3].

Proposition 5.22. *For $\mu\in X_*(T)^+$ and $w\in W$, there is an isomorphism*

$$\mathrm{CT}_{B^-}^{t(\mu)w}(\nabla_{t(\mu)}(\mathbf{Z})\star -)\cong \mathrm{CT}_{B^-}^w(-)[\langle 2\rho,\mu\rangle] \tag{5.3}$$

of functors $\mathrm{DTM}_{\mathcal{I}}(\mathrm{Fl})\to \mathrm{DTM}_{L^+T}(S)$.

Proof. The proof is analogous to [ALWY23, Proposition 3.23]. Let $i_w\colon \mathcal{S}_w\to \mathrm{Fl}$ be the inclusion, $f_w\colon \mathcal{S}_w\to S$ the structure map, and $\mathcal{F}\in \mathrm{DTM}_{\mathcal{I}}(\mathrm{Fl})$ any object. Fix also a lift $z^\mu\in N_G(T)(\mathbf{Z}(\!(t)\!))\subseteq LG(\mathbf{Z})$ of $t(\mu)\in W$, and denote its image in $\mathrm{Fl}(\mathbf{Z})$ the same way. By hyperbolic localization, we have

$$\mathrm{CT}_{B^-}^{t(\mu)w}(\nabla_{t(\mu)}(\mathbf{Z})\star \mathcal{F})\cong f_{t(\mu)w,*}i_{t(\mu)w}^!(m'_*(\mathbf{Z}[\langle 2\rho,\mu\rangle]\widetilde{\boxtimes}\mathcal{F})),$$

where $m'\colon \mathrm{Fl}^{t(\mu)}\widetilde{\times}\mathrm{Fl}\to \mathrm{Fl}$ is the restriction of the convolution map. Now, we have $\mathcal{S}_{t(\mu)}\supseteq \mathrm{Fl}^{t(\mu)}$: for this we can assume S is the spectrum of a field, in which case it can be shown as in [ALWY23, Lemma 2.12]. As $\mathrm{Fl}^{t(\mu)}=L^+Uz^\mu$ (again by [ALWY23, Lemma 2.12]), this implies the fiber product $\mathcal{S}_{t(\mu)w}\times_{\mathrm{Fl}}(\mathrm{Fl}^{t(\mu)}\widetilde{\times}\mathrm{Fl})$ agrees with $L^+Uz^\mu L^+U\overset{L^+U}{\times}\mathcal{S}_w$.

Now, consider the following diagram with the right square cartesian:

By base change, it suffices to compute

$$f_{t(\mu)w,*}m''_*\widetilde{i}^!(\mathbf{Z}[\langle 2\rho,\mu\rangle]\widetilde{\boxtimes}\mathcal{F}).$$

By definition, we have an isomorphism

$$b^!\widetilde{i}^!(\mathbf{Z}[\langle 2\rho,\mu\rangle]\widetilde{\boxtimes}\mathcal{F})\cong a^!(\mathbf{Z}[\langle 2\rho,\mu\rangle]\boxtimes i_w^!(\mathcal{F})).$$

Replacing L^+U by a suitable split unipotent truncation, purity yields a similar isomorphism when replacing $a^!,b^!$ by a^*,b^*. In that case, a and b are torsors under split unipotent groups, hence Zariski-locally trivial by [RS20, Proposition A.6]. We can then compute

$$\mathrm{CT}_{B^-}^{t(\mu)w}(\nabla_{t(\mu)}(\mathbf{Z})\star \mathcal{F})\cong f_{t(\mu)w,*}m''_*b_*b^*\widetilde{i}^!(\mathbf{Z}[\langle 2\rho,\mu\rangle]\widetilde{\boxtimes}\mathcal{F})$$

$$\cong (h_{t(\mu)}\times f_w)_*a_*a^*(\mathbf{Z}[\langle 2\rho,\mu\rangle]\boxtimes i_w^!(\mathcal{F}))\cong \mathrm{CT}_{B^-}^w(\mathcal{F})[\langle 2\rho,\mu\rangle],$$

where the last isomorphism uses homotopy invariance and $\mathrm{Fl}^{t(\mu)}\cong \mathbf{A}^{l(t(\mu))}$. $\qquad\square$

Since the maximal reductive quotient of $\mathcal{I}$ is naturally isomorphic to the torus T, [RS20, Proposition 2.2.11] gives us canonical equivalences

$$\mathrm{DM}(\mathcal{I}\backslash S)\cong \mathrm{DM}(T\backslash S)\cong \mathrm{DM}(L^+T\backslash S),$$

identifying the subcategories of Tate motives.

Corollary 5.23. *Let $\mu\in X_*(T)$. Then $\mathrm{CT}_{B^-}^{\mathbf{a}_0}\circ \mathbf{J}_\mu^B$ takes values in motives supported on the connected component of $(\mathrm{Fl})^0$ corresponding to $t(\mu)$, and there is an isomorphism*

$$\mathrm{CT}_{B^-}^{t(\mu)}\circ \mathbf{J}_\mu^B\cong \mathrm{id}(\langle 2\rho,\mu\rangle)[\langle 2\rho,\mu\rangle]$$

of functors $\mathrm{DTM}_{\mathcal{I}}(S)\to \mathrm{DTM}_{\mathcal{I}}(S)$.

Proof. Let $\nu\in X_*(T)^+$ be such that $\nu\unrhd_B\mu$, and $\mathcal{F}\in \mathrm{DTM}_{\mathcal{I}}(S)$. By Remark 5.8 and Proposition 5.22, there exist isomorphisms

$$\mathrm{CT}_{B^-}^w(\mathbf{J}_\mu^B(\mathcal{F}))\cong \mathrm{CT}_{B^-}^{t(\nu-\mu)w}(\mathbf{J}_\nu^B(\mathbf{Z})\star \Delta_e(\mathcal{F}))(-\langle 2\rho,\nu-\mu\rangle)[-\langle 2\rho,\nu-\mu\rangle]$$

$$\cong \mathrm{CT}_{B^-}^{t(-\mu)w}(\Delta_e(\mathcal{F}))(\langle 2\rho,\mu\rangle)[\langle 2\rho,\mu\rangle].$$

Since $\mathrm{Fl}^e\subset \mathcal{S}_e$, the rightmost term vanishes unless $w=t(\mu)$, and we clearly have $\mathrm{CT}_{B^-}^e\circ \Delta_e\cong \mathrm{id}$. $\qquad\square$

With all the preparations out of the way, we can now prove the main theorem of this section. As usual, let $\deg := \langle 2\rho, -\rangle\colon \mathrm{Gr}_T \to \mathbf{Z}$ be the (locally constant) degree function.

Theorem 5.24. *There is a natural isomorphism of functors*

$$\mathrm{Grad} \circ \mathsf{Z}_G \cong \mathrm{CT}^{\mathbf{f}_0}_{B^-}(-\deg)[-\deg].$$

In particular, $\mathsf{Z}_G = \mathsf{Z}_{\mathbf{a}_0}$ is t-exact.

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Nilp}_{\mathbf{Q}}\, \mathrm{DTM}_{L^+T}(\mathrm{Fl}^0) & \xleftarrow{\ \mathrm{CT}^{\mathbf{a}_0}_{B^-}[-\deg]\ } & \mathrm{Nilp}_{\mathbf{Q}}\, \mathrm{Wak} \\[2pt]
\mathsf{Z}_T \uparrow & & \uparrow \mathsf{Z}_G \\[2pt]
\mathrm{DTM}_{L^+T}(\mathrm{Gr}_T) & \xleftarrow[\ \mathrm{CT}^{\mathbf{f}_0}_{B^-}[-\deg]\ ]{} & \mathrm{DTM}_{L^+G}(\mathrm{Gr}_G)
\end{array}
\tag{5.4}
$$

arising from Proposition 3.17 and Corollary 5.15. By Corollary 3.14, Z_T agrees with the pushforward along the open and closed immersion (on reduced loci) $\mathrm{Gr}_T \subseteq \mathrm{Fl}^0$ (with trivial monodromy). Moreover, the lower arrow is t-exact by [CvdHS22, Proposition 5.9] (since λ is anti-dominant).

