Problem 3.24

a) A uniform line charge density \( \rho_l \) lies along the \( z \) axis. Show that \( \nabla \cdot \mathbf{D} = 0 \) everywhere except on the line charge.

We know that \( \mathbf{D} = \rho_l / (2\pi \rho) \mathbf{a}_\rho \), and so

\[
\nabla \cdot \mathbf{D} = \frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{\rho_l}{2\pi \rho} \right) = 0
\]

The line charge itself represents a singularity, and so we cannot define the divergence there. (The derivative is not defined at \( \rho = 0 \)).

b) Replace the line charge with a uniform volume charge density \( \rho_v \) for \( 0 < \rho < a \). Relate \( \rho_v \) to \( \rho_l \) so that the charge per unit length is the same. Then find \( \nabla \cdot \mathbf{D} \) everywhere.

The equal charge per unit length requirement requires that \( \rho_l z = \pi a^2 \rho_v z \). We apply Gauss’s law to the interior of the volume charge, with the gaussian surface being a cylinder of radius \( \rho \) and of unit length. We find inside the charge:

\[
2\pi \rho (1) D_\rho = \pi a^2 (1) \rho_v \Rightarrow D_{\text{in}} = \frac{\rho_v}{2} a_z = \frac{\rho_l}{2\pi \rho} a_z
\]

Outside the charge, the field is found through

\[
2\pi \rho (1) D_\rho = \pi a^2 (1) \rho_v \Rightarrow D_{\text{out}} = \frac{a^2 \rho_v}{2\rho} = \frac{\rho_l}{2\pi \rho}
\]

We find \( \nabla \cdot \mathbf{D}_{\text{out}} = 0 \) as found in part a. Inside, we find

\[
\nabla \cdot \mathbf{D}_{\text{in}} = \frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{\rho_v}{2} \right) = \frac{\rho_v}{\rho}
\]
Problem 3.30

Let \( \mathbf{D} = 20\rho^2 \mathbf{a}_\rho \) nC/m^2.

a) What is the volume charge density at the point \( P(0.5, 60^\circ, 2) \)?

This will be

\[
\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{\rho} \frac{d}{d\rho} \left( \rho(20\rho^2) \right) = \frac{1}{60\rho} = 30\text{nC/m}^3 \text{ at } \rho = 0.5
\]

b) Use two different methods to find the amount of charge lying within the closed surface bounded by \( \rho = 3, 0 \leq z \leq 2 \).

Method 1: volume integral of the charge density.

\[
Q = \int_0^\pi \int_0^3 \int_0^2 60\rho \rho d\rho d\phi dz = 4\pi \times 60 \rho^3/3 \bigg|_0^3 = 2.16 \mu\text{C}
\]

Method 2: Flux integral of \( \mathbf{D} \).

\[
Q = \int_0^\pi \int_0^3 20(3)^2 \mathbf{a}_\rho \cdot \mathbf{a}_\phi \, 3d\phi dz = 4\pi \times 20 \times 27 = 2.16 \mu\text{C}
\]

Solution must show all these quantities at some point of the calculation explicitly. Otherwise, individual points should be taken off for each missing part.
Problem 4.32

Using Eq. (36), a) find the energy stored in the dipole field in the region \( r > a \):

We start with

\[
E(r, \theta) = \frac{qd}{4\pi \varepsilon_0 r^3} [2 \cos \theta, a_r + \sin \theta a_\theta]
\]

Then the energy will be

\[
W_e = \int_{\text{vol}} \frac{1}{2} \varepsilon_0 \mathbf{E} \cdot \mathbf{E} \, dv = \int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{(qd)^2}{32\pi^2 \varepsilon_0 r^6} \left[ \frac{4 \cos^2 \theta + \sin^2 \theta}{3 \cos \theta + 1} \right] r^2 \sin \theta \, dr \, d\theta \, d\phi
\]

\[
= \frac{-2\pi(qd)^2}{32\pi^2 \varepsilon_0^2} \frac{1}{3} \int_0^\infty \int_0^{2\pi} \int_0^\pi \left[ 3 \cos^2 \theta + 1 \right] \sin \theta \, d\theta \, d\phi
\]

\[
= \frac{(qd)^2}{12\pi \varepsilon_0 a^3}
\]

b) Why can we not let \( a \) approach zero as a limit? From the above result, a singularity in the energy occurs as \( a \to 0 \). More importantly, \( a \) cannot be too small, or the original far-field assumption used to derive Eq. (36) \((a \gg d)\) will not hold, and so the field expression will not be valid.
Problem 4.34

