Problem 1.22

A sphere of radius $a$, centered at the origin, rotates about the $z$ axis at angular velocity $\Omega$ rad/s. The rotation direction is clockwise when one is looking in the positive $z$ direction.

$a)$ Using spherical components, write an expression for the velocity field, $\mathbf{v}$, which gives the tangential velocity at any point within the sphere:

As in problem 1.20, we find the tangential velocity as the product of the angular velocity and the perpendicular distance from the rotation axis. With clockwise rotation, we obtain

$$
\mathbf{v}(r, \theta) = \Omega \sin \theta \mathbf{a}_\phi (r < a)
$$

$b)$ Convert to rectangular components:

From here, the problem is the same as part $c$ in Problem 1.20, except the rotation direction is reversed. The answer is

$$
\mathbf{v}(x, y) = \Omega [-y \mathbf{a}_x + x \mathbf{a}_y], \quad \text{where} \ (x^2 + y^2 + z^2)^{1/2} < a.
$$
Problem 1.23

The surfaces \( \rho = 3 \), \( \rho = 5 \), \( \phi = 100^\circ \), \( \phi = 130^\circ \), \( z = 3 \), and \( z = 4.5 \) define a closed surface.

\[
\text{Vol} = \int_0^4 \int_0^{30^\circ} \int_0^{30^\circ} \rho d\rho d\phi dz = 6.28
\]

\( a) \) Find the enclosed volume:

\[
\text{Area} = 2 \int_0^{30^\circ} \int_0^{30^\circ} \rho d\rho d\phi + \int_0^{4.5} \int_0^{130^\circ} 3 d\phi dz
\]

\[
\int_0^4 \int_0^{30^\circ} \rho d\rho d\phi = 20.7
\]

\( c) \) Find the total length of the twelve edges of the surfaces:

\[
\text{Length} = 4 \times 1.5 + 4 \times 2 + 2 \times \left[ \frac{30^\circ}{360^\circ} \times 2\pi \times 3 + \frac{30^\circ}{360^\circ} \times 2\pi \times 5 \right] = 22.4
\]

\( d) \) Find the length of the longest straight line that lies entirely within the volume: This will be between the points \( A(\rho = 3, \phi = 100^\circ, z = 3) \) and \( B(\rho = 5, \phi = 130^\circ, z = 4.5) \). Performing point transformations to cartesian coordinates, these become \( A\left(x = -0.52, y = 2.95, z = 3\right) \) and \( B(x = -3.21, y = 3.83, z = 4.5) \). Taking \( A \) and \( B \) as vectors directed from the origin, the requested length is

\[
\text{Length} = |\mathbf{B} - \mathbf{A}| = |(-2.69, 0.88, 1.5)| = 3.21
\]

Endpoints
Problem 1.27

The surfaces \( r = 2 \) and \( 4 \), \( \theta = 30^\circ \) and \( 50^\circ \), and \( \phi = 20^\circ \) and \( 60^\circ \) identify a closed surface.

\( a) \) Find the enclosed volume: This will be
\[
\text{Vol} = \frac{1}{3} \int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{2}} \int_{0}^{2} r^2 \sin \theta \, dr \, d\theta \, d\phi = 2.91
\]
where degrees have been converted to radians.

\( b) \) Find the total area of the enclosing surface:
\[
\text{Area} = \int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{2}} (4^2 + 2^2) \sin \theta \, d\theta \, d\phi + \int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{2}} r(\sin 30^\circ + \sin 50^\circ) \, dr \, d\phi
\]
\[+ 2 \int_{0}^{\frac{\pi}{2}} \int_{0}^{r} r^2 \, dr \, d\theta = 12.61 \]

\( c) \) Find the total length of the twelve edges of the surface:
\[
\text{Length} = 4 \int_{2}^{4} r^2 \, dr + 2 \int_{30^\circ}^{50^\circ} (4^2 + 2^2) \, d\theta + \int_{30^\circ}^{50^\circ} (4 \sin 50^\circ + 4 \sin 30^\circ + 2 \sin 50^\circ + 2 \sin 30^\circ) \, d\phi
\]
\[= 17.49
\]