Recall from Corollary 5.17 that every $\mathcal{F} \in \mathrm{Wak}$ is an extension of $\mathbf{J}^B_\mu(\mathrm{Grad}_\mu(\mathcal{F}))$ for varying μ. Since we know $\mathrm{CT}^{\mathbf{a}_0}_{B^-}(\mathbf{J}^B_\mu(\mathrm{Grad}_\mu(\mathcal{F})))$ is supported on $t(\mu) \in \mathrm{Fl}^0$ with value $\mathrm{Grad}_\mu(\mathcal{F})(\langle 2\rho, \mu\rangle)[\langle 2\rho, \mu\rangle]$ by Corollary 5.23, the natural isomorphism $\mathrm{Grad} \circ \mathsf{Z}_G \cong \mathrm{CT}^{\mathbf{f}_0}_{B^-}(-\deg)[-\deg]$ follows from (5.4) (additionally using Corollary 5.18).

Again using the fact that Z_G is an extension of $\mathbf{J}^B_\mu(\mathrm{Grad}_\mu(\mathsf{Z}_G(-)))$'s, each of which is t-exact by t-exactness of $\mathrm{CT}^{\mathbf{f}_0}_{B^-}(-\deg)[-\deg]$, we conclude that Z_G itself is t-exact. $\qquad\square$

5.4. **The case of general parahorics.** Until now, this section was only concerned with the case where $\mathbf{f} = \mathbf{a}_0$ is an alcove. Following an idea of Achar, we now use these results to show that the central functor $\mathsf{Z}_{\mathbf{f}}$ is in fact t-exact for general facets. For the rest of this section, fix a facet $\mathbf{f}$ in the closure of $\mathbf{a}_0$. We start with the analogues of Definition 5.1 and Proposition 5.2, and use similar notations $\iota_{\overline{w}}$ and $h_{\overline{w}}$ for $\overline{w} \in W/W_{\mathbf{f}}$.

Definition 5.25. For $\overline{w} \in W/W_{\mathbf{f}}$, the standard functor is

$$\Delta^{\mathbf{f}}_{\overline{w}}\colon \mathrm{DM}(\mathcal{I}\backslash S) \to \mathrm{DM}(\mathcal{I}\backslash \mathrm{Fl}_{\mathbf{f}})\colon \mathcal{F} \mapsto \iota_{\overline{w}!} h_{\overline{w}}^* \mathcal{F}[l(\overline{w})],$$

while the costandard functor is

$$\nabla^{\mathbf{f}}_{\overline{w}}\colon \mathrm{DM}(\mathcal{I}\backslash S) \to \mathrm{DM}(\mathcal{I}\backslash \mathrm{Fl}_{\mathbf{f}})\colon \mathcal{F} \mapsto \iota_{\overline{w}*} h_{\overline{w}}^* \mathcal{F}[l(\overline{w})].$$

These functors preserve Tate motives by [CvdHS22, Proposition 3.7]. In the following proposition, which is an adaptation of [AR16, Proposition 4.7] to the motivic setting, we consider Tate motives on $\mathrm{Fl}_{\mathbf{f}}$ with respect to the stratification by $\mathcal{I}$-orbits.

Proposition 5.26. *For any $\overline{w} \in W/W_{\mathbf{f}}$, the functors $\Delta^{\mathbf{f}}_{\overline{w}}, \nabla^{\mathbf{f}}_{\overline{w}}\colon \mathrm{DTM}_{\mathcal{I}}(S) \to \mathrm{DTM}_{\mathcal{I}}(\mathrm{Fl}_{\mathbf{f}})$ are t-exact.*

Proof. We will consider the case of standard functors, as costandard functors can be handled similarly. Note also that standard functors are automatically right t-exact, and that we may forget about the $\mathcal{I}$-equivariance.

Since $\mathbf{f}$ lies in the closure of $\mathbf{a}_0$, we can consider the projection $\pi\colon \mathrm{Fl} \to \mathrm{Fl}_{\mathbf{f}}$, which is a Zariski-locally trivial fibration with typical fiber $L^+G_{\mathbf{f}}/\mathcal{I}$ [RS20, Proposition 4.3.13 (i)] (the Zariski local triviality follows from the fact that $LG \to \mathrm{Fl}_{\mathbf{f}}$ has Zariski local sections). Let d be the relative dimension of the smooth S-scheme $L^+G_{\mathbf{f}}/\mathcal{I}$. Since π is $\mathcal{I}$-equivariant and smooth surjective, $\pi^*[d]$ preserves stratified Tate motives and is conservative and t-exact by [RS20, Proposition 4.3.9 (i)].

Now, let $w_{\mathbf{f}} \in W_{\mathbf{f}}$ be the longest element, and consider the two maps $\mathrm{Fl} \overset{\mathcal{I}}{\times} \mathrm{Fl}^{\leq w_{\mathbf{f}}} \to \mathrm{Fl}$, given by the projection on the first factor, and the convolution respectively. These maps agree after composition with π, so they induce a map

$$\mathrm{Fl} \overset{\mathcal{I}}{\times} \mathrm{Fl}^{\leq w_{\mathbf{f}}} \to \mathrm{Fl} \times_{\mathrm{Fl}_{\mathbf{f}}} \mathrm{Fl},$$

which is an isomorphism (compare [AR16, Proof of Lemma 4.3]). Let $w \in W$ be the smallest length representative of $\overline{w}$, so that $\pi_! \circ \Delta_w \cong \Delta^{\mathbf{f}}_{\overline{w}}$. Then base change and Proposition 5.3(1) yield an isomorphism

$$\pi^*[d](\Delta^{\mathbf{f}}_{\overline{w}})(-) \cong \Delta_w(\mathbf{Z}) \star \pi^*[d](\Delta^{\mathbf{f}}_{\overline{e}})(-).$$

Now $\Delta^{\mathbf{f}}_{\overline{e}}(-)$ is t-exact, thus so is $\pi^*[d](\Delta^{\mathbf{f}}_{\overline{e}})(-)$, and hence $\pi^*[d](\Delta^{\mathbf{f}}_{\overline{w}})(-)$ is left t-exact by Lemma 5.4. This implies that $\Delta^{\mathbf{f}}_{\overline{w}}$ is left t-exact by t-exactness and conservativity of $\pi^*[d]$. $\qquad\square$

Now, consider the convolution product

$$- \star -\colon \mathrm{DM}(\mathcal{I}\backslash \mathrm{Fl}) \times \mathrm{DM}(\mathcal{I}\backslash \mathrm{Fl}_{\mathbf{f}}) \to \mathrm{DM}(\mathcal{I}\backslash \mathrm{Fl}_{\mathbf{f}}),$$

which preserves stratified Tate motives by the same argument as in [RS21, Theorem 3.17]. At least on homotopy categories, it clearly determines an action of the monoidal category $\mathrm{DTM}_{\mathcal{I}}(\mathrm{Fl})$ on $\mathrm{DTM}_{\mathcal{I}}(\mathrm{Fl}_{\mathbf{f}})$.