A sphere of radius $a$ contains volume charge of uniform density $\rho_0 \text{ C/m}^3$. Find the total stored energy by applying

a) Eq. (43): We first need the potential everywhere inside the sphere. The electric field inside and outside is readily found from Gauss's law:

$$E_i = \frac{\rho_0 r}{3\varepsilon_0} a, \quad r \leq a \quad \text{and} \quad E_o = \frac{\rho_0 a^3}{3\varepsilon_0 r^2} a, \quad r \geq a$$

The potential at position $r$ inside the sphere is now the work done in moving a unit positive point charge from infinity to position $r$:

$$V(r) = -\int_a^r E_i \cdot dr - \int_0^a E_i \cdot da = -\int_a^r \frac{\rho_0 a^3}{3\varepsilon_0 r^2} dr - \int_0^a \frac{\rho_0}{6\varepsilon_0} (3a^2 - r^2)$$

Now using this result in (43) leads to the energy associated with the charge in the sphere:

$$W_e = \frac{1}{2} \int_a^r \int_0^{2\pi} \frac{\rho_0 a^3}{6\varepsilon_0} (3a^2 - r^2) r^2 \sin \theta dr d\theta d\phi = \frac{\pi \rho_0}{3\varepsilon_0} \left[ - \frac{1}{r^2} \right]_{r=0}^{r=a}$$

b) Eq. (45): Using the given fields we find the energy densities

$$w_{ei} = \frac{1}{2} \varepsilon_0 E_i \cdot E_i = \frac{\rho_0^2 r^2}{18\varepsilon_0} \quad r \leq a \quad \text{and} \quad w_{eo} = \frac{1}{2} \varepsilon_0 E_o \cdot E_o = \frac{\rho_0^2 a^6}{18\varepsilon_0 r^4} \quad r \geq a$$

We now integrate these over their respective volumes to find the total energy:

$$W_e = \frac{1}{2} \int_a^r \int_0^{2\pi} \frac{\rho_0 a^3}{18\varepsilon_0} r^2 \sin \theta dr d\theta d\phi + \int_a^r \int_0^{2\pi} \frac{\rho_0^2 a^6}{18\varepsilon_0 r^4} r^2 \sin \theta dr d\theta d\phi = \frac{4\pi \rho_0^2 a^5}{15\varepsilon_0}$$

NOTE:

In both part (a) and part (b) the stored energy can be interpreted as including the energy in the region outside the charged sphere. This region gives:

$$W_e = \frac{1}{2} \varepsilon_0 \int \int \int \frac{\rho_0^2}{3\varepsilon_0} r^2 \sin \theta dr d\theta d\phi = \frac{1}{2} \varepsilon_0 \left( \frac{8\pi a^3}{3\varepsilon_0} \right) \int_0^{\pi} \int_0^{2\pi} \int_0^a r^2 \sin \theta dr d\theta d\phi$$

$$= \frac{1}{2} \varepsilon_0 \left( \frac{8\pi a^3}{3\varepsilon_0} \right) \frac{4\pi}{2} \left[ - \frac{1}{r} \right]_{r=0}^{r=a} = \frac{2\pi^2 \varepsilon_0 a^5}{3\varepsilon_0}$$

We must include this in the total:

$$W_e = \frac{4\pi a^5 \rho_0^2}{15\varepsilon_0} + \frac{2\pi^2 a^5}{9\varepsilon_0}$$

as indicated above.

The potential does not appear to include this contribution since the integral on $r$ is from 0 to $a$ only. However it really does, because the first method gives the work required to assemble the charges that produce an E field throughout space.
Problem 5.5

Let

\[
\mathbf{J} = \frac{25}{\rho} \mathbf{a}_\rho - \frac{20}{\rho^2 + 0.01} \mathbf{a}_z \text{A/m}^2
\]

a) Find the total current crossing the plane \( z = 0.2 \) in the \( \mathbf{a}_z \) direction for \( \rho < 0.4 \): Use

\[
I = \iiint [\mathbf{J} \cdot \mathbf{n}]_{z=0.2} \, dV = \int_0^{0.2} \int_0^{2\pi} \frac{-20}{\rho^2 + 0.01} \rho \, d\rho \, d\phi = \left. \left( \frac{1}{2} \right) 20 \ln(0.1 + \rho^2) \right|_0^0 = -20\pi \ln(0.1) = -178.0 \text{A} - 1
\]

b) Calculate \( \frac{\partial \rho}{\partial t} \) : This is found using the equation of continuity:

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho J_\rho) + \frac{\partial J_z}{\partial z} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (25) + \frac{\partial}{\partial z} \left( \frac{-20}{\rho^2 + 0.01} \right) = 0
\]

c) Find the outward current crossing the closed surface defined by \( \rho = 0.01 \), \( \rho = 0.4 \), \( z = 0 \), and \( z = 0.2 \) : This will be

\[
I = \iiint [\mathbf{J} \cdot \mathbf{n}]_{z=0.2} \, dV + \iiint [\mathbf{J} \cdot \mathbf{n}]_{z=0} \, dV = \int_0^{0.2} \int_0^{2\pi} \frac{-20}{\rho^2 + 0.01} \rho \, d\rho \, d\phi + \int_0^{2\pi} \int_0^{0.4} \frac{-20}{\rho^2 + 0.01} \rho \, d\rho \, d\phi = 0
\]

since the integrals will cancel each other.

d) Show that the divergence theorem is satisfied for \( \mathbf{J} \) and the surface specified in part b. In part c, the net outward flux was found to be zero, and in part b, the divergence of \( \mathbf{J} \) was found to be zero (as will be its volume integral). Therefore, the divergence theorem is satisfied.
Problem 5.10

A solid wire of conductivity $\sigma_1$ and radius $a$ has a jacket of material having conductivity $\sigma_2$, and whose inner radius is $a$ and outer radius is $b$. Show that the ratio of the current densities in the two materials is independent of $a$ and $b$.

A constant voltage between the two ends of the wire means that the field within must be constant throughout the wire cross-section. Calling this field $E$, we have

$$E = \frac{J_1}{\sigma_1} = \frac{J_2}{\sigma_2} \Rightarrow \frac{J_1}{J_2} = \frac{\sigma_1}{\sigma_2}$$

which is independent of the dimensions.

\[ \frac{3}{1} \]
Problem 5.12

Two identical conducting plates, each having area $A$, are located at $z = 0$ and $z = d$. The region between plates is filled with a material having $z$-dependent conductivity, $\sigma(z) = \sigma_0 e^{-zd}$, where $\sigma_0$ is a constant. Voltage $V_0$ is applied to the plate at $z = d$; the plate at $z = 0$ is at zero potential. Find, in terms of the given parameters:

a) the resistance of the material: We start with the differential resistance of a thin slab of the material of thickness $dz$, which is

$$dR = \frac{dz}{\sigma A} = \frac{\sigma_0 e^{-zd} dz}{\sigma_0 A}$$

so that

$$R = \int dR = \left[ \frac{d}{\sigma_0 A} (e^{-zd}) \right]_0^d = \frac{d}{\sigma_0 A} (e^{-d} - 1) = \frac{1.72d}{\sigma_0 A} \Omega$$

b) the total current flowing between plates: We use

$$I = \frac{V_0}{R} = \frac{\sigma_0 A V_0}{1.72d}$$

c) the electric field intensity $E$ within the material: First the current density is

$$J = \frac{I}{A} \mathbf{a}_z = \frac{-\sigma_0 V_0}{1.72d} \mathbf{a}_z$$

so that

$$E = \frac{J}{\sigma(z)} = \frac{-V_0 e^{-zd}}{1.72d} \mathbf{a}_z V/m$$
Problem 5.20  

(By the method of images)

Two point charges of $-100\pi$ $\mu$C are located at (2,-1,0) and (2,1,0). The surface $x=0$ is a conducting plane.

a) Determine the surface charge density at the origin. I will solve the general case first, in which we find the charge density anywhere on the $y$ axis. With the conducting plane in the $yz$ plane, we will have two image charges, each of $+100\pi$ $\mu$C, located at (-2, -1, 0) and (-2, 1, 0). The electric flux density on the $y$ axis from these four charges will be

$$D(y) = -\frac{100\pi}{4\pi} \left[ \frac{[(y-1)a_y - 2a_z]}{[(y-1)^2 + 4]^{3/2}} + \frac{[(y+1)a_y - 2a_z]}{[(y+1)^2 + 4]^{3/2}} \right] \mu\text{C/m}^2$$

$$+ \left[ \frac{[(y-1)a_y + 2a_z]}{[(y-1)^2 + 4]^{3/2}} + \frac{[(y+1)a_y + 2a_z]}{[(y+1)^2 + 4]^{3/2}} \right] \mu\text{C/m}^2$$