\( d) \) Find the length of the longest straight line that lies entirely within the surface: This will be from \( A(r = 2, \theta = 50^\circ, \phi = 20^\circ) \) to \( r = 4, \theta = 30^\circ, \phi = 60^\circ \) or
\[
A(x = 2 \sin 50^\circ \cos 20^\circ, y = 2 \sin 50^\circ \sin 20^\circ, z = 2 \cos 50^\circ)
\]
to
\[
B(x = 4 \sin 30^\circ \cos 60^\circ, y = 4 \sin 30^\circ \sin 60^\circ, z = 4 \cos 30^\circ)
\]
or finally \( A(1.44, 0.52, 1.29) \) to \( B(1.00, 1.73, 3.46) \). Thus \( B - A = (-0.44, 1.21, 2.18) \) and
\[
\text{Length} = \|B - A\| = 2.53
\]
Problem 2.4

Eight identical point charges of $Q$ C each are located at the corners of a cube of side length $a$, with one charge at the origin, and with the three nearest charges at $(a,0,0)$, $(0,a,0)$, and $(0,0,a)$. Find an expression for the total vector force on the charge at $P(a,a,a)$, assuming free space:

The total electric field at $P(a,a,a)$ that produces a force on the charge there will be the sum of the fields from the other seven charges. This is written below, where the charge locations associated with each term are indicated:

$$E_{net}(a,a,a) = \frac{q}{4\pi\varepsilon_0 a^2} \left[ \frac{a_x + a_y + a_z}{3\sqrt{3}} + \frac{a_x + a_y}{2\sqrt{2}} + \frac{a_y + a_z}{2\sqrt{2}} + \frac{a_z + a_x}{2\sqrt{2}} + \frac{a_x}{(0,0,a)} + \frac{a_y}{(a,0,a)} + \frac{a_z}{(a,a,0)} \right]$$

The force is now the product of this field and the charge at $(a,a,a)$. Simplifying, we obtain

$$F(a,a,a) = qE_{net}(a,a,a) = \frac{q^2}{4\pi\varepsilon_0 a^2} \left[ \frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{2}} + 1 \right] (a_x + a_y + a_z) = \frac{1.90 q^2}{4\pi\varepsilon_0 a^2} (a_x + a_y + a_z)$$

in which the magnitude is $|F| = 3.29 \frac{q^2}{(4\pi\varepsilon_0 a^2)}$.  

Exact 2
Problem 2.6

Three point charges, each $5 \times 10^{-9} \text{ C}$, are located on the $x$ axis at $x = -1$, 0, and 1 in free space.

a) Find $E$ at $x = 5$: At a general location, $x$,

$$E(x) = \frac{q}{4\pi \varepsilon_0} \left[ \frac{1}{(x+1)^2} + \frac{1}{x^2} + \frac{1}{(x-1)^2} \right] a_x$$

At $x = 5$, and with $q = 5 \times 10^{-9} \text{ C}$, this becomes $E(x = 5) = 5.8 a_x \text{ V/m}$.

b) Determine the value and location of the equivalent single point charge that would produce the same field at very large distances: For $x \gg 1$, the above general field in part a becomes

$$E(x \gg 1) \approx \frac{3q}{4\pi \varepsilon_0 x^2} a_x$$

Therefore, the equivalent charge will have value $3q = 1.5 \times 10^{-8} \text{ C}$, and will be at location $x = 0$.

c) Determine $E$ at $x = 5$, using the approximation of b). Using $3q = 1.5 \times 10^{-8} \text{ C}$ and $x = 5$ in the part b result gives $E(x = 5) \approx 5.4 a_x \text{ V/m}$, or about 7% lower than the exact result.
Problem 2.8.

A crude device for measuring charge consists of two small insulating spheres of radius $a$, one of which is fixed in position. The other is movable along the $x$ axis, and is subject to a restraining force $kx$, where $k$ is a spring constant. The uncharged spheres are centered at $x = 0$ and $x = d$, the latter fixed. If the spheres are given equal and opposite charges of $Q$ coulombs:

a) Obtain the expression by which $Q$ may be found as a function of $x$:

The spheres will attract, and so the movable sphere at $x = 0$ will move toward the other until the spring and Coulomb forces balance. This will occur at location $x$ for the movable sphere. With equal and opposite forces, we have

$$\frac{Q^2}{4\pi\varepsilon_0 (d-x)^2} = kx$$

from which $Q = 2(d-x)\sqrt{\frac{k}{\pi\varepsilon_0}}$.

b) Determine the maximum charge that can be measured in terms of $\varepsilon_0$, $k$, and $d$, and state the separation of the spheres then: With increasing charge, the spheres move toward each other until they just touch at $x_{\text{max}} = d - 2a$. Using the part a result, we find the maximum measurable charge: $Q_{\text{max}} = 4a\sqrt{\frac{\pi\varepsilon_0 k}{2a}}$. Presumably some form of stop mechanism is placed at $x = x_{\text{max}}$ to prevent the spheres from actually touching.

c) What happens if a larger charge is applied?

