Matrix pencils and entanglement classification

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Quantum entanglement plays a central role in quantum information processing. A main objective of the theory is to classify different types of entanglement according to their interconvertibility through manipulations that do not require additional entanglement to perform. While bipartite entanglement is well understood in this framework, the classification of entanglements among three or more subsystems is inherently much more difficult. In this paper, we study pure state entanglement in systems of dimension $2^m \otimes 2^n$. Two states are considered equivalent if they can be reversibly converted from one to the other with a nonzero probability using only local quantum resources and classical communication (SLOCC). We introduce a connection between entanglement manipulations in these systems and the well-studied theory of matrix pencils. All previous attempts to study general SLOCC equivalence in such systems have relied on somewhat contrived techniques which fail to reveal the elegant structure of the problem that can be seen from the matrix pencil approach. Based on this method, we report the first polynomial-time algorithm for deciding when two $2^m \otimes 2^n$ states are SLOCC equivalent. We then proceed to present a canonical form for all $2^m \otimes 2^n$ states based on the matrix pencil construction such that two states are equivalent if and only if they have the same canonical form. Besides recovering the previously known 26 distinct SLOCC equivalence classes in $2^3 \otimes 2^n$ systems, we also determine the hierarchy between these classes. © 2010 American Institute of Physics. [doi:10.1063/1.3459069]

I. INTRODUCTION

The feature that most distinguishes multipartite quantum systems from their classical counterpart is their ability to be in the so-called entangled states. Not only does quantum entanglement enable apparent “spooky action at a distance” between separated systems,1 but it also has the potential to fundamentally change and dramatically improve the current information processing and cryptographic technologies.2 It becomes natural then to treat entanglement in a multipartite system as an information processing resource that takes on different forms as the system realizes different states. Much effort has been devoted to formally quantifying the amount of entanglement contained in a given quantum state with the motivating intuition being that states having more entanglement possess a greater degree of computational and communication power than those with a lesser amount.

Under this interpretation, one may reasonably argue that a state $|\phi\rangle$ possesses no less entanglement than another state $|\phi\rangle$ of the same system if the system can be converted from $|\phi\rangle$ to $|\phi\rangle$ “free of charge,” or without needing any further entanglement to facilitate the transformation. The physical operations implementing such transformations are the celebrated class of local operations
with classical communication (LOCC), which, as described by its name, consists of all operations in which each multiparticle subsystem is manipulated locally but perhaps in a manner globally correlated through classical communication. Thus, LOCC has become a major framework for quantifying and classifying entanglement (see, e.g., the recent surveys on quantum entanglement by Horodecki et al. and Gühne and Tóth).

Following the notation of Bennett et al., we write \( |\psi\rangle \preceq_{\text{LOCC}} |\phi\rangle \) if \( |\phi\rangle \) can be converted to \( |\psi\rangle \) through LOCC. Natural questions immediately arise such as: when can a state be converted to another, what is the largest ratio of conversion for multiple copies of one state versus multiple copies of another, and when is there a state maximum in the sense that all other states in the system can be obtained from this state? In two-party systems, many questions of this sort can be answered. For example, a remarkable theorem by Nielsen states that for bipartite states, \( |\psi\rangle \preceq_{\text{LOCC}} |\phi\rangle \) if and only if the spectrum of \( |\phi\rangle \)'s reduced density operator (i.e., the Schmidt numbers) is majorized by that of \( |\psi\rangle \). While for multipartite states the LOCC partial ordering still remains unknown, Bennett et al. made the important observation that if any two states are equivalent under LOCC, they are related by a local unitary (LU) transformation. Thus, LOCC equivalence classes are simply the orbits of local unitary operations in multipartite systems. However, such a partitioning is too fine for most interests: even in the two-qubit case, there exists an infinite number of LU equivalence classes.

If the required success probability of both the forward and reverse transformations is reduced to be simply nonzero, a much coarser partitioning is achieved. General LOCC transformations occurring with a nonzero probability are called stochastic (SLOCC) and denoted by \( |\psi\rangle \preceq_{\text{SLOCC}} |\phi\rangle \) if the transformation is from \( |\phi\rangle \) to \( |\psi\rangle \). It turns out that SLOCC equivalence classes are precisely the orbits under local invertible linear transformations. Similar to the situation with LOCC, bipartite entanglement is well understood under SLOCC. Indeed, for bipartite pure states, \( |\psi\rangle \preceq_{\text{SLOCC}} |\phi\rangle \) if and only if the rank of the reduced density operator of \( |\psi\rangle \) (i.e., the Schmidt rank) is no larger than that of \( |\phi\rangle \). Thus, two states are SLOCC equivalent if and only if they have the same Schmidt rank. The optimal success probability can also be computed easily from the Schmidt numbers.

In contrast, entanglement among three or more parties behaves fundamentally different from bipartite entanglement. For example, while there is a maximum SLOCC equivalence class for bipartite systems of any dimension, there exists two maximal equivalence classes for the simplest tripartite system of three qubits. In contrast to Nielsen’s theorem and the rank criterion for bipartite SLOCC conversion, deciding SLOCC convertibility, in general, encodes many difficult computational problems. For the general tripartite conversions, the problem is NP-hard (observed in Ref. 10 using a NP-hardness result by Håstad on computing tensor rank). For certain tripartite asymptotic conversions, the optimal conversion ratio is exactly the exponent of matrix multiplication. For converting a multipartite state to a bipartite state, it is equivalent to the important problem of polynomial identity testing. In view of these results, a simple criterion or an efficient algorithm for checking SLOCC convertibility or equivalence seems obtainable only for systems of restricted dimensions.

