Problem 10.4

Conductor surfaces are located at $\rho = 1\, \text{cm}$ and $\rho = 2\, \text{cm}$ in free space. The volume $1\, \text{cm} < \rho < 2\, \text{cm}$ contains the fields $H_\phi = (2/\rho) \cos(6 \times 10^8 \pi t - 2\pi z)$ A/m and $E_\rho = (240\pi/\rho) \cos(6 \times 10^8 \pi t - 2\pi z)$ V/m.

a) Show that these two fields satisfy Eq. (6), Sec. 10.1: Have

$$\nabla \times \mathbf{E} = \frac{\partial E_\rho}{\partial z} \mathbf{a}_\phi = \frac{2\pi(240\pi)}{\rho} \sin(6 \times 10^8 \pi t - 2\pi z) \mathbf{a}_\phi = \frac{480\pi^2}{\rho^2} \sin(6 \times 10^8 \pi t - 2\pi z) \mathbf{a}_\phi
$$

Then

$$- \frac{\partial \mathbf{B}}{\partial t} = \frac{2\mu_0 (6 \times 10^8)^2 \pi}{\rho} \sin(6 \times 10^8 \pi t - 2\pi z) \mathbf{a}_\phi
$$

$$= \frac{(8\pi \times 10^{-7})(6 \times 10^8)^2 \pi}{\rho} \sin(6 \times 10^8 \pi t - 2\pi z) = \frac{480\pi^2}{\rho} \sin(6 \times 10^8 \pi t - 2\pi z) \mathbf{a}_\phi$$

b) Evaluate both integrals in Eq. (4) for the planar surface defined by $\phi = 0$, $1\, \text{cm} < \rho < 2\, \text{cm}$, $0 < z < 0.1\, \text{m}$, and its perimeter, and show that the same results are obtained: we take the normal to the surface as positive $\mathbf{a}_\phi$, so the loop surrounding the surface (by the right hand rule) is in the negative $\mathbf{a}_\rho$ direction at $z = 0$, and is in the positive $\mathbf{a}_\rho$ direction at $z = 0.1$. Taking the left hand side first, we find

$$\int \mathbf{E} \cdot d\mathbf{L} = \int_{0}^{0.1} \frac{240\pi}{\rho} \cos(6 \times 10^8 \pi t) \mathbf{a}_\rho \cdot d\rho$$

$$+ \int_{0.1}^{0.2} \frac{240\pi}{\rho} \cos(6 \times 10^8 \pi t - 2\pi(0.1)) \mathbf{a}_\rho \cdot d\rho$$

$$= 240\pi \cos(6 \times 10^8 \pi t) \ln \left( \frac{1}{2} \right) + 240\pi \cos(6 \times 10^8 \pi t - 0.2\pi) \ln \left( \frac{2}{1} \right)$$

$$= 240\pi \left[ \cos(6 \times 10^8 \pi t - 0.2\pi) - \cos(6 \times 10^8 \pi t) \right]\ (\checkmark)
$$

Now for the right hand side. First,

$$\int \mathbf{B} \cdot d\mathbf{S} = \int_{0}^{0.1} \int_{0}^{0.1} \frac{8\pi \times 10^{-7}}{\rho} \cos(6 \times 10^8 \pi t - 2\pi z) \mathbf{a}_\phi \cdot d\rho dz$$

$$= \int_{0}^{0.1} (8\pi \times 10^{-7}) \ln 2 \cos(6 \times 10^8 \pi t - 2\pi z) dz$$

$$= -4 \times 10^{-7} \ln 2 \left[ \sin(6 \times 10^8 \pi t - 0.2\pi) - \sin(6 \times 10^8 \pi t) \right]$$

Then

$$- \frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S} = 240\pi \ln 2 \left[ \cos(6 \times 10^8 \pi t - 0.2\pi) - \cos(6 \times 10^8 \pi t) \right]$$

