Final Exam - Math 217
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Winter 2008

Name: __________________________

Please circle your answers. Cross out any work that you do not want graded.

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Credit for problems 1-4 is based on your answers only.

(1) Mark the following statements as either “True” or “False.”

- If $A$ and $B$ are invertible square matrices, then $(AB)^{-1}$ must be equal to $A^{-1}B^{-1}$.

- Suppose that $S$ is an orthonormal set of vectors in $\mathbb{R}^5$. Then $S$ cannot contain more than 5 elements.

- There does not exist a linear transformation $T: \mathbb{R}^5 \to \mathbb{R}^4$ that is both one-to-one and onto.

- The determinant of a diagonal matrix is always equal to the product of its diagonal entries.

- Suppose that $V$ is a vector space and $\mathcal{B} = \{b_1, \ldots, b_n\}$ is a basis for $V$. Then the coordinate mapping $V \to \mathbb{R}^n$ (given by $v \mapsto [v]_\mathcal{B}$) must be an isomorphism.
(2) Mark the following statements as either “True” or “False.”

- If \( A \) is an \( m \times n \) matrix, and \( b \) is a vector in \( \mathbb{R}^m \), then the set of solutions to the equation \( Ax = b \) must be a subspace of \( \mathbb{R}^n \).

- For any \( n \times n \) matrix \( A \) and any real number \( c \),
  \[
  \det(cA) = c^n \det(A).
  \]

- Let \( \mathbf{w} \) and \( \mathbf{v} \) be vectors in \( \mathbb{R}^n \). If \( \mathbf{v} \) is orthogonal to \( \mathbf{w} \), then the length of \( (\mathbf{w} + \mathbf{v}) \) must be the same as the length of \( \mathbf{w} \).

- Let \( M_{n \times n} \) denote the vector space of \( n \times n \) matrices (with the usual rules for addition and scalar multiplication). Then
  \[
  \dim M_{n \times n} = n
  \]
  for any \( n \).

- Let \( A \) be a \( n \times n \) matrix that has one (and only one) real eigenvalue. Then \( A \) must be equal to a scalar multiple of the identity matrix.
(3) Let

\[
A = \begin{bmatrix}
1 & 2 & 2 & 1 \\
0 & 1 & 3 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\].

(a) Compute \( A^{-1} \).

(b) Compute \( \det A \).
(4) Please circle your answers for the following problems.

(a) Let $S: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which flips vectors across the origin. Find the standard matrix for $S$.

(b) Let

$$v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$ 

Find a vector $w$ of length 1 which is orthogonal to $v$.

(c) Let $T$ be a triangle in $\mathbb{R}^2$ whose vertices all have integer coordinates. What is the smallest possible area of the interior of $T$? (Assume that the vertices of $T$ do not all lie on a single line.)
Credit for problems 5-8 is based on answers and on work shown.

(5) Let

\[ A = \begin{bmatrix} 1 & 2 \\ -2 & -3 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ -4 \\ -4 \end{bmatrix}. \]

Find a vector \( x \) which satisfies the equation \( Ax = b \).
(6) Let

\[ H = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \]

Compute the dimension of \( H \). (Be sure to show work that supports your answer.)
(7) Let

\[ A = \begin{bmatrix} 1 & -2 \\ 4 & 7 \end{bmatrix}. \]

(a) Find a diagonal matrix \( D \) and an invertible matrix \( P \) such that \( A = PDP^{-1} \).

(b) Let \( \mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), and consider the sequence of vectors

\( \mathbf{v}, A\mathbf{v}, A^2\mathbf{v}, \ldots \)

Which of the following statements best describes the geometric behavior of this sequence? (Circle one.)

- The sequence tends towards the origin.
- The sequence tends towards the vector \( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \).
- The sequence tends towards the vector \( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).
- The sequence tends away from the origin.
Let
\[ H = \text{Span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\} \]
and \[ v = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}. \]
Find the vector \( w \in H \) which makes the distance \( \|w - v\| \) as small as possible.
(9) Let $A$ be a $10 \times 10$ matrix such that $A^2$ is the zero matrix. Prove that the dimension of Nul $A$ must be at least 5.