This article studies the SLOCC equivalence classes of tripartite pure states in systems of dimensions \( 2 \otimes m \otimes n \). Dür et al. presented the first major result in the study of multipartite SLOCC equivalence classes by showing there to be six different classes in \( 2 \otimes 2 \otimes 2 \) systems. Their work was extended to four qubit systems by Verstraete and co-workers in which already an infinite number of equivalence classes exist. For an arbitrary number of subsystems, Miyake has shown how multidimensional determinate theory can be used to obtain general properties and results concerning SLOCC equivalence. Specific to tripartite \( 2 \otimes 2 \otimes n \) systems, Miyake and Verstraete have also completely characterized the equivalence class hierarchy and found that for \( n \geq 4 \) exactly nine different classes exist. Using the method of successive Schmidt decompositions, Cornelio and Piza obtained partial results concerning the equivalence classes in \( 2 \otimes m \otimes n \) systems. Chen et al. completed the finite orbit picture by enumerating all 26 equivalence classes in \( 2 \otimes 3 \otimes 6 \) systems, and showed that for the \( 3 \otimes 3 \otimes 3 \) and \( 2 \otimes 4 \otimes 4 \) systems (and all systems of higher dimensions), there are an infinite number of SLOCC equivalence classes. They used a
technique called “the range criterion,”\textsuperscript{17} which states that two states are SLOCC equivalent if and only if the ranks of the reduced density operators are identical and their supports are related by local invertible linear operations. While these results are quite interesting, the tools used to obtain them appear rather \textit{ad hoc} and neither the range criterion nor any previous technique provide an efficient algorithm (or any algorithm at all) for determining SLOCC equivalence. The noninvertible hierarchy among the 26 classes has also remained an open problem. In another work, Cheng \textit{et al.}\textsuperscript{18} tackled the restricted problem of $2 \otimes n \otimes n$ equivalence by an approach that most closely resembles the one used in this article. However, the authors err in their analysis and we correct their oversight here while encompassing the style of their analysis in a much broader framework.

The main insight of this article is that the theory of matrix pencils is the perfect tool for analyzing SLOCC equivalence in $2 \otimes m \otimes n$ systems. For two matrices $R, S \in \mathbb{C}^{m \times n}$, the linear matrix polynomial $\lambda R + \mu S$ is called a matrix pencil. A fundamental result is the existence of a canonical form, discovered by Kronecker (see, e.g., Gantmacher\textsuperscript{19}). The theory of matrix pencils remains an important subject of study for its applications in control and system theory. An example is the computation of the generalized eigenvalues.\textsuperscript{2} The efficient computation of the Kronecker canonical forms, other canonical forms, and related problems is still an active field of research, e.g., see Ref. 20 and following articles. We also note that the underlying object of our study, the representations of $GL_2(\mathbb{C}) \times GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$, has been investigated in the mathematical literature with an emphasis on the geometry of equivalent matrix pencils (see Ref. 21 and references within).

The connection with our problem is that each quantum pure state in a $2 \otimes m \otimes n$ space can be represented as a matrix pencil (see Sec. III, for details). In short, just as any $m \otimes n$ bipartite pure state corresponds to a matrix or bilinear form $R : \mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}; (x, y) \mapsto x^T R y$, every $2 \otimes m \otimes n$ tripartite pure state corresponds to a matrix pencil or trilinear form $\mu R + \lambda S : \mathbb{C}^2 \times \mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}; (a, b, x, y) \mapsto x^T (a R + b S) y$. The local operations on the second and third subsystems map the corresponding pencil to an equivalent one. While actions on the first subsystem may bring the pencil to an inequivalent one, we show that if two states are SLOCC equivalent, the operation on the first subsystem can be selected from a small number of choices. As a consequence, we derive the first efficient algorithm for determining SLOCC equivalence in general $2 \otimes m \otimes n$ quantum systems. We present a canonical form for $2 \otimes m \otimes n$ pure states and an algorithm to calculate it such that two states are equivalent if and only if they have the same canonical form. For the systems having a finite number of equivalence classes, we rederive the previously known equivalence orbits and then represent them using our defined canonical forms. We also determine all possible noninvertible transformations among these equivalence classes.

The rest of this article begins with a brief introduction to some main results in matrix pencil theory. In Sec. III we develop the relationship between tripartite pure states and matrix pencils which then allows us to derive necessary and sufficient conditions for the SLOCC convertibility of $2 \otimes m \otimes n$ states. Section IV presents a polynomial-time algorithm for deciding when two states are SLOCC equivalent and establishes a unique canonical form to which every $2 \otimes m \otimes n$ pure state is SLOCC equivalent. Sections V and VI discuss the SLOCC hierarchy of all tripartite systems possessing a finite number of SLOCC orbits and its explicit presentation is included in the Appendix. The article closes with some brief concluding remarks.

II. MATRIX PENCILS

The theory of matrix pencils was first developed by Kronecker over a century ago. A completely thorough treatment of the subject can be found in Gantmacher’s two volume texts\textsuperscript{19} from which we will here only cite the main definitions and results. For a more modern treatment, see Ref. 22. Given two complex $m \times n$ matrices $R$ and $S$, we form the homogeneous matrix polynomial $P_{(R, S)} = \mu R + \lambda S$ in variables $\mu$ and $\lambda$. For concreteness, we will use the pencil
as an example. Two pencils \( \mathcal{P}(R,S) \) and \( \mathcal{P}(R',S') \) are strictly equivalent if there exists invertible matrices \( B \) and \( C \) independent of \( \mu \) and \( \lambda \), such that \( \mu R' + \lambda S' = B(\mu R + \lambda S)C^T \). It immediately follows that \( \mathcal{P}(R,S) \) and \( \mathcal{P}(R',S') \) are strictly equivalent if and only if there exists invertible \( B \) and \( C \), such that \( BRC^T = R' \) and \( BSC^T = S' \).