Lemma 5.27. *Let $v \in W$ be any element. Then left-convolving with $\Delta_v(\mathbf{Z})$ (resp. $\nabla_v(\mathbf{Z})$) is a left (resp. right) t-exact endofunctor of $\mathrm{DTM}_{\mathcal{I}}(\mathrm{Fl}_{\mathbf{f}})$.*

Proof. As usual, we only prove the case of standard functors. Let $\mathcal{F} \in \mathrm{DTM}_{\mathcal{I}}(\mathrm{Fl}_{\mathbf{f}})$ be any object. The diagram

$$
\begin{array}{ccc}
\mathrm{Fl} \,\widetilde{\times}\, \mathrm{Fl} & \xrightarrow{\ m\ } & \mathrm{Fl} \\
{\scriptstyle \mathrm{id}\,\widetilde{\times}\,\pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
\mathrm{Fl} \,\widetilde{\times}\, \mathrm{Fl}_{\mathbf{f}} & \xrightarrow{\ m\ } & \mathrm{Fl}_{\mathbf{f}}
\end{array}
$$

is cartesian. As in the proof of Proposition 5.26, π is smooth and $\pi^*[d]$ is t-exact and conservative, where d is the relative dimension. Then $\pi^*(\Delta_v(\mathbf{Z}) \star \mathcal{F})[d] \cong \Delta_v(\mathbf{Z}) \star \pi^*(\mathcal{F})[d]$, so the proposition follows from Lemma 5.4. $\qquad\square$

The idea of the following proof was explained to us by Lourenço.

Lemma 5.28. *There exists a Borel B containing T such for any dominant μ (with respect to this Borel), $t(\mu)$ is minimal in its right $W_{\mathbf{f}}$-orbit.*

Proof. It is equivalent to require that the length functions on W and $W/W_{\mathbf{f}}$ assign the same value to $t(\mu)$ and $\overline{t(\mu)}$, respectively. If $\mathbf{f}_0$ lies in the closure of $\mathbf{f}$, then by [RS20, Remark 4.2.16] we may take B to be the unique Borel whose associated Weyl chamber contains $\mathbf{a}_0$. In general, it suffices to consider the case where $\mathbf{f}$ is a vertex. Let $\Sigma^+(\mathbf{f})$ be the set of affine roots α which vanish at $\mathbf{f}$ and such that $\alpha(\mathbf{a}_0) > 0$. By [Ric13, (1.8), (1.10)], we need to find B such that if μ is B-dominant, then $\alpha(\mathbf{a}_0 + t(\mu)) > 0$ for all $\alpha \in \Sigma^+(\mathbf{f})$ (note that the normalization of the Kottwitz map in op. cit. differs from ours by a sign).

Each $\alpha \in \Sigma^+(\mathbf{f})$ vanishes on some affine hyperplane in the standard apartment $\mathscr{A}$. The complement of this hyperplane is a disjoint union of connected components $\mathbb{H}^+_\alpha \sqcup \mathbb{H}^-_\alpha$, where $\mathbb{H}^+_\alpha$ is determined by the requirement that it contains $\mathbf{a}_0$. Then $\cap_{\alpha \in \Sigma^+(\mathbf{f})} \mathbb{H}^+_\alpha$ is an open affine cone with vertex at $\mathbf{f}$, and we need to choose B so that its Weyl chamber C satisfies $\mathbf{a}_0 + C \subset \cap_{\alpha \in \Sigma^+(\mathbf{f})} \mathbb{H}^+_\alpha$. Let τ be the translation of $\mathscr{A}$ which sends $\mathbf{f}$ to $\mathbf{f}_0$. Then the boundary of the closure of $\tau(\mathbb{H}^+_\alpha)$ is a hyperplane equal to the vanishing locus of some non-affine root in the root system associated (G, T). Thus, for B we may pick any Borel whose Weyl chamber is contained in $\cap_{\alpha \in \Sigma^+(\mathbf{f})} \tau(\mathbb{H}^+_\alpha)$. $\qquad\square$

Remark 5.29. If $\mathbf{f}$ is a non-special vertex then the gradients of the affine roots vanishing at $\mathbf{f}$ do not give all the (non-affine) roots associated to (G, T), even up to scaling. Hence there will be at least two possible Borels in Lemma 5.28 in this case. For example, if $\mathbf{f}$ is a right-angle vertex in the B_2 affine root system, then there are two possible Borels since each Weyl chamber has angle $\pi/4$.

These results allow us to generalize (part of) Theorem 5.24 to general facets.

Theorem 5.30. *The central functor $\mathsf{Z}_{\mathbf{f}}\colon \mathrm{DTM}_{L^+G}(\mathrm{Gr}) \to \mathrm{DTM}_{L^+ G_{\mathbf{f}}}(\mathrm{Fl}_{\mathbf{f}})$ is t-exact.*

Proof. Since $\mathbf{f}$ lies in the closure of $\mathbf{a}_0$, we can consider the projection $\pi\colon \mathrm{Fl} \to \mathrm{Fl}_{\mathbf{f}}$. Let us choose B as in Lemma 5.28; this is harmless since $\mathsf{Z}_{\mathbf{f}}$ does not depend on a choice of Borel. Then for μ dominant, $\mathrm{Fl}^{t(\mu)}$ has the same dimension as its image in $\mathrm{Fl}_{\mathbf{f}}$ by Proposition 4.4(3). This implies that $\pi_* \nabla_{t(\mu)} \cong \nabla^{\mathbf{f}}_{\overline{t(\mu)}}$, which is t-exact by Proposition 5.26. In particular, for any $\lambda \in X_*(T)$ and $\mu \in X_*(T)^+$ with $\mu \trianglerighteq_B \lambda$, pushing forward the Wakimoto functors gives

$$\pi_* \mathbf{J}^B_\lambda \cong \pi_*(\Delta_{t(\lambda-\mu)}(\mathbf{Z}) \star \nabla_{t(\mu)}(\langle 2\rho, \mu\rangle)) \cong \Delta_{t(\lambda-\mu)}(\mathbf{Z}) \star \nabla^{\mathbf{f}}_{\overline{t(\mu)}}(\langle 2\rho, \mu\rangle),$$

which is left t-exact by Lemma 5.27.

Since we have $\mathsf{Z}_{\mathbf{f}} \cong \pi_* \circ \mathsf{Z}_{\mathbf{a}_0}$ (after forgetting the equivariance), the t-exact Wakimoto filtration on $\mathsf{Z}_{\mathbf{a}_0}$ from Theorem 5.24 shows that $\mathsf{Z}_{\mathbf{f}}$ is left t-exact, at least when considering the stratification of $\mathrm{Fl}_{\mathbf{f}}$ by $\mathcal{I}$-orbits. By replacing B by its opposite Borel B^- (cf. Remark 4.1), an argument dual to the above shows that $\mathsf{Z}_{\mathbf{f}}$ is right t-exact. It remains to observe that by [RS20, Proposition 3.2.22], if a motive in $\mathcal{F} \in \mathrm{DTM}_{L^+ G_{\mathbf{f}}}(\mathrm{Fl}_{\mathbf{f}})$ is mixed Tate when viewed as a motive in $\mathrm{DTM}_{\mathcal{I}}(\mathrm{Fl}_{\mathbf{f}})$, then we already have $\mathcal{F} \in \mathrm{MTM}_{L^+ G_{\mathbf{f}}}(\mathrm{Fl}_{\mathbf{f}})$. $\qquad\square$

6. Geometrization of generic Hecke algebras

In this section, we introduce generic parahoric Hecke algebras as decategorifications, i.e., Grothendieck groups, of appropriate categories of Tate motives. We show how this specializes to classical Hecke algebras, and also interpolates between generic Iwahori–Hecke algebras [Vig06, Vig16] and generic spherical Hecke algebras as considered in [CvdHS22, §6.3]. We finally identify the centers of these generic parahoric Hecke algebras in many cases.