No further motion is possible, so nothing happens.

... well, actually there is a loud bang as the charges neutralize one another and discharge (resulting in two uncharged spheres) through the air at some point!
Problem 2.14

The charge density varies with radius in a cylindrical coordinate system as 
\[ \rho_v = \frac{\rho_0}{(\rho^2 + a^2)^2} \text{ C/m}^3. \] Within what distance from the z axis does half the total charge lie?

Choosing a unit length in z, the charge contained up to radius \( \rho \) is

\[ Q(\rho) = \int_0^a \int_0^{2\pi} \frac{\rho_0}{(\rho^2 + a^2)^2} \rho \, d\rho \, d\phi \, dz = 2\pi \rho_0 \left[ \frac{-1}{2(a^2 + \rho^2)^2} \right]_0^\rho = \frac{\pi \rho_0}{a^2} \left[ 1 - \frac{1}{1 + \rho^2/a^2} \right]. \]

The total charge is found when \( \rho \to \infty \), or \( Q_{\text{net}} = \pi \rho_0/a^2 \). It is seen from the \( Q(\rho) \) expression that half of this occurs when \( \rho = a \).

**Note:** One mark also for charge expressions that integrate over some other z length like \( \int dz = L \) or that leave the z integral as

\[ \int_0^\infty dz \] (on both sides of equation) used to determine \( Q \)

However, the final expression for the half-charge radius must be independent of z.
Problem 2.16

Within a region of free space, charge density is given as \( \rho_r = \rho_0 r/a \) C/m\(^3\), where \( \rho_0 \) and \( a \) are constants. Find the total charge lying within:

a) the sphere, \( r \leq a \): This will be

\[
Q_s = \frac{\rho_0}{a} \int_0^a \int_0^{2\pi} r^2 \sin \theta \, d\theta \, d\phi = 4\pi \int_0^a \frac{\rho_0 r^3}{a} \, dr = \pi \rho_0 a^3
\]

b) the cone, \( r \leq a, 0 \leq \theta \leq 0.1\pi \):

\[
Q_c = \int_0^{2\pi} \int_0^{0.1\pi} \int_0^a \frac{\rho_0 r^2}{a} \sin \theta \, dr \, d\theta \, d\phi = 2\pi \frac{\rho_0 a^3}{4} [1 - \cos(0.1\pi)] = 0.024 \pi \rho_0 a^3
\]

c) the region, \( r \leq a, 0 \leq \theta \leq 0.1\pi, 0 \leq \phi \leq 0.2\pi \).

\[
Q_e = \int_0^{2\pi} \int_0^{0.1\pi} \int_0^a \frac{\rho_0 r^2}{a} \sin \theta \, dr \, d\theta \, d\phi = 0.024 \pi \rho_0 a^3 \left( \frac{0.2\pi}{2\pi} \right) = 0.0024 \pi \rho_0 a^3
\]
Problem 2.22

Two identical uniform sheet charges with $\rho_s = 100 \ \text{nC/m}^2$ are located in free space at $z = \pm 2.0 \ \text{cm}$. What force per unit area does each sheet exert on the other?

The field from the top sheet is $E = -\rho_s/(2\varepsilon_0)\mathbf{a}_z$ V/m. The differential force produced by this field on the bottom sheet is the charge density on the bottom sheet times the differential area there, multiplied by the electric field from the top sheet: $dF = \rho_s da E$.

The force per unit area is then just

$$\vec{F} = \rho_s \vec{E} = (100 \times 10^{-9})(-100 \times 10^{-9})/(2\varepsilon_0)\mathbf{a}_z = -5.6 \times 10^{-4} \mathbf{a}_z \text{N/m}^2.$$