The rank of \( \mathcal{P}(R,S) \) is the largest \( r \), such that there exists an \( r \)-minor of \( \mathcal{P}(R,S) \) not identically zero (not equaling zero upon every complex substitution for \( \mu \) and \( \lambda \)). For \( i = r \) we let \( D_i(\mu, \lambda) \) denote the highest degree polynomial, monic with respect to \( \lambda \), that divides each of \( \mathcal{P}(R,S)_i \)'s \( i \)-minors. It can be seen that \( D_i \) is rank 3 with all 3-minors being divisible by \( \mu(\mu + \lambda) \).

The invariant polynomials of pencil \( \mathcal{P}(R,S) \) are the homogeneous polynomials \( E_i(\mu, \lambda) = D_i(\mu, \lambda)/D_i-1(\mu, \lambda) \) for \( i = 1 \ldots, r \), where \( D_i(\mu, \lambda) = 1 \). There will be a unique factorization of \( D_i(\mu, \lambda) \) as \( D_i(\mu, \lambda) = \mu^{-r}p_1 \ldots p_k \), where \( p_i \) is of the form \( \mu x_i + \lambda \) for \( x_i \in \mathbb{C} \), and the invariant polynomials will likewise have a factorization in terms of these \( p_i \) and powers of \( \mu \). In fact, we can identify each distinct \( p_i \) (suppose there are \( f \) of them), with its \( \mu \)-coefficient \( x_i \), and the factor \( \mu \) with the element \( x_{f+1} = +\infty \) so that the degree 1 divisors of all the invariant polynomials are specified by elements from \( \mathbb{C}^* = \mathbb{C} \cup \{+\infty\} \). These distinct \( x_i \) values from \( \mathbb{C}^* \) generated by the factorization of \( D_i(\mu, \lambda) \) are called the eigenvalues of \( \mathcal{P}(R,S) \). To each eigenvalue \( \lambda \), there is a corresponding eigenvalue size signature which is a sequence of integers \( X' = (n'_1, \ldots, n'_f) \), such that \( n'_i \) is the highest power of \( \mu x_i + \lambda \) that divides invariant polynomial \( E_i(\mu, \lambda) \) if \( x_i \neq +\infty \), and \( n'_i \) is the highest power of \( \mu \) that divides \( E_i(\mu, \lambda) \) if \( x_i = +\infty \). The elementary divisors of the pencil are then the multiset \( \{(\mu x_i + \lambda)|_{x_i=+\infty}|_{1 \leq i \leq f}\} \cup \{(\mu x_i + \lambda)|_{x_i=-\infty}|_{1 \leq i \leq f}\} \). In our example pencil \( \mathcal{P}_0 \), \( E_0(\mu, \lambda) = \mu \), \( E_1(\mu, \lambda) = \mu + \lambda \), and \( E_1(\mu, \lambda) = 1 \). Thus, \( \mathcal{P}_0 \) has eigenvalues \( \{1, +\infty\} \) with respective size signatures \((0, 1, 0)\) and \((0, 0, 1)\).

While each pencil has a unique set of eigenvalues \( \{x_1, \ldots, x_f\} \) and corresponding size signatures \( \{X_1, \ldots, X_f\} \), it will be desirable to have some canonical way of uniquely specifying these values. To do this, we first fix some total ordering for sequences of integers and some total ordering for elements of \( \mathbb{C}^* \). Next, eigenvalues \( \{x_1, \ldots, x_f\} \) are ordered nondecreasingly according to their size signatures, and finally for eigenvalues having the same size signatures, they are ordered nondecreasingly according to their position in \( \mathbb{C}^* \). Every pencil \( \mathcal{P}(R,S) \) then has a unique canonical sequence of eigenvalues \( \hat{x} = (x_1, \ldots, x_f) \) with their size signatures forming a nondecreasing sequence \( \hat{X} = (X_1, \ldots, X_f) \).

For pencil \( \mathcal{P}(R,S) \), consider the set of polynomial vectors \( x_i(\mu, \lambda) \), such that \( (\mu R + \lambda S)x_i(\mu, \lambda) = 0 \). This set is a finitely generated module over the ring of polynomials, and any homogeneous basis will have elements of the form

\[
x_i(\mu, \lambda) = \sum_{j=0}^{e_i} x_{ij}\mu^{e_i-j}\lambda^j,
\]

with degrees \( e_1 \leq \ldots \leq e_f \). An important property of any \( x_i(\mu, \lambda) \) belonging to such a basis is that the \( x_{ij} \) are linearly independent. Likewise, the set of polynomial vectors satisfying \( \mathcal{P}(R,S)x_i(\mu, \lambda) = 0 \) will have a homogeneous basis whose elements have degrees \( \nu_1 \leq \ldots \leq \nu_p \). The values \( e_1, \ldots, e_f \) and \( \nu_1, \ldots, \nu_p \) are called the minimal indices of \( \mathcal{P}(R,S) \). In particular, the number of \( e_i \) that are zero will be called the zero index number, and the number of \( \nu_i \) that are zero will be called the transpose zero index number. For the pencil \( \mathcal{P}_0 \), it can be readily verified that \( \mathcal{P}_0[1,0,1,0]^T(\mu + [0, -1, 0, 1]^T) = 0 \), and \( \mathcal{P}_0 \) has a single minimal indices of \( e_1 = 1 \). With this overview, we can now state the main theorem characterizing strictly equivalent pencils.