Recall that for a monoidal triangulated category C, its Grothendieck group $K_0(C)$ is a ring.

6.1. **Generic parahoric Hecke algebras.** Let $\mathbf{f}$ be a facet in the Bruhat-Tits building of G, which we assume to lie in the closure of $\mathbf{a}_0$. Recall from Lemma 4.18 that the convolution product turns the ∞-category $\mathrm{DTM}_{L^+ G_{\mathbf{f}}}(\mathrm{Fl}_{\mathbf{f}})^{\mathrm{lc,anti}}$ of anti-effective locally compact stratified Tate motives on the partial affine flag variety of $\mathbf{f}$ into a monoidal triangulated category.

Definition 6.1. The *generic parahoric Hecke algebra* at $G_{\mathbf{f}}$-level is the Grothendieck ring
$$\mathcal{H}^{\mathbf{f}}(\mathbf{q}) := K_0(\mathrm{DTM}_{L^+ G_{\mathbf{f}}}(\mathrm{Fl}_{\mathbf{f}})^{\mathrm{lc,anti}}).$$

This is naturally a $\mathbf{Z}[\mathbf{q}]$-algebra, where the variable $\mathbf{q}$ acts via the negative Tate twist (-1). As a $\mathbf{Z}[\mathbf{q}]$-module, $\mathcal{H}^{\mathbf{f}}$ is free with basis indexed by $W_{\mathbf{f}} \backslash W / W_{\mathbf{f}}$ (Corollaries 2.13 and 2.15). (In particular, it is independent of the base scheme S fixed in Notation 2.11.)

Example 6.2. If $\mathbf{f} = \mathbf{f}_0$ is hyperspecial, we write $\mathcal{H}^{\mathbf{f}_0}(\mathbf{q}) =: \mathcal{H}^{\mathrm{sph}}(\mathbf{q})$, and refer to it as the *generic spherical Hecke algebra*. According to [CvdHS22, Theorem 6.32 and Corollary 6.37], this recovers the definition in terms of generators and relations in [CvdHS22, Definition 6.35].

On the other extreme, we have the case where $\mathbf{f} = \mathbf{a}_0$ is an alcove, for which we also write $\mathcal{H}^{\mathbf{a}_0}(\mathbf{q}) = \mathcal{H}^{\mathcal{I}}(\mathbf{q})$. We now relate this case of Definition 6.1 to the generic Hecke algebra defined by Vignéras [Vig16, Theorem 2.1], which was defined as the free $\mathbf{Z}[\mathbf{q}]$-module with basis $\{T_w\}_{w \in W}$ endowed with the unique $\mathbf{Z}[\mathbf{q}]$-algebra structure satisfying:

- (the braid relations) $T_w T_{w'} = T_{ww'}$ if $w, w' \in W$ and $l(ww') = l(w) + l(w')$.
- (the quadratic relations) $T_s^2 = \mathbf{q} + (\mathbf{q} - 1) T_s$ if $s \in S_{\mathrm{aff}}$.

Recall the standard sheaves $\Delta_w(\mathbf{Z})$ from Definition 5.1.

Proposition 6.3. *The generic Iwahori–Hecke algebra $\mathcal{H}^{\mathcal{I}}(\mathbf{q})$, with basis $(-1)^{l(w)}[\Delta_w(\mathbf{Z})]$ satisfies the braid relations and the quadratic relations. In particular, it agrees with Vignéras' definition.*

Proof. The braid relations are satisfied by Proposition 5.3. For the quadratic relations, let $s \in S_{\mathrm{aff}}$. Since $s^2 = \mathrm{id} < s$, the reduced fibers of the convolution map $m \colon \mathrm{Fl}^{\leq s} \tilde{\times} \mathrm{Fl}^{\leq s} \to \mathrm{Fl}^{\leq s}$ are all isomorphic to $\mathbf{P}^1$. This follows from the arguments in the proof of [Fal03, Lemma 9] since m can be viewed as a Demazure map for the non-reduced word ss. We must determine the fibers of the restricted convolution map $m' \colon \mathrm{Fl}^s \tilde{\times} \mathrm{Fl}^s \to \mathrm{Fl}^{\leq s}$. We have $(m')^{-1}(\{e\}) = m^{-1}(\{e\}) \setminus (\{e\}, \{e\}) \cong \mathbf{A}^1$ and $(m')^{-1}(\{s\}) = m^{-1}(\{s\}) \setminus (\{e\}, \{s\}) \cup (\{s\}, \{e\}) \cong \mathbf{G}_{\mathrm{m}}$. Thus $\iota_e^*(\Delta_s(\mathbf{Z}) * \Delta_s(\mathbf{Z})) \cong \mathbf{Z}(-1)[0]$ and $\iota_s^*(\Delta_s(\mathbf{Z}) * \Delta_s(\mathbf{Z})) \cong \mathbf{Z}[1] \oplus \mathbf{Z}(-1)[0]$. By a straightforward computation this implies that our map respects the quadratic relations. $\square$

Recall that the *classical Hecke algebra* $\mathcal{H}^{\mathbf{f}} := \mathcal{H}^{\mathbf{f}}(q)$ is defined as the set of bi-$G_{\mathbf{f}}(\mathbf{F}_q[\![t]\!])$-equivariant compactly supported functions on $G(\mathbf{F}_q(\!(t)\!))$ with values in $\mathbf{Z}$, equipped with the convolution product.

Proposition 6.4. *There is a canonical ring isomorphism*
$$\mathcal{H}^{\mathbf{f}}(\mathbf{q}) \otimes_{\mathbf{Z}[\mathbf{q}], \mathbf{q} \mapsto q} \mathbf{Z} \xrightarrow{\cong} \mathcal{H}^{\mathbf{f}}.$$

Proof. To construct a map $\mathcal{H}^{\mathbf{f}}(\mathbf{q}) \to \mathcal{H}^{\mathbf{f}}$, we may choose $S = \mathrm{Spec}\, \mathbf{F}_q$. We pass to motives with rational coefficients (this does not even affect the Grothendieck groups by Remark 2.16), which have étale descent. To any motive $M \in \mathrm{DTM}_{L^+ G_{\mathbf{f}}}(\mathrm{Fl}_{\mathbf{f}})^{\mathrm{lc,anti}}$, we associate a compactly supported function on $G_{\mathbf{f}}(\mathcal{O}_F) \backslash G(F) / G_{\mathbf{f}}(\mathcal{O}_F)$, by *-pulling back M to the basepoints of the strata in $\mathrm{Fl}_{\mathbf{f}}$, and then applying the trace of Frobenius (e.g., cf. [Cis21]). Since the trace of Frobenius on $\mathbf{Z}(-1)$ is q, these functions indeed take values in $\mathbf{Z}$. The resulting map is a ring morphism by a similar argument as in [Zhu17b, Lemma 5.6.1]. Clearly, $\mathbf{q} - q$ is contained in the kernel of this morphism.

