TEACHING STATEMENT
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During the winter of 2009 I had the privilege of teaching a course called “Introduction to Coding Theory.” This was a course on the mathematics of electronic communication. It was populated mainly by math and computer science students. On the first day of class, I started my lecture by drawing a simple conceptual model of communication. I discussed the basic challenge of coding theory (to get a message across a noisy channel quickly and accurately). And I said: “in this course we’re concerned with two things: optimization and insight. We have a goal, and we want to accomplish it efficiently and correctly. That’s optimization. At the same time, we want a clear conceptual understanding of what we’re doing. We want a theory which supports our methods, which organizes our thoughts and allows us to think clearly about what we’re doing. We want insight. These two goals often dovetail in interesting ways. What we will try to do in this course is develop them in parallel.”

That introduction reflects my general philosophy of teaching.

What is mathematics? There are countless answers to this question. But there are two points of view in particular which support and influence the way that I teach.

One point of view is that mathematics is the study of rigorous conceptual constructions. In this point of view, math is built around objects like “matrices,” “functions,” “random variables,” “permutations,” and so forth. (These are what Yuri Manin calls “ideas which can be handled as if they were real things.”) Each has a certain level of complexity and difficulty, and each has a theory which supports it, and each is codependent with many other constructions. So one could say teaching mathematics is a process of building, developing, and clarifying these ideas in students’ minds.

Another point of view, and one which is maybe closer to the point of view that students are likely to bring to my classes, is that math is the study of problem-solving techniques. Math gives us tools which allow us to solve problems in the sciences. It gives us techniques to find conclusions that we couldn’t reach with our everyday physical intuition. In this view, the process of teaching math is empowering students, getting them to higher and higher levels of achievement, and enabling them to be good scientists.

The second point of view is fairly important to me. (It’s become particularly important since I’ve started doing interdisciplinary research.) But I think the second point of view is incomplete without the first. This incompleteness is reflected in the kind of preparation that students often bring to my classes. I often talk to students who are used to thinking of mathematics as a bundle of problem-solving methods: complete the square, find the line of best-fit, apply the second derivative test, etc. They know how to apply a procedure to a fixed class of problems. But they’ve had little practice at using math concepts outside the context of a particular procedure. This kind of understanding is fine as far as it goes, but I think it has limited value. It is not a good basis for further learning. I think what a lot
of students need is the opportunity to develop a durable conceptual understanding which supports their problem-solving strategies.

When I plan a course, I ask myself two questions: “what are the central concepts?” and “what are the central problems?” I make a short list of answers to these questions. This list is the first thing that I look at for the first few weeks whenever I’m preparing lecture notes. My list for the linear algebra class I taught (Winter 2008) included problems like “Describe the solution set for a linear system,” “Determine the behavior of an iterated matrix transformation,” and concepts like “matrices,” “eigenvalues,” and “the determinant.”

The central problems are important because they are the most prominent external features of the course. On the first day of a class I usually state a general problem: “our first goal in this class will be to solve problem X.” (In my algebra class this semester, the problem was: “solve the equation $ax + by = 1$.”) The central problems give the class a sense of momentum, and also give the students and I yardstick by which we can measure our progress. They also give us connections to subjects outside of math.

The central concepts give me the inner workings of the course. Each one is a nexus for the material that we’re studying; a point at which the students and I can stand and look at the theory that we are developing. The central concepts give me a way of untangling the course material and drawing out the most important lessons. I think one of my best talents as an instructor is my ability to find an easy point of view on a difficult subject. I’ve often gotten compliments from students about my ability to offer clear explanations and simplify tough material.

From this conceptual point of view, one my biggest challenges when I’m teaching is helping students to learn to use their mathematical intuition. This isn’t something that can be forced. I think students often come upon mathematical intuition unexpectedly. Maybe the fourth or fifth time they see me use a concept in class, they think to themselves, “Ah! Now I get it.” (This is why repetition is a good thing.) Beyond developing intuition in the first place, though, there’s also the challenge of sustaining the intuition that students already have. This isn’t always an easy thing. I’ve taught classes—like “Math 312: Applied Modern Algebra”—in which students are supposed to take a familiar construct, like the set of rational numbers, and learn a rigorous logical definition for it. This is good exercise, and it is important. But in any theory class there’s the danger that the theory will clutter up and obscure students’ original intuition for the constructs that they’re dealing with. When we’re formalizing a concept, it’s important that we return frequently to our original intuition. This is a guideline that I try to use in my lessons.

Once students have a good conceptual understanding, they’re better able to solve problems. They can face a wider array of problems, and they solve problems with more confidence. They’re less likely to ask, “... is that right?” They get automatic reinforcement from their understanding.

When I talk about problems with students during office hours, I try to help them develop an independent sense of what they’re doing. Typically, a student who comes to office hours will bring me a homework or quiz problem that they haven’t been able to solve. The way I usually respond is to ask them to tell me their best attempt at the problem. Usually their attempts—however misguided they might be—are clues to some sort of mathematical intuition. I try to find a place for their intuition in the context of the problem, and then lead
them in the right direction so that they can find a solution for themselves. I let the student take charge of solving the problem; my role is just to provide conceptual background.

The dual perspective that I use in planning a course—thinking in terms of both concepts and problem-solving strategies—is also the one that I use when I’m assessing what the students have learned. I try to ask questions that test conceptual understanding. Examples are: “Describe the structure of the finite field $F_9$.” “Suppose that $A$ is a $10 \times 10$ matrix such that $A^2 = 0$. Prove that $\text{rank}(A) \leq 5$.” A good conceptual problem is one that is in a form that the students have never seen before, and one that can’t be answered by students who have only a superficial understanding. And I also include procedural problems that are in standard form (e.g.: “Diagonalize the following matrix: [...]”) since these are important for measuring students’ achievement.

It’s gratifying to look back at old exams and see how the students have progressed. On conceptual problems I get to see how the students have learned to think more clearly and more deeply about the course material. Also on procedural problems, what I usually see is the students writing more cleanly, becoming more confident in what they’re doing, and following a straighter path from set-up to solution. It’s nice for me to see the exams of the strongest students who have mastered all of the course material. And it’s also pleasing to see exams from students who have progressed far in the course, even if the more difficult course-concepts are still out of reach for them at the end of the semester.

When I first started as an instructor at the University of Michigan, I wrote the following in my journal: “I’d like to show students that math is not dry and rigid, but fluid.” To me this means that even though mathematics deals with absolutes, it is ultimately based on human insight, which is something that is complex and unpredictable. I try to work this sense of fluidity into my classes. When I show students a problem-solving method in lecture, I also try to show them the mathematical insights that lead to the method. I point out that what I’m showing them is by no means the “only” correct approach—just one of the best of the known approaches. I try to cultivate a sense of the creativity in mathematics.

What I would hope most for students is that they come away from my class with a higher level of problem-solving ability, and a stronger sense of the value of their own insight.