Lemma 1: (Kronecker's) Two matrix pencils are strictly equivalent if and only if they have the same rank, elementary divisors, and minimal indices. Moreover, suppose that \( \mathcal{P}(R,S) \) has a canonical sequence of eigenvalues \( \hat{x} = (x_1, \ldots, x_f) \) with size signatures \( \hat{X} = (X_1, \ldots, X_f) \), minimal indices
$\varepsilon_1 \leq \ldots \leq \varepsilon_p$ and $\nu_1 \leq \ldots \leq \nu_q$, a zero index number of $g$, and a transpose zero index number of $h$. Then, $P_{(R,S)}$ is strictly equivalent to the canonical block-form diagonal pencil,

$$\{F^0, L_{\varepsilon_1}, \ldots, L_{\varepsilon_p}, L_{(h_1)}^T, \ldots, L_{(h_q)}^T, J\}$$

(1)

where $F^0$ is the $h \times g$ zero matrix,

$$L_\varepsilon = \begin{pmatrix} \lambda & \mu & 0 & \ldots & 0 \\ 0 & \lambda & \mu & \ldots & 0 \\ \ldots \\ 0 & 0 & \ldots & \lambda & \mu \end{pmatrix},$$

$J = \odot_{i=1}^p M^i$, where

$$M^i = \begin{cases} \oplus_{j=1}^p [((\mu \alpha_j + \lambda)I_j^i + \mu H^i_j)] & \text{if } x_i \neq \pm \infty \\ \oplus_{j=1}^p [\mu I_j^i + \lambda H^i_j] & \text{if } x_i = \pm \infty, \end{cases}$$

$I'$ being the $t \times t$ identity matrix and $H'$ the $t \times t$ matrix whose only nonzero elements are ones on the superdiagonal.

Note that by our ordering of the eigenvalues and minimal indices, every pencil is equivalent to one and only one Kronecker canonical form (KCF) as defined above. Our example pencil $P_0$ has the KCF of

$$\begin{pmatrix} \lambda & \mu & 0 & 0 \\ 0 & 0 & \mu + \lambda & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}.$$

III. CONNECTION TO 2 $\otimes$ m $\otimes$ n PURE STATES

Any $2 \otimes m \otimes n$ state can be expressed in bra-ket form as $|\psi\rangle = |0\rangle_A (R_{BC} | + |1\rangle_A |S_{BC}\rangle$. By choosing local bases $\{|i\rangle_B\}_{i=0,\ldots,m-1}$ and $\{|i\rangle_C\}_{i=0,\ldots,n-1}$ for Bob and Charlie, respectively, we can express the state as

$$|\psi\rangle = (|0\rangle_A (R + 1)| + |1\rangle_A (S \otimes 1)) |\Phi_m\rangle = (|0\rangle_A (1 \otimes R^T) + |1\rangle_A (1 \otimes S^T)) |\Phi_m\rangle,$$

(2)

where $R_{ij} = \alpha_{ij}$, $S_{ij} = \beta_{ij}$, and $|\Phi_m\rangle = \sum_{i=0}^{m-1} |i\rangle_B |i\rangle_C$. Thus, there is a one-to-one correspondence between a $2 \otimes m \otimes n$ pure state $|\psi\rangle = |0\rangle_A |R_{BC}\rangle + |1\rangle_A |S_{BC}\rangle$ and the pair of matrices $(R,S)$, so that to every $|\psi\rangle$ and choice of indeterminates $\mu, \lambda$, we can uniquely associate the pencil $P_{(R,S)}$ which we shall equivalently denote as $P_{\psi}$.

There exists a nice relationship between the structure of $P_{\psi}$ and the local ranks of each subsystem. The reduced states of Bob and Charlie are obtained by performing a partial trace on the matrix $|\psi\rangle\rangle$. From above, then, it follows that

$$\rho_B = tr_{AC}(|\psi\rangle\rangle) = RR^T + SS^T,$$

$$\rho_C = tr_{AB}(|\psi\rangle\rangle) = R^T R + S^T S.$$

(3)

Here, “$T$” denotes the matrix transpose with respect to the basis $|i\rangle_{BC}\rangle$ and “$\cdot$” the complex conjugate of its entries. Also note that since Alice has a two dimensional system, her subsystem will either have full rank or $|\psi\rangle$ is a product state of the form $|\psi\rangle = |0\rangle_A |\phi\rangle_{BC}$. In this case, we say Alice is separated from Bob and Charlie. Combining these facts, we can prove the following.
Lemma 2: (i) Bob and Charlie share pure entanglement (Alice separated) if and only if $\mathcal{P}_d$ can be expressed as a matrix polynomial in one indeterminate $\lambda$: i.e.,

$$\mu R + \lambda S = \hat{\lambda}\hat{S},$$

and (ii) Bob and Charlie’s local ranks are $m-h$ and $n-g$, respectively, where $g$ is the zero index number of $\mathcal{P}_d$ and $h$ its transpose zero index number.