The resulting map $\mathcal{H}^{\mathbf{f}}(\mathbf{q}) \otimes_{\mathbf{Z}[\mathbf{q}], \mathbf{q} \mapsto q} \mathbf{Z} \to \mathcal{H}^{\mathbf{f}}$ sends the standard motive $\iota_{w!}(\mathbf{Z}[l(w)])$ (where $\iota_w \colon \mathrm{Fl}_w^\circ(\mathbf{f}, \mathbf{f}) \to \mathrm{Fl}_{\mathbf{f}}$ is the inclusion) to the characteristic function of the double coset of w, up to a factor of ± 1. That is, the map preserves bases (of free $\mathbf{Z}$-modules) and is therefore an isomorphism. $\square$

Remark 6.5. One could also use the recent result of Haines on the cellularity of the fibers of convolution morphisms [Hai23, §9] to define generic Hecke algebras at parahoric level in terms of generators and relations. However, maps such as (6.1) below become much less clear in such an approach.

6.2. **Construction of central elements.** Recall that the central functor $\mathrm{Z}_{\mathbf{f}}$ from Definition 4.15 preserves locally compact anti-effective Tate motives (Theorem 4.37). By Theorems 4.41, 4.49, passing to Grothendieck groups, it induces a ring homomorphism
$$\varphi^{\mathbf{f}} \colon \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \to Z(\mathcal{H}^{\mathbf{f}}(\mathbf{q})), \tag{6.1}$$
where Z denotes the center of a ring. By Theorem 4.43, $\varphi^{\mathbf{f}}$ is injective for any $\mathbf{f}$. Using results on centers of classical Hecke algebras, we now show this is an isomorphism in case $\mathbf{f} = \mathbf{a}_0$.

Recall that classically, $\mathcal{H}^{\mathcal{I}}$ contains a (non-unique) maximally commutative subalgebra generated by the Bernstein elements, which are in turn categorified by the Wakimoto sheaves. Using the motivic Wakimoto functor from Definition 5.7, we adapt this as follows.

Definition 6.6. For $\lambda \in X_*(T)$, we define the *Bernstein element*

$$\theta_\lambda := [\mathbf{J}_\lambda^B(\mathbf{Z})(-\frac{\langle 2\rho, \lambda \rangle + l(t(\lambda))}{2})] \in \mathcal{H}^{\mathcal{I}}(\mathbf{q}).$$

We denote by $\mathcal{A}(\mathbf{q}) \subseteq \mathcal{H}^{\mathcal{I}}(\mathbf{q})$ the sub-$\mathbf{Z}[\mathbf{q}]$-algebra generated by θ_λ for $\lambda \in X_*(T)$.

Note that since $\mathcal{A}(\mathbf{q})$ is defined via Wakimoto motives, it depends on the choice of Borel B. Recall from Remark 5.8 that the Wakimoto functors for dominant cocharacters involved a twist, to ensure that the full Wakimoto functor is monoidal. The drawback of this is that $\mathbf{J}_\mu^B(\mathbf{Z})$ is not anti-effective in general, which is why we reversed the twist in the definition above. This implies that the θ_λ are not invertible in $\mathcal{H}_G^{\mathcal{I}}(\mathbf{q})$, only after inverting $\mathbf{q}$.

Remark 6.7. In order to compare our results to [Vig06], we set all parameters q_s to $\mathbf{q}$ in loc. cit. Then θ_λ agrees with the element $(-1)^{l(t(\lambda))} E_\lambda$ from [Vig06, Théorème 1.5], as long as one replaces the fixed Borel B in loc. cit. by its opposite B^-. Indeed, for dominant and anti-dominant cocharacters this follows from the definitions, and for general cocharacters it follows from the equality

$$\theta_\lambda \cdot \theta_\mu = \mathbf{q}^{\frac{l(t(\lambda)) + l(t(\mu)) - l(t(\lambda+\mu))}{2}} \cdot \theta_{\lambda+\mu}$$

induced by Proposition 5.3, and similar relations for the E_λ in [Vig06]. In particular, $\mathcal{A}(\mathbf{q})$ agrees with the algebra $A \cap H$ from [Vig06, (1.6.1)]. This also shows that we indeed have $\theta_\lambda \in \mathcal{H}^{\mathcal{I}}(\mathbf{q})$, without having to invert $\mathbf{q}$.

Proposition 6.8. *The set $\{\theta_\lambda \mid \lambda \in X_*(T)\}$ forms a $\mathbf{Z}[\mathbf{q}]$-basis of $\mathcal{A}(\mathbf{q})$, which in turn is a maximal commutative subalgebra of $\mathcal{H}^{\mathcal{I}}(\mathbf{q})$.*

Proof. Recall from Proposition 5.3 that $\Delta_s(\mathbf{Z}) \star \nabla_s(\mathbf{Z}) = \Delta_e(-1)$, for any $s \in S_{\text{aff}}$. Hence, writing $[\nabla_s(\mathbf{Z})]$ as a linear combination of $[\Delta_e(\mathbf{Z})]$ and $[\Delta_s(\mathbf{Z})]$, the quadratic relations from Proposition 6.3 yield $[\nabla_s(\mathbf{Z})] = [\Delta_s(\mathbf{Z})] + \mathbf{q} - 1$. Applying Proposition 5.3 again, this allows us to determine the classes of all costandard objects, and we see that $[\nabla_w(\mathbf{Z})]$ is a linear combination of $\{T_v \mid v \leq w\}$, and that the coefficient of T_w appearing is ± 1. Hence the linear independence of $\{\theta_\lambda \mid \lambda \in X_*(T)\}$ follows from Proposition 6.3. The fact that $\{\theta_\lambda \mid \lambda \in X_*(T)\}$ also span $\mathcal{A}(\mathbf{q})$ follows essentially by definition, since $\{\theta_\lambda \mid \lambda \in X_*(T)\}$ is closed under multiplication, up to a power of $\mathbf{q}$.

Since $\theta_\lambda \cdot \theta_\mu = \theta_\mu \cdot \theta_\lambda$, we see that $\mathcal{A}(\mathbf{q})$ is a commutative algebra, and we want to show it is maximal with respect to this property. Now, for any generic object, which is denoted by $(\mathbf{q})$, and any prime power q, we denote by (q) the specialization along $\mathbf{q} \mapsto q$. Then we can consider the diagram

$$\begin{array}{ccc} \mathcal{A}(\mathbf{q}) & \longrightarrow & \mathcal{H}^{\mathcal{I}}(\mathbf{q}) \\ \downarrow & & \downarrow \\ \prod_q \mathcal{A}(q) & \longrightarrow & \prod_q \mathcal{H}^{\mathcal{I}}(q), \end{array}$$

where $\mathcal{A}(q) \subseteq \mathcal{H}^{\mathcal{I}}(q)$ is the subalgebra generated by the classical Bernstein elements E_λ from [Vig06, Théorème 1.5], and the vertical maps are the product of all specialization maps, obtained by mapping $\mathbf{q}$ to a prime power q. Then all maps above are injective, and the diagram is cartesian as $\{\theta_\lambda \mid \lambda \in X_*(T)\}$ can be extended to a basis of $\mathcal{H}^{\mathcal{I}}(\mathbf{q})$ (similar to [Vig06, Théorème 1]). On the other hand, all specialization maps $\mathcal{A}(\mathbf{q}) \to \mathcal{A}(q)$ and $\mathcal{H}^{\mathcal{I}}(\mathbf{q}) \to \mathcal{H}^{\mathcal{I}}(q)$ are surjective. Hence, the fact that $\mathcal{A}(\mathbf{q}) \subseteq \mathcal{H}^{\mathcal{I}}(\mathbf{q})$ is maximal commutative will follow from the similar assertion for usual Hecke algebras with integral coefficients. However, since the images of θ_λ can be extended to a basis of $\mathcal{H}^{\mathcal{I}}(q)$, it even suffices to show it for Iwahori–Hecke algebras with complex coefficients. This is well-known, and follows from the explicit description of the multiplication of the basis elements above, such as in [Vig16, Corollary 5.47]. $\qquad\square$

Having decategorified the Wakimoto motives, we can now identify the center of $\mathcal{H}^{\mathcal{I}}(\mathbf{q})$.

