Problem 1.1

Given the vectors $\mathbf{M} = -10\mathbf{a}_x + 4\mathbf{a}_y - 8\mathbf{a}_z$ and $\mathbf{N} = 8\mathbf{a}_x + 7\mathbf{a}_y - 2\mathbf{a}_z$, find:

**Sol.**

a) a unit vector in the direction of $-\mathbf{M} + 2\mathbf{N}$.

$-\mathbf{M} + 2\mathbf{N} = 10\mathbf{a}_x - 4\mathbf{a}_x + 8\mathbf{a}_z + 16\mathbf{a}_z + 14\mathbf{a}_y - 4\mathbf{a}_y = (26, 10, 4)$

Thus

$$\mathbf{a} = \frac{(26,10,4)}{|(26,10,4)|} = (0.92, 0.36, 0.14)$$

b) the magnitude of $5\mathbf{a}_x + \mathbf{N} - 3\mathbf{M}$:

$(5, 0, 0) + (8, 7, -2) - (-30, 12, -24) = (43, -5, 22)$, and $|(43, -5, 22)| = 48.6$.

c) $|\mathbf{M}||2\mathbf{N}|(\mathbf{M} + \mathbf{N})$:

$|(-10, 14, -8)|(|16, 14, -4||(-2, 11, -10) = (13.4)(21.6)(-2, 11, -10)$

$= (-580.5, 3193, -2902)$
Problem 1.2

The three vertices of a triangle are located at \( A(-1, 2, 5), \) \( B(-4, -2, -3), \) and \( C(1, 3, -2). \)

\[ \text{Sol.} \]

2) Find the length of the perimeter of the triangle: Begin with \( \mathbf{AB} = (-3, -4, -8), \)
\( \mathbf{BC} = (5, 5, 1), \) and \( \mathbf{CA} = (-2, -1, 7). \) Then the perimeter will
be \( \ell = |\mathbf{AB}| + |\mathbf{BC}| + |\mathbf{CA}| = \sqrt{9 + 16 + 64} + \sqrt{25 + 25 + 1} + \sqrt{4 + 1 + 49} = 23.9. \)

b) Find a unit vector that is directed from the midpoint of the side \( AB \) to the midpoint of side
\( BC. \) The vector from the origin to the midpoint of \( AB \) is \( \mathbf{M_{AB}} = \frac{1}{2}(\mathbf{A} + \mathbf{B}) = \frac{1}{2}(-5\mathbf{a}_x + 2\mathbf{a}_z). \)
The vector from the origin to the midpoint of \( BC \) is \( \mathbf{M_{BC}} = \frac{1}{2}(\mathbf{B} + \mathbf{C}) = \frac{1}{2}(-3\mathbf{a}_x + \mathbf{a}_y - 5\mathbf{a}_z). \)
The vector from midpoint to midpoint is now \( \mathbf{M_{AB}} - \mathbf{M_{BC}} = \frac{1}{2}(-2\mathbf{a}_x + \mathbf{a}_y - 7\mathbf{a}_z). \) The unit
vector is therefore
\[ a_{MM} = \frac{\mathbf{M_{AB}} - \mathbf{M_{BC}}}{|\mathbf{M_{AB}} - \mathbf{M_{BC}}|} = \frac{(-2\mathbf{a}_x + \mathbf{a}_y - 7\mathbf{a}_z)}{7.35} = -0.27\mathbf{a}_x - 0.14\mathbf{a}_y + 0.95\mathbf{a}_z \]
where factors of \( \frac{1}{2} \) have cancelled

3) Show that this unit vector multiplied by a scalar is equal to the vector from \( A \) to \( C \)
and that the unit vector is therefore parallel to \( AC. \) First we find \( \mathbf{AC} = 2\mathbf{a}_x + \mathbf{a}_y - 7\mathbf{a}_z, \)
which we recognize as \(-7.35 a_{MM}. \) The vectors are thus parallel (but oppositely-directed).
Problem 1.3

The vector from the origin to the point $A$ is given as $(6, -2, -4)$, and the unit vector directed from the origin toward point $B$ is $(2, -2, 1)/3$. If points $A$ and $B$ are ten units apart, find the coordinates of point $B$.