Proof: (i) Alice is unentangled if and only if up to an overall phase, the state can be written as $|0\rangle(|R\rangle+|S\rangle)+|\alpha\rangle(|R\rangle+|S\rangle)$ which happens if and only if its associated pencil is $(\mu+\lambda\alpha)R+(\mu+\lambda\alpha)S=\lambda\hat{S}$. (ii) By definition, the zero index number is the number of linearly independent constant vectors $v_j$, such that $R[v_j]=S[v_j]=0$. In this case, we must also have $R[\hat{v}_j]=S[\hat{v}_j]=0$. It follows from (3) that $\rho_c[\hat{v}_j]=0$ if and only if $R[\hat{v}_j]=S[\hat{v}_j]=0$, and since complex conjugation does not affect linear dependence, we have $\text{rank}(\rho_c)=n-g$. An analogous argument shows that $\text{rank}(\rho_B)=m-h$.

We now want to observe the effect of local invertible operators implemented by Alice, Bob, and Charlie; i.e., a SLOCC transformation. Any such operation can be decomposed as $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$, where Bob and Charlie first act, and then Alice follows alone. When Bob and Charlie perform the invertible operator $\mathcal{B} \otimes \mathcal{C}$, it is easy to check that the transformation $|R\rangle_{BC} \rightarrow \mathcal{B} \otimes \mathcal{C}|R\rangle_{BC}$ corresponds to $R \rightarrow BRBC^T$ and likewise for $S$. Thus, the action of Bob and Charlie initiates the matrix pencil transformation $\mu R + \lambda S \rightarrow (\mu R + \lambda S)C$. In other words, local invertible operators of Bob and Charlie map matrix pencils to strictly equivalent ones.

Any invertible operation by Alice can be represented by a matrix $(a_{ij})$ with $ad-bc \neq 0$. Then the most general action by Alice will transform the state $|\psi\rangle$ as

$$|0\rangle_A|R\rangle_{BC} + |1\rangle_A|S\rangle_{BC} \rightarrow |0\rangle_A(a|R\rangle_{BC} + c|S\rangle_{BC}) + |1\rangle_A(b|R\rangle_{BC} + d|S\rangle_{BC}).$$

Hence, the corresponding pencil transformation is $\mu R + \lambda S \rightarrow (\mu a + \lambda b)R + (\mu c + \lambda d)S = \hat{\mu}R + \hat{\lambda}S$,

where $\hat{\mu} = \mu a + \lambda b$ and $\hat{\lambda} = \mu c + \lambda d$.

While obviously such a coordinate change does not alter the rank, what concerns us is how the transformation $(\mu, \lambda) \rightarrow (\hat{\mu}, \hat{\lambda})$ affects the eigenvalues and minimal indices of a given pencil. For the latter, care must be taken since minimal indices are defined by the degree of polynomials in variables $\mu$ and $\lambda$. Nevertheless, the following lemma shows minimal indices to be a SLOCC invariant in $2 \otimes m \otimes n$ systems.

Lemma 3: The minimal indices of a given pencil remain invariant under the action of Alice.

Proof: Let $r$ denote the rank of the given pencil. Under an invertible transformation $(\mu, \lambda) \rightarrow (\hat{\mu}, \hat{\lambda}) = (a\mu + b\lambda, c\mu + d\lambda)$, a polynomial $r$-component vector $p(\mu, \lambda) = \sum_{\ell=0}^{r-1}\sum_{k=0}^{r-1}x_{\ell k}\mu^\ell\lambda^k$ is identically zero if and only if $p(\hat{\mu}, \hat{\lambda}) = 0$. To see this, we can introduce the standard basis $\{e_i\}_{i=1,..,r}$

and consider $p(\mu, \lambda)$ as an $rmn$-component vector in the space spanned by basis $\mu^\ell\lambda^k e_i$. Then the transformation $(\mu, \lambda) \rightarrow (\hat{\mu}, \hat{\lambda})$ induces a homomorphism on this space which thus cannot map any nonzero vector to zero. Consequently, for any set of polynomial vectors $\{x_i(\mu, \lambda)\}_{i=1,..,n}$ (a) $(\mu R + \lambda S)x_i(\mu, \lambda) = 0$ if and only if $(\hat{\mu} R + \hat{\lambda} S)x_i(\hat{\mu}, \hat{\lambda}) = 0$, and (b) $\{x_i(\mu, \lambda)\}_{i=1,..,n}$ is linearly independent if and only if $\{x_i(\hat{\mu}, \hat{\lambda})\}_{i=1,..,n}$ is linearly independent, where linear independence means that for polynomials $p_i(\mu, \lambda) = \sum_{\ell=0}^{r-1}\sum_{k=0}^{r-1}x_{\ell k}\mu^\ell\lambda^k = 0 \Rightarrow p_i(\mu, \lambda) = 0$ for all $i$. Next, we claim that (c) for any set of linearly independent scalar vectors $\{x_{ij}\}_{j=0,..,n}$ with $x_{ie} \neq 0$, the highest degree of $\lambda$ having a nonzero vector coefficient in $x_i(\mu, \lambda) = \sum_{j=0}^{r-1}x_{ij}\lambda^j$ is the same as that in $x_i(\hat{\mu}, \hat{\lambda}) = \sum_{j=0}^{r-1}x_{ij}\hat{\lambda}^j$. This follows because the coefficient of $\lambda^e$ in $\sum_{j=0}^{r-1}x_{ij}\lambda^j = \sum_{j=0}^{r-1}x_{ij}\hat{\lambda}^j$ is $\sum_{j=0}^{r-1}x_{ij}\lambda^j$, which is nonvanishing due to the linear independence of $\{x_{ij}\}_{j=0,..,n}$.