Theorem 6.9. *The functor $\varphi^{\mathbf{a}_0}$ in (6.1) is an isomorphism.*

Proof. By (a suitable variation of) [Vig06, (1.6.5)], the elements $\sum_{\lambda \in W_0(\mu)} \theta_\lambda$ for $\mu \in X_*(T)^+$ form a basis of $Z(\mathcal{H}^{\mathcal{I}}(\mathbf{q}))$. On the other hand, by the motivic Satake equivalence [CvdHS22, Theorem 6.32] and the generic Satake isomorphism [CvdHS22, Corollary 6.37], the classes of the rational IC-motives $[\text{IC}_{\mu,\mathbf{Q}}] \in K_0(\text{DTM}_{L+G}(\text{Gr}_G)^{\text{lc,anti}})$ for $\mu \in X_*(T)^+$ form a basis of $\mathcal{H}^{\text{sph}}(\mathbf{q})$. Now by Corollary 5.15, $[\mathbf{Z}_G(\text{IC}_{\mu,\mathbf{Q}})]$ is a $\mathbf{Z}[\mathbf{q}^{\pm 1}]$-linear combination of the θ_ν for ν in the convex hull of $W_0(\mu)$. It suffices to show this actually a $\mathbf{Z}[\mathbf{q}]$-linear combination, and that the coefficient of θ_ν equals 1 for all $\nu \in W_0(\mu)$.

By Theorem 5.24, $\text{Grad}_\nu(\mathbf{Z}_G(\text{IC}_{\mu,\mathbf{Q}})) \cong \text{CT}_{B^-}^{t(\nu)}(\text{IC}_{\mu,\mathbf{Q}})(-\langle 2\rho, \nu \rangle)[-\langle 2\rho, \nu \rangle]$, where λ is a regular antidominant cocharacter. Let $\mathcal{S}_\nu^+$ be the attractor associated to ν in Gr_G and let $n_{\mu,\nu}$ be the number of irreducible components of $\mathcal{S}_\nu^+ \cap \text{Gr}_G^\mu$. Note that $n_{\mu,\nu} = 1$ if $\nu \in W_0(\mu)$, and $n_{\mu,\nu}$ is constant for ν in the same W_0-orbit. Now by [CvdHS22,

Theorem 5.39] we have $\mathrm{CT}_{B^-}^{t(\nu)}(\mathrm{IC}_{\mu,\mathbf{Q}})[-\langle 2\rho, \nu\rangle] \cong \mathbf{Q}^{\oplus n_{\mu,\nu}}(-\dim(\mathcal{S}_\nu^+ \cap \mathrm{Gr}^\mu))$. Using that $\dim(\mathcal{S}_\nu^+ \cap \mathrm{Gr}^\mu) = \langle \rho, \mu - v\rangle$ [MV07, Theorem 3.2], a straightforward computation shows

$$[\mathbf{J}_\nu^B(\mathrm{Grad}_\nu(\mathsf{Z}_G(\mathrm{IC}_{\mu,\mathbf{Q}})))] = n_{\mu,\nu}\mathbf{q}^{\frac{l(\mu)-l(\nu)}{2}}\theta_\nu.$$

These coefficients clearly satisfy the desired properties. $\qquad\square$

References

[AB09]   Sergey Arkhipov and Roman Bezrukavnikov. Perverse sheaves on affine flags and Langlands dual group. *Israel J. Math.*, 170:135–183, 2009. With an appendix by Bezrukavnikov and Ivan Mirković. doi:10.1007/s11856-009-0024-y. 1, 2, 3, 4, 42

[AEWH15]   Giuseppe Ancona, Stephen Enright-Ward, and Annette Huber. On the motive of a commutative algebraic group. *Doc. Math.*, 20:807–858, 2015. 11

[AGLR22]   Johannes Anschütz, Ian Gleason, João Lourenço, and Timo Richarz. On the *p*-adic theory of local models, 2022. arXiv:2201.01234. 33, 36, 37, 48

[AGV22]   Joseph Ayoub, Martin Gallauer, and Alberto Vezzani. The six-functor formalism for rigid analytic motives. *Forum Math. Sigma*, 10:Paper No. e61, 182, 2022. doi:10.1017/fms.2022.55. 15

[AHI23]   Toni Annala, Marc Hoyois, and Ryomei Iwasa. Algebraic cobordism and a Conner-Floyd isomorphism for algebraic K-theory, 2023. arXiv:2303.02051. 18

[ALWY23]   Johannes Anschütz, João Lourenço, Zhiyou Wu, and Jize Yu. Gaitsgory's central functor and the Arkhipov-Bezrukavnikov equivalence in mixed characteristic, 2023. arXiv:2311.04043. 3, 4, 12, 42, 47, 49

[AR]   Pramod N. Achar and Simon Riche. Central sheaves on affine flag varieties. URL: https://lmbp.uca.fr/~riche/central.pdf. 3, 4, 21, 22, 23, 24, 26, 27, 28, 29, 30, 31, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49

[AR16]   Pramod N. Achar and Simon Riche. Modular perverse sheaves on flag varieties, II: Koszul duality and formality. *Duke Mathematical Journal*, 165(1):161–215, 2016. doi:10.1215/00127094-3165541. 4, 44, 50

[Ayo07a]   Joseph Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I. *Astérisque*, (314):x+466 pp., 2007. 2, 15, 18, 20

[Ayo07b]   Joseph Ayoub. The motivic vanishing cycles and the conservation conjecture. In *Algebraic cycles and motives. Vol. 1*, volume 343 of *London Math. Soc. Lecture Note Ser.*, pages 3–54. Cambridge Univ. Press, Cambridge, 2007. doi:10.1017/CBO9780511721496.002. 2, 17, 18, 20

[Ayo10]   Joseph Ayoub. Note sur les opérations de Grothendieck et la réalisation de Betti. *J. Inst. Math. Jussieu*, 9(2):225–263, 2010. doi:10.1017/S1474748009000127. 20

[Ayo14]   Joseph Ayoub. La réalisation étale et les opérations de Grothendieck. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(1):1–145, 2014. doi:10.24033/asens.2210. 20

[Ayo15]   Joseph Ayoub. Motifs des variétés analytiques rigides. *Mém. Soc. Math. Fr. (N.S.)*, (140-141):vi+386, 2015. doi:10.24033/msmf.449. 15

[BB73]   A. Białynicki-Birula. Some theorems on actions of algebraic groups. *Ann. of Math. (2)*, 98:480–497, 1973. doi:10.2307/1970915. 35

[BB76]   A. Białynicki-Birula. Some properties of the decompositions of algebraic varieties determined by actions of a torus. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 24(9):667–674, 1976. 35

[Bea23]   Jonathan Beardsley. On bialgebras, comodules, descent data and Thom spectra in ∞-categories. *Homology Homotopy Appl.*, 25(2):219–242, 2023. 7

[Ber84]   J. N. Bernstein. Le "centre" de Bernstein. In P. Deligne, editor, *Representations of reductive groups over a local field*, Travaux en Cours, pages 1–32. Hermann, Paris, 1984. 1

[Bez16]   Roman Bezrukavnikov. On two geometric realizations of an affine Hecke algebra. *Publ. Math. Inst. Hautes Études Sci.*, 123:1–67, 2016. doi:10.1007/s10240-015-0077-x. 1, 2, 4