**Sol.**

With $A = (6, -2, -4)$ and $B = \frac{1}{3}B(2, -2, 1)$, we use the fact that $|B - A| = 10$, or

$$|\left(6 - \frac{2}{3} B\right)x + \left(-2 - \frac{2}{3} B\right)y + \left(-4 + \frac{1}{3} B\right)z| = 10$$

Expanding, obtain

$$36 - 8B + \frac{4}{3}B^2 + 4 - \frac{8}{3}B + \frac{4}{3}B^2 + 16 + \frac{8}{3}B + \frac{4}{3}B^2 = 100$$

or $B^2 - 8B - 44 = 0$. Thus $B = \frac{8 \pm \sqrt{64 - 4 \cdot 44}}{2} = 11.75$ (taking positive option) and so

$$B = \frac{2}{3}(11.75)x - \frac{2}{3}(11.75)y + \frac{1}{3}(11.75)z = 7.83x - 7.83y + 3.92z$$
Problem 1.4

A circle, centered at the origin with a radius of 2 units, lies in the \( xy \) plane. Determine the unit vector in rectangular components that lies in the \( xy \) plane, is tangent to the circle at \((\sqrt{3}, 1, 0)\), and is in the general direction of increasing values of \( y \):

**Sol.**

A unit vector tangent to this circle in the general increasing \( y \) direction is \( \mathbf{t} = a_\phi \). Its \( x \) and \( y \) components are \( t_x = a_\phi \cdot a_x = -\sin \phi \), and \( t_y = a_\phi \cdot a_y = \cos \phi \). At the point \((\sqrt{3}, 1)\), \( \phi = 30^\circ \), and so \( \mathbf{t} = -\sin 30^\circ \mathbf{a}_x + \cos 30^\circ \mathbf{a}_y = 0.5(-\mathbf{a}_x + \sqrt{3}\mathbf{a}_y) \).
Problem 1.5

A vector field is specified as \( \mathbf{G} = 24xy \mathbf{a}_x + 12(x^2 + 2) \mathbf{a}_y + 18z^2 \mathbf{a}_z \). Given two points, \( P(1,2,-1) \) and \( Q(-2,1,3) \), find:

**Sol.**

a) \( \mathbf{G} \) at \( P : \mathbf{G}(1,2,-1) = (48,36,18) \)

b) a unit vector in the direction of \( \mathbf{G} \) at \( Q : \mathbf{G}(-2,1,3) = (48,72,162) \), so

\[ \mathbf{a}_o = \frac{(48,72,162)}{|(48,72,162)|} = \frac{(-0.26, 0.39, 0.88)}{2} \]

c) A unit vector directed from \( Q \) toward \( P \):

\[ \mathbf{a}_{QP} = \frac{\mathbf{P} - \mathbf{Q}}{|\mathbf{P} - \mathbf{Q}|} = \frac{(3,-1,4)}{\sqrt{26}} = \frac{(0.59, 0.20, -0.78)}{3} \]

d) The equation of the surface on which \( |\mathbf{G}| = 60 \): We write 60 = \( |(24xy, 12(x^2 + 2), 18z^2)| \), or 10 = \( |(4xy, 2x^2 + 4, 3z^2)| \), so the equation is

\[ 100 = 16x^2y^2 + 4x^4 + 16x^2 + 16 + 9z^4 \]
Problem 1.12

Show that the vector fields $A = \rho \cos \phi a_\rho + \rho \sin \phi a_\phi + \rho a_z$
and $B = \rho \cos \phi a_\rho + \rho \sin \phi a_\phi - \rho a_z$ are everywhere perpendicular to each other:

**Sol.**

We find $A \cdot B = \rho^2 (\sin^2 \phi + \cos^2 \phi) - \rho^2 = |A||B| \cos \theta$. Therefore $\cos \theta = 0$ or $\theta = 90^\circ$. 
Problem 1.14

Show that the vector fields \( \mathbf{A} = a_r \left( \sin 2\theta \right)/r^2 + 2a_\theta \left( \sin \theta \right)/r^2 \) and \( \mathbf{B} = r \cos \theta a_r + r a_\theta \) are everywhere parallel to each other:

\[ \mathbf{A} \times \mathbf{B} = \left( \frac{\sin 2\theta}{r} - \frac{2\sin \theta \cos \theta}{r} \right) a_\phi = 0 = |\mathbf{A}| |\mathbf{B}| \sin \theta n \]

Identify \( n = a_\phi \), and so \( \sin \theta = 0 \), and therefore \( \theta = 0 \) (they're parallel).
Problem 1.18

Transform the vector field $\mathbf{H} = \left(\frac{A}{\rho}\right)\mathbf{a}_\phi$, where $A$ is a constant, from cylindrical coordinates to spherical coordinates:

**Sol.**

First, the unit vector does not change, since $\mathbf{a}_\phi$ is common to both coordinate systems. We only need to express the cylindrical radius, $\rho$, as $\rho = r \sin \theta$, obtaining

$$\mathbf{H}(r, \theta) = \frac{A}{r \sin \theta} \mathbf{a}_\phi$$