(\checkmark)
Problem 10.6

A perfectly conducting filament containing a small 500-Ω resistor is formed into a square, as illustrated in Fig. 10.6. Find \(i(t)\) if

a) \(\mathbf{B} = 0.3 \cos(120\pi t - 30') \mathbf{a}_z\) T: First the flux through the loop is evaluated, where the unit normal to the loop is \(\mathbf{a}_z\). We find

\[
\Phi = \oint_{\partial \text{loop}} \mathbf{B} \cdot d\mathbf{S} = (0.3)(0.5)^2 \cos(120\pi t - 30') \text{Wb}
\]

Then the current will be

\[
I(t) = \frac{\text{emf}}{R} = -\frac{1}{R} \frac{d\Phi}{dt} = \frac{(120\pi)(0.3)(0.25)}{500} \sin(120\pi t - 30') = \frac{9\pi}{(120\pi - 30')} \text{mA}
\]

b) \(\mathbf{B} = 0.4 \cos[\pi(c t - y)] \mathbf{a}_x \mu T\) where \(c = 3 \times 10^8\) m/s: Since the field varies with \(y\), the flux is now

\[
\Phi = \oint_{\text{loop}} \mathbf{B} \cdot d\mathbf{S} = (0.5)(0.4) \int_0^\pi \cos(\pi c t - \pi y) \, dy = \frac{0.2}{\pi} \left[ -\sin(\pi c t - \pi/2) + \sin(\pi c t) \right] \text{mWb}
\]

The current is then

\[
I(t) = \frac{\text{emf}}{R} = -\frac{1}{R} \frac{d\Phi}{dt} = \frac{-0.2c}{500} \left[ \cos(\pi c t - \pi/2) - \cos(\pi c t) \right] \mu A
\]

\[
= \frac{-0.2(3 \times 10^8)}{500} \left[ \sin(\pi c t) - \cos(\pi c t) \right] \mu A = \frac{120}{\pi} \left[ \cos(\pi c t) - \sin(\pi c t) \right] \mu A
\]
Problem 10.12

Show that the displacement current flowing between the two conducting cylinders in a lossless coaxial capacitor is exactly the same as the conduction current flowing in the external circuit if the applied voltage between conductors is $V_0 \cos \omega t$ volts.

From Chapter 7, we know that for a given applied voltage between the cylinders, the electric field is

$$E = \frac{V_0 \cos \omega t}{\rho \ln(b/a)} \hat{a}_n \text{N/m} \Rightarrow D = \frac{\varepsilon_0 V_0 \cos \omega t}{\rho \ln(b/a)} \hat{a}_n \text{C/m}^2$$

Then the displacement current density is

$$\frac{\partial D}{\partial t} = \frac{\varepsilon_0 \omega V_0 \sin \omega t}{\rho \ln(b/a)} \hat{a}_n$$

Over a length $\ell$, the displacement current will be

$$I_d = \oint \frac{\partial D}{\partial t} \cdot d\mathbf{S} = 2\pi \rho \ell \frac{\partial D}{\partial t} = \frac{2\pi \varepsilon_0 \omega V_0 \sin \omega t}{\ln(b/a)} \ell = \frac{dV}{dt} = I_c$$

where we recall that the capacitance is given by $C = 2\pi \varepsilon_0 \ell / \ln(b/a)$.

**Alternate solution:**

$$\nabla \times \mathbf{E} = \frac{\partial E_y}{\partial z} \hat{\phi} - \frac{1}{\varepsilon} \frac{\partial E_x}{\partial \phi} \hat{\phi} = 0$$

and

$$\mathbf{H} = \left(\frac{1}{j \omega \mu_0}\right) \nabla \times \mathbf{E} = 0$$

In view of the Maxwell relation $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$, we conclude

$$\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = 0 \quad \text{or} \quad \mathbf{J} = -\frac{\partial \mathbf{D}}{\partial t}$$

So the conduction current density $\mathbf{J}$ equals the displacement current $\frac{\partial \mathbf{D}}{\partial t}$ (apart from phase angle).