[BGT13]   Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada. A universal characterization of higher algebraic *K*-theory. *Geom. Topol.*, 17(2):733–838, 2013. doi:10.2140/gt.2013.17.733. 4

[BGV23]   Federico Binda, Martin Gallauer, and Alberto Vezzani. Motivic monodromy and p-adic cohomology theories, 2023. arXiv:2306.05099. 7, 11, 15

[BT84]   F. Bruhat and J. Tits. Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée. *Inst. Hautes Études Sci. Publ. Math.*, (60):197–376, 1984. URL: http://www.numdam.org/item?id=PMIHES_1984__60__5_0. 26

[BZFN10]   David Ben-Zvi, John Francis, and David Nadler. Integral transforms and Drinfeld centers in derived algebraic geometry. *J. Amer. Math. Soc.*, 23(4):909–966, 2010. 6

[Cam18]   Justin Campbell. Nearby cycles of Whittaker sheaves on Drinfeld's compactification. *Compos. Math.*, 154(8):1775–1800, 2018. doi:10.1112/s0010437x18007285. 2, 13, 15

[CGP10]   Brian Conrad, Ofer Gabber, and Gopal Prasad. *Pseudo-reductive groups*, volume 17 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2010. doi:10.1017/CBO9780511661143. 32

[Che23]   Lin Chen. Nearby cycles on Drinfeld-Gaitsgory-Vinberg interpolation Grassmannian and long intertwining functor. *Duke Math. J.*, 172(3):447–543, 2023. doi:10.1215/00127094-2022-0042. 13

[Cis21]   Denis-Charles Cisinski. Cohomological methods in intersection theory. In *Homotopy theory and arithmetic geometry—motivic and Diophantine aspects*, volume 2292 of *Lecture Notes in Math.*, pages 49–105. Springer, Cham, [2021] ©2021. doi:10.1007/978-3-030-78977-0_3. 52

[CvdHS22] Robert Cass, Thibaud van den Hove, and Jakob Scholbach. The geometric Satake equivalence for integral motives, 2022. arXiv:2211.04832. 1, 2, 3, 4, 8, 9, 10, 16, 17, 19, 27, 30, 31, 33, 34, 36, 37, 38, 39, 40, 41, 42, 43, 44, 48, 50, 51, 52, 53, 54

[CX22]   Robert Cass and Yujie Xu. Geometrization of the Satake transform for mod *p* Hecke algebras, 2022. arXiv:2211.16226. 35, 36

[DG02]   W. G. Dwyer and J. P. C. Greenlees. Complete modules and torsion modules. *Amer. J. Math.*, 124(1):199–220, 2002. URL: http://muse.jhu.edu/journals/american_journal_of_mathematics/v124/124.1dwyer.pdf. 8

[DG14]   V. Drinfeld and D. Gaitsgory. On a theorem of Braden. *Transform. Groups*, 19(2):313–358, 2014. `doi:10.1007/s00031-014-9267-8`. 18, 19

[Dri18]   Vladimir Drinfeld. On algebraic spaces with an action of $\mathbb{G}_m$. 2018. `arXiv:1308.2604`. 36

[EK19]   Jens Niklas Eberhardt and Shane Kelly. Mixed motives and geometric representation theory in equal characteristic. *Selecta Math. (N.S.)*, 25(2):Paper No. 30, 54, 2019. `doi:10.1007/s00029-019-0475-x`. 4, 43

[ES23]   Jens Niklas Eberhardt and Jakob Scholbach. Integral motivic sheaves and geometric representation theory. *Adv. Math.*, 412:Paper No. 108811, 42, 2023. `doi:10.1016/j.aim.2022.108811`. 2, 8, 20

[Fal03]   Gerd Faltings. Algebraic loop groups and moduli spaces of bundles. *J. Eur. Math. Soc. (JEMS)*, 5(1):41–68, 2003. `doi:10.1007/s10097-002-0045-x`. 25, 28, 35, 36, 52

[FS21]   Laurent Fargues and Peter Scholze. Geometrization of the local Langlands correspondence, 2021. `arXiv:2102.13459`. 1

[Gai01]   D. Gaitsgory. Construction of central elements in the affine Hecke algebra via nearby cycles. *Invent. Math.*, 144(2):253–280, 2001. `doi:10.1007/s002220100122`. 1, 34

[Gai20]   Dennis Gaitsgory. The local and global versions of the Whittaker category. *Pure Appl. Math. Q.*, 16(3):775–904, 2020. `doi:10.4310/PAMQ.2020.v16.n3.a14`. 12, 30

[GKR20]   Richard Garner, Magdalena Kędziorek, and Emily Riehl. Lifting accessible model structures. *J. Topol.*, 13(1):59–76, 2020. `doi:10.1112/topo.12123`. 6

[GR17]   Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry. Vol. I. Correspondences and duality*, volume 221 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017. 5, 8, 12, 24

[Hai23]   Thomas J. Haines. Cellular pavings of fibers of convolution morphisms, 2023. `arXiv:2309.14218`. 52

[Hau21]   Rune Haugseng. On lax transformations, adjunctions, and monads in $(\infty, 2)$-categories. *High. Struct.*, 5(1):244–281, 2021. 17

[HK06]   Annette Huber and Bruno Kahn. The slice filtration and mixed Tate motives. *Compos. Math.*, 142(4):907–936, 2006. `doi:10.1112/S0010437X06002107`. 9, 10

[HN02]   Thomas J. Haines and Bao Châu Ngô. Alcoves associated to special fibers of local models. *Amer. J. Math.*, 124(6):1125–1152, 2002. URL: `http://muse.jhu.edu/journals/american_journal_of_mathematics/v124/124.6haines.pdf`. 36

[Hov99]   Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999. 6

[Hoy17]   Marc Hoyois. The six operations in equivariant motivic homotopy theory. *Adv. Math.*, 305:197–279, 2017. `doi:10.1016/j.aim.2016.09.031`. 8

[HR20]   Thomas J. Haines and Timo Richarz. The test function conjecture for local models of Weil-restricted groups. *Compos. Math.*, 156(7):1348–1404, 2020. `doi:10.1112/s0010437x20007162`. 25, 26, 27, 28, 29, 32, 33, 34

[HR21]   Thomas J. Haines and Timo Richarz. The test function conjecture for parahoric local models. *J. Amer. Math. Soc.*, 34(1):135–218, 2021. `doi:10.1090/jams/955`. 3, 32, 33, 48

[JY21a]   Fangzhou Jin and Enlin Yang. Künneth formulas for motives and additivity of traces. *Adv. Math.*, 376:Paper No. 107446, 83, 2021. `doi:10.1016/j.aim.2020.107446`. 18, 36

[JY21b]   Fangzhou Jin and Enlin Yang. Some results on the motivic nearby cycle, 2021. `arXiv:2107.08603`. 18, 20

[Kha16]   Adeel Khan. Motivic homotopy theory in derived algebraic geometry, 2016. PhD thesis, Universität Duisburg-Essen. URL: `https://duepublico.uni-duisburg-essen.de/servlets/DerivateServlet/Derivate-42021/Diss_Khan.pdf`. 8

[KP21]   Dmitry Kubrak and Artem Prikhodko. *p*-adic Hodge theory for Artin stacks, 2021. `arXiv:2105.05319`. 18

[Kra10]   Henning Krause. Localization theory for triangulated categories. In *Triangulated categories*, volume 375 of *London Math. Soc. Lecture Note Ser.*, pages 161–235. Cambridge Univ. Press, Cambridge, 2010. `doi:10.1017/CBO9781139107075.005`. 45