In the expression, all $y$ components cancel, and we are left with

$$D(y) = 100 \left[ \frac{1}{[(y-1)^2 + 4]^{3/2}} + \frac{1}{[(y+1)^2 + 4]^{3/2}} \right] a_z \mu\text{C/m}^2$$

We now find the charge density at the origin:

$$\rho_s(0,0,0) = D \cdot a_z \bigg|_{y=0} = 17.9 \mu\text{C/m}^2$$

b) Determine $\rho_s$ at $P(0,h,0)$. This will be

$$\rho_s(0,h,0) = D \cdot a_z \bigg|_{y=h} = 100 \left[ \frac{1}{[(h-1)^2 + 4]^{3/2}} + \frac{1}{[(h+1)^2 + 4]^{3/2}} \right] \mu\text{C/m}^2$$

*See next page for standard approach.*
Solutions
5.20

\[ Q = Q_1 = Q_2 = -100 \pi \times 10^{-6} \ (C) \]

(a) At the origin the \( y \)-components of the fields produced by \( Q_1 \) and \( Q_2 \) are equal and opposite. Only the components directed along the negative \( x \)-axis are non-zero and add.

\[ \cos \alpha = \frac{2}{\sqrt{5}} \quad \text{and} \quad r = \sqrt{5} \]

\[ \varepsilon_2 E_{z2n} - \varepsilon_1 E_{zn} = 0 \]

Since Region (1) is a conductor, \( E_{zn} = 0 \)

\[ \Rightarrow \varepsilon_2 E_{z2n} = 0 \]

Now \( E_{z2n} = 2 \cos \alpha \cdot \frac{Q}{4 \pi \varepsilon_2 r^2} = 2 \cdot \frac{2}{\sqrt{5}} \cdot \frac{(-100 \pi \times 10^{-6})}{4 \pi \varepsilon_2 \cdot (\sqrt{5})^2} \]
Therefore, at the origin the surface charge density would seem to be

\[ \sigma_s(0) = \frac{\varepsilon_0}{\varepsilon_r} \cdot \left( \frac{100 \times 10^{-6}}{2 \cdot 5} \right) \]

In the coordinate system in which our boundary conditions were derived, the normal to the boundary in region 1 (\(\hat{n}\)) is negatively directed. Hence the orientation of the surface in region 1 has a minus sign associated with it. The charge density (which must be multiplied by the surface area to get the whole charge) is therefore positive with respect to the sources in Region 1. That is, in the same coordinate system as region 1 we find \(\sigma_s\) is positive.

\[ \sigma_s(0) = +8.94 \times 10^{-6} \text{ C} \]

When \(P\) is not at the origin, the tangential components of the field at the interface are still zero. This is
not because the contributions from $Q_1$ and $Q_2$ cancel one another but because $E_{1t} = 0$ in a conductor and $E_{2t} = E_{1t}$.

Hence we have only to add the normal components to find the total normal field in the same way as before.

\[ E_{2n} = E_{2n}^{(Q_1)} \sin \beta_1 + E_{2n}^{(Q_2)} \sin \beta_2 \]

1. \[ \sin \beta_1 = \frac{2}{\sqrt{(2)^2 + (h+1)^2}} \quad \sin \beta_2 = \frac{2}{\sqrt{(2)^2 + (h-1)^2}} \]

and \[ r_1 = \sqrt{(2)^2 + (h+1)^2} \quad r_2 = \sqrt{(2)^2 + (h-1)^2} \]

Hence in this case we find

\[ \sigma_6(h) = +0 \left[ \frac{100 \times 10^{-6}}{4 \pi r_1^2 \sin \beta_1} + \frac{100 \times 10^{-6}}{4 \pi r_2^2 \sin \beta_2} \right] \]

\[ = \left[ \frac{2 \times 2.5}{(\sqrt{(2)^2 + (h+1)^2})^3} + \frac{2 \times 2.5}{(\sqrt{(2)^2 + (h-1)^2})^3} \right] -4 \times 10^{-4} \]

\[ \sigma_5(h) = \left[ \frac{1}{(4 + [h+1]^2)^{3/2}} + \frac{1}{(4 + [h-1]^2)^{3/2}} \right] -3 \times 10^{-3} \]