From (a), (c), and the linear independence of $\{x_{ij}\}_{j=0,..,n}$ noted in the introductory discussion for any fundamental set of vectors, $x_i(\mu, \lambda)$ is a minimum degree polynomial in the null space of $\mu R + \lambda S$ if and only if $x_i(\hat{\mu}, \hat{\lambda})$ is a minimum degree polynomial in the null space of $\hat{\mu} R + \hat{\lambda} S$. Now
suppose that \( \{x_i(\mu, \lambda)\}_{i=1..n} \) are the first \( n \) vectors in a fundamental set for \( \mu R + \lambda S \) if and only if \( \{x_i(\hat{\mu}, \hat{\lambda})\}_{i=1..n} \) are the first \( n \) vectors in a fundamental set for \( \hat{\mu} R + \hat{\lambda} S \). Then by (c), \( \mu R + \lambda S \) and \( \hat{\mu} R + \hat{\lambda} S \) will have the same first \( n \) minimal indices. From (a)–(c) again, \( x_{n+1}(\hat{\mu}, \hat{\lambda}) \) will be the next vector in the same fundamental set for \( \mu R + \lambda S \) if and only if \( x_{n+1}(\hat{\mu}, \hat{\lambda}) \) is likewise for \( \hat{\mu} R + \hat{\lambda} S \). Hence by induction and by running the exact same argument on \( \mu R + \lambda S \), the lemma is proven. 

As for the elementary divisors, Alice’s transformation does have an effect. By direct substitution, it follows immediately that after normalization, the divisors transform as:

\[
\mu^n \rightarrow \left( \frac{a}{b} + \lambda \right)^n \quad \text{if} \quad b \neq 0 \\
\mu^n \quad \text{if} \quad b = 0,
\]

and

\[
(\mu x_i + \lambda)^n \rightarrow \left( \frac{a x_i + c}{b x_i + d} + \lambda \right)^n \quad \text{if} \quad b x_i + d \neq 0 \\
\mu^n \quad \text{if} \quad b x_i + d = 0.
\]

We see that eigenvalues transform as

\[
x_i \rightarrow \begin{cases} 
  \frac{a x_i + c}{b x_i + d} & \text{if } x_i \neq -d/b, + \infty \\
  + \infty & \text{if } x_i = + \infty \\
  + \infty & \text{if } x_i = -d/b \\
  a/b & \text{if } x_i = + \infty,
\end{cases}
\]

which exactly defines a linear fractional transformation on the extended complex line \( \mathbb{C}^* \). As a result, we have the following.

**Theorem 1:** Two \( 2 \otimes m \otimes n \) states \( |\psi\rangle \) and \( |\phi\rangle \) are SLOCC equivalent if and only if their corresponding pencils are of the same rank, have the same minimal indices, and there exists a linear fractional transformation (LFT) relating the eigenvalues \( x_i \) of \( P_\psi \) to the eigenvalues \( y_i \) of \( P_\phi \) such that \( x_i \) and \( y_i \) have the same size signatures; i.e., for all \( i \),

\[
\frac{a x_i + c}{b x_i + d} = y_i (ad - bc \neq 0),
\]

with eigenvalue size signatures remaining invariant.

A nice property of LFTs is the following which we will rely on heavily.

**Proposition 1:** Given any two trios \( \{x_1, x_2, x_3\} \) and \( \{y_1, y_2, y_3\} \) each with distinct values, there always exists a unique LFT relating the sets.\(^{23}\) The form of the transformation is given by the determinants,

\[
a = \begin{vmatrix} x_1 y_1 & 1 \\ x_2 y_2 & 2 \\ x_3 y_3 & 3 \end{vmatrix}, \quad b = \begin{vmatrix} x_1 y_1 & x_1 & 1 \\ x_2 y_2 & x_2 & 2 \\ x_3 y_3 & x_3 & 3 \end{vmatrix}, \quad c = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 3 \end{vmatrix}, \quad \text{and} \quad d = \begin{vmatrix} x_1 y_1 & x_1 & 1 \\ x_2 y_2 & x_2 & 2 \\ x_3 y_3 & x_3 & 3 \end{vmatrix}.
\]

A corollary of the uniqueness in a LFT relating three distinct numbers is the infinitude of SLOCC equivalence classes among states with four distinct elementary divisors. Indeed, for each \( x_i \), there exist infinite values for \( x_2 \), such that states having eigenvalues \( (0, 1, 2, x_2) \) and \( (0, 1, 2, x_3) \) are inequivalent; this follows because there are only a finite number of LFTs that map the first three numbers onto themselves. As a result, we can always find an infinite set \( \{x_1, x_2, \ldots\} \), such that states with eigenvalues \( (0, 1, 1, x_2) \) and \( (0, 1, 2, x_2) \) are inequivalent for each \( x_i \). Since states of \( 2 \otimes m \otimes n \) systems can have at least four distinct elementary divisors if and only if \( \min\{m, n\} \geq 4 \), this leads to the following.

**Corollary 1:** Systems of dimension \( 2 \otimes m \otimes n \) have an infinite number of SLOCC equivalence classes if and only if \( \min\{m, n\} \geq 4 \).
By slightly generalizing our current analysis, we can strengthen the previous result and prove that for any $p \otimes m \otimes n$ quantum system, a finite number of orbits requires at least one of the subsystems to be a qubit. To accomplish this, it is sufficient to study the dimensions $3 \otimes 3 \otimes 3$ in which a third indeterminate can be introduced to represent any state by the pencil 

The following lemma proves the infinitude of SLOCC equivalence classes in such systems by using a family of states introduced in Ref. 17 where an alternative argument is given for the following result.