[Lou23]   João Lourenço. Grassmanniennes affines tordues sur les entiers. *Forum Math. Sigma*, 11:Paper No. e12, 65, 2023. `doi:10.1017/fms.2023.4`. 26, 27, 32

[Lur09]   Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009. `doi:10.1515/9781400830558`. 5

[Lur17]   Jacob Lurie. Higher Algebra. 2017. 4, 5, 6, 7, 11, 21

[LZ11]   Yifeng Liu and Weizhe Zheng. Enhanced six operations and base change theorem for higher Artin stacks. 2011. `arXiv:1211.5948`. 8

[Man22]   Lucas Mann. A *p*-adic 6-functor formalism in rigid-analytic geometry, 2022. `arXiv:2206.02022`. 8, 12

[May70]   J. Peter May. A general algebraic approach to Steenrod operations. In *The Steenrod Algebra and its Applications (Proc. Conf. to Celebrate N. E. Steenrod's Sixtieth Birthday, Battelle Memorial Inst., Columbus, Ohio, 1970)*, volume Vol. 168 of *Lecture Notes in Math.*, pages 153–231. Springer, Berlin-New York, 1970. 11

[Mor18]   Sophie Morel. Beilinson's construction of nearby cycles and gluing, 2018. URL: `http://perso.ens-lyon.fr/sophie.morel/`. 17, 20

[MR18]   Carl Mautner and Simon Riche. Exotic tilting sheaves, parity sheaves on affine Grassmannians, and the Mirković–Vilonen conjecture. *Journal of the European Mathematical Society*, 20(9):2259–2332, July 2018. `doi:10.4171/JEMS/812`. 1

[MRR23]   Arnaud Mayeux, Timo Richarz, and Matthieu Romagny. Néron blowups and low-degree cohomological applications. *Algebr. Geom.*, 10(6):729–753, 2023. `doi:10.14231/ag-2023-026`. 26

[MV07]   I. Mirković and K. Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. of Math. (2)*, 166(1):95–143, 2007. `doi:10.4007/annals.2007.166.95`. 1, 54

[PR08]   G. Pappas and M. Rapoport. Twisted loop groups and their affine flag varieties. *Adv. Math.*, 219(1):118–198, 2008. With an appendix by T. Haines and Rapoport. `doi:10.1016/j.aim.2008.04.006`. 26, 28, 35

[PS23]   Cédric Pepin and Tobias Schmidt. Generic and mod *p* Kazhdan-Lusztig Theory for GL_2. *Represent. Theory*, 27:1142–1193, 2023. `doi:10.1090/ert/656`. 4

[PZ13]   G. Pappas and X. Zhu. Local models of Shimura varieties and a conjecture of Kottwitz. *Invent. Math.*, 194(1):147–254, 2013. `doi:10.1007/s00222-012-0442-z`. 25, 26, 27, 28, 32

[Ric13]   Timo Richarz. Schubert varieties in twisted affine flag varieties and local models. *J. Algebra*, 375:121–147, 2013. `doi:10.1016/j.jalgebra.2012.11.013`. 25, 26, 51

[Ric14]   Timo Richarz. A new approach to the geometric Satake equivalence. *Doc. Math.*, 19:209–246, 2014. 1

[Ric16]   Timo Richarz. Affine Grassmannians and geometric Satake equivalences. *Int. Math. Res. Not. IMRN*, (12):3717–3767, 2016. `doi:10.1093/imrn/rnv226`. 26, 28

[Ric19]   Timo Richarz. Spaces with $\mathbb{G}_m$-action, hyperbolic localization and nearby cycles. *J. Algebraic Geom.*, 28(2):251–289, 2019. `doi:10.1090/jag/710`. 3, 18, 19, 32, 33, 37

[Rob14]   Marco Robalo. Théorie Homotopique Motivique des Espaces Noncommutatifs, 2014. PhD thesis at the Université Montpellier 2. URL: `https://webusers.imj-prg.fr/~marco.robalo/these.pdf`. 12, 45

[RS]   Timo Richarz and Jakob Scholbach. Corrigendum to "The intersection motive of the moduli stack of shtukas". URL: `https://jakobscholbach.github.io/`. 10, 34

[RS20]   Timo Richarz and Jakob Scholbach. The intersection motive of the moduli stack of shtukas. *Forum Math. Sigma*, 8:Paper No. e8, 99, 2020. `doi:10.1017/fms.2019.32`. 8, 9, 10, 12, 24, 25, 26, 27, 29, 30, 34, 40, 43, 45, 49, 50, 51

[RS21]   Timo Richarz and Jakob Scholbach. The motivic Satake equivalence. *Math. Ann.*, 380(3-4):1595–1653, 2021. `doi:10.1007/s00208-021-02176-9`. 1, 2, 8, 12, 25, 30, 44, 51

[Spi16]   Markus Spitzweck. Notes for a mini-course on "Mixed Tate Motives and Fundamental Groups" given in Bonn, 2016. URL: `https://www.him.uni-bonn.de/fileadmin/him/Spitzweck-minicourse-HIM.pdf`. 9

[Spi18]   Markus Spitzweck. A commutative $\mathbb{P}^1$-spectrum representing motivic cohomology over Dedekind domains. *Mém. Soc. Math. Fr. (N.S.)*, (157):110, 2018. `doi:10.24033/msmf.465`. 2, 8

[SS00]   Stefan Schwede and Brooke E. Shipley. Algebras and modules in monoidal model categories. *Proc. London Math. Soc. (3)*, 80(2):491–511, 2000. `doi:10.1112/S002461150001220X`. 6

[Sta24]   Stacks Project Authors. Stacks Project. `http://stacks.math.columbia.edu`, 2024. 10, 33

[Tot18]   Burt Totaro. Adjoint functors on the derived category of motives. *J. Inst. Math. Jussieu*, 17(3):489–507, 2018. `doi:10.1017/S1474748016000104`. 16

[Vig06]   Marie-France Vignéras. Algèbres de Hecke affines génériques. *Represent. Theory*, 10:1–20, 2006. `doi:10.1090/S1088-4165-06-00185-3`. 4, 51, 53

[Vig16]   Marie-France Vigneras. The pro-p-Iwahori Hecke algebra of a reductive p-adic group I. *Compos. Math.*, 152(4):693–753, 2016. `doi:10.1112/S0010437X15007666`. 51, 52, 53

[XZ22]   Daxin Xu and Xinwen Zhu. Bessel F-isocrystals for reductive groups. *Invent. Math.*, 227(3):997–1092, 2022. `doi:10.1007/s00222-021-01079-5`. 1

[Zhu14]   Xinwen Zhu. On the coherence conjecture of Pappas and Rapoport. *Ann. of Math. (2)*, 180(1):1–85, 2014. `doi:10.4007/annals.2014.180.1.1`. 1, 26, 27

[Zhu15]   Xinwen Zhu. The geometric Satake correspondence for ramified groups. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(2):409–451, 2015. `doi:10.24033/asens.2248`. 1

[Zhu17a] Xinwen Zhu. Affine Grassmannians and the geometric Satake in mixed characteristic. *Ann. of Math. (2)*, 185(2):403–492, 2017. `doi:10.4007/annals.2017.185.2.2`. 1

[Zhu17b] Xinwen Zhu. An introduction to affine Grassmannians and the geometric Satake equivalence. In *Geometry of moduli spaces and representation theory*, volume 24 of *IAS/Park City Math. Ser.*, pages 59–154. Amer. Math. Soc., Providence, RI, 2017. 52

[Zhu20]   Xinwen Zhu. A note on Integral Satake isomorphisms, 2020. `arXiv:2005.13056`. 4