**Lemma 4:** Consider the family of states,

$$|\psi_t\rangle = |0\rangle(|00\rangle + |12\rangle) + |1\rangle(|11\rangle + |02\rangle) + |2\rangle(|22\rangle + |10\rangle).$$

For every distinct $t, s \in C$, $|\psi_t\rangle$ and $|\psi_s\rangle$ are SLOCC inequivalent.

**Proof:** Suppose the contrary and that $A \otimes B \otimes C |\psi_s\rangle = |\psi_t\rangle$ for some invertible operations. The corresponding matrix pencil of $|\psi_t\rangle$ is

$$\mathcal{P}_\phi(\mu, \lambda, \omega) = \begin{pmatrix} \mu & \lambda \\ \omega & \lambda \\ \mu & t\omega \end{pmatrix}. \quad (9)$$

Alice’s operation induces the mapping $x=(\mu, \lambda, \omega)^T \rightarrow Ax=(\mu, \lambda, \omega)^T$, and after her action, the pencils $\mathcal{P}_\phi(\mu, \lambda, \omega)$ and $\mathcal{P}_\phi(\mu, \hat{\lambda}, \hat{\omega})$ are strictly equivalent. Thus, by invariance of the determinant, we must have

$$\hat{\mu} \hat{\lambda} \hat{\omega} = g \mu \lambda \omega, \quad (10)$$

where $g$ is some constant. Equation (10) can be true only if, up to an overall permutation, $x$ transforms as $(\hat{\mu}, \hat{\lambda}, \hat{\omega})^T = (a \mu, b \lambda, c \omega)$, where $abc = g \neq 0$. As a result, $B$ and $C$ satisfy

$$B \begin{pmatrix} a \mu & b \lambda \\ c \omega & b \lambda \\ a \mu & c \omega \end{pmatrix} = \begin{pmatrix} \mu & \lambda \\ \omega & \lambda \\ \mu & s \omega \end{pmatrix} C^{-1}. \quad (11)$$

Since the action of $B$ and $C^{-1}$ is to perform elementary row and column operations, respectively, Eq. (11) implies the existence of some constants $\{r_i, c_i\}_{i=1,3}$, such that

$$ar_1 = c_1, \quad ar_2 = c_2, \quad br_1 = c_3$$

$$cr_2 = c_1, \quad br_2 = c_2, \quad tcr_3 = sc_3.$$ 

However, these equations can all be simultaneously satisfied only if $t=s$. \hfill \blacksquare

**IV. EQUIVALENCE ALGORITHM AND THE STATE KROECKER CANONICAL FORM**

**A. Equivalence algorithm**

Using Theorem 1, we can construct an algorithm for determining whether two general $2 \otimes m \otimes n$ pure states $|\psi\rangle$ and $|\phi\rangle$ are SLOCC equivalent.

(I) Input pencils $\mathcal{P}_\psi$ and $\mathcal{P}_\phi$ determine their rank, minimal indices, eigenvalues, and corresponding eigenvalue size signatures. If the rank, minimal indices, and set of eigenvalue size signatures are not the same, $|\psi\rangle$ and $|\phi\rangle$ are inequivalent. If they are and $\mathcal{P}_\phi$ has less than three eigenvalues, $|\psi\rangle$ and $|\phi\rangle$ are equivalent. If $\mathcal{P}_\phi$ has three or more eigenvalues, proceed to the next step.

(II) Fix any three eigenvalues $x_i$ of $\mathcal{P}_\psi$. Choose any sequence of three eigenvalues $y_j$ belonging to $\mathcal{P}_\phi$ having the same size signatures as the $x_i$. Determine the LFT relating $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ according to (8). Choose a new $x_i$ and determine if the LFT relates it to any...
remaining $y_i$ with the same size signature. By uniqueness of the LFT, if there is no such $y_i$, the states are not equivalent. If there is, choose another $x_i$ and repeat the search on the remaining $y_i$.

(III) If a perfect matching exists for all $x_i$ and $y_i$, then the states are equivalent. If not, repeat step (II) by choosing another ordered trio of the $y_i$. If no LFT exists for all possible trios, the states are not equivalent.

The Kronecker canonical form of an $m \times n$ pencil can be computed in time $O(m^2n)$ (see the algorithm by Beelen and Van Dooren[26]). For sets of $t$ elementary divisors, step (II) of this algorithm will require at most $O(t^3)$ steps. Thus, the total running time is $O(m^2n + \min\{m,n\}^3)$. Furthermore, the algorithm is constructive in nature because if two states are SLOCC equivalent, we determine the specific $a,b,c,d$ constituting Alice’s operator in the transformation $|\psi\rangle \rightarrow |\phi\rangle$. The operators Bob and Charlie are to perform can be determined from the invertible matrices that bring pencils $P_{\psi}$ and $P_{\phi}$ to their canonical forms of (1) and are so-obtained by a Gaussian elimination procedure. Hence, not only does our algorithm determine whether two states are equivalent but it also provides the necessary operators achieving the transformations.

B. The state Kronecker canonical form

We now wish to define a unique canonical form, which we will call the state Kronecker canonical form (SKCF), to which every state in dimensions $2 \otimes m \otimes n$ is SLOCC equivalent. To compute the SKCF of a state $|\psi\rangle$, denoted as $F_{\psi}$, we first compute the KCF of pencil $P_{\psi}$, which according to Lemma 1 is specified completely by its minimal indices and its canonical sequence of eigenvalues $\hat{x}$ with corresponding size signature sequence $\hat{X}$. Under a SLOCC transformation, both the minimal indices and the sequence $\hat{X}$ remain invariant. Thus, we just need some canonical way to choose the eigenvalues with respect to a LFT; the following algorithm suffices.

- Input canonical eigenvalue sequence $\hat{x} = (x_1, \ldots, x_f)$ of pencil $P_{\psi}$. If $f < 3$, then let $\theta_0$ be the LFT that maps $(x_1, \ldots, x_f)$ to the first $f$ elements of $(0, 1, \infty)$. Apply $\theta_0$ to $P_{\psi}$ and output $F_{\psi}$ as the KCF of this resultant pencil.
- If $f \geq 3$, let $B_i$ be the set of all eigenvalues with the same size signature as $x_i$. Let

$$T_{\psi} = \{(\theta(x_1), \ldots, \theta(x_f)) : \theta \text{ is an LFT mapping distinct elements } (x_i, x_j, x_k)$$

$$\rightarrow (0, 1, \infty) \text{ for } x_i \in B_1, x_j \in B_2, x_k \in B_3\}.$$ 

Note that this set must be of finite size. With respect to some fixed total ordering of sequences of complex numbers, let $\theta_0$ be the LFT that realizes the minimal element in $T_{\psi}$. Apply $\theta_0$ to $P_{\psi}$ and output $F_{\psi}$ as the KCF of this resultant pencil.

**Theorem 2:** States $|\psi\rangle$ and $|\phi\rangle$ are SLOCC equivalent if and only if they have the same state Kronecker canonical form $F_{\psi}$ and $F_{\phi}$.

**Proof:** If the states have the same SKCF $F$, then for both $P_{\psi}$ and $P_{\phi}$ there is a LFT that maps each to some form strictly equivalent to $F$. Since the set of LFTs is closed under composition and inversion, there exists a SLOCC transformation relating $|\psi\rangle$ and $|\phi\rangle$.

Conversely, suppose $|\psi\rangle$ and $|\phi\rangle$ are SLOCC equivalent. Let $P_{\psi}$ and $P_{\phi}$ have respective canonical sequences of eigenvalues $(x_1, \ldots, x_f)$ and $(y_1, \ldots, y_f)$, and note that they will have the same corresponding size signature sequence $(X^1, \ldots, X^f)$. The additional invariance in minimal indices then implies that the states will have the same SKCF if $T_{\psi} = T_{\phi}$. Now, there is some invertible LFT $\theta$ so that $(y_1, \ldots, y_f) = (\theta(x_1), \ldots, \theta(x_f))$, and thus $x_i$ will have the same size signature as $x_j$ if and only if $\theta(x_i)$ has the same size signature as $y_j$. Then again since the set of LFTs form a group under composition, we have $(\theta(y_1), \ldots, \theta(y_f)) \in T_{\psi} \Rightarrow (\theta(\theta(x_1), \ldots, \theta(x_f)) \in T_{\phi}$ and likewise $(\theta(x_1), \ldots, \theta(x_f)) \in T_{\phi} \Rightarrow (\theta^{-1}(y_1), \ldots, \theta^{-1}(y_f)) \in T_{\psi}$. 


V. ALL TRIPARTITE SYSTEMS WITH A FINITE SLOCC EQUIVALENCE PARTITIONING

As a result of Corollary 1 and Lemma 4, for tripartite systems, a finite number of SLOCC equivalence classes exists only in systems of low dimensions. To count and characterize all the orbits, we just need to find what combination of minimal indices and elementary divisors fit in an $m \times n$ matrix of form (1). A few simplifications assist in this process. First, since any $m \times n$ pencil is simply the matrix transpose of an $n \times m$ one, it is enough to just consider $m \leq n$. Next, for a given dimension, we must only study the equivalence classes with Bob and Charlie having maximal local ranks since any rank deficient case will correspond to a class of maximum local ranks in a smaller dimension. To this end, Lemma 2 allows us to immediately determine the local ranks associated with each equivalence class. Furthermore, as evident from the Schmidt decomposition of any state with respect to bipartition AB:C, Charlie’s local rank cannot exceed the product of Alice and Bob’s. Consequently, if $n \geq 2m$, any state of a $2 \otimes m \otimes n$ system is the same as one in a $2 \otimes m \otimes 2m$ system up to a local change of basis on Charlie’s part. This means that for the task of finite enumeration, we only need to consider systems up to dimensions $2 \otimes 2 \otimes 4$ and $2 \otimes 3 \otimes 6$.

One further property of each equivalence class that we are able to study is the tensor rank. The tensor rank of a state is the minimum number of product states whose linear span contains the state, and this quantity turns out to be invariant under invertible SLOCC transformations. For tensor rank of a state is the minimum number of product states whose linear span contains the state, and this quantity turns out to be invariant under invertible SLOCC transformations. For bipartite systems, the tensor rank is equivalent to the Schmidt rank, and a nonincrease in Schmidt rank is also a sufficient condition for SLOCC convertibility between two such states; SLOCC equivalence classes are characterized completely by the Schmidt rank. Interestingly, in three qubit systems, tensor rank is also sufficient to distinguish between the various equivalence classes. However, we find that even for systems having a finite partitioning, the tensor rank is an insufficient measure for determining SLOCC equivalence. Our results follow from previous research on the tensor rank of matrix pencils done by Ja'Ja'24 and rederived in Ref.25.

**Lemma 5:** (References 24 and 25) Let $P_{(R,S)}$ be a pencil with no infinite divisors in canonical form (1) with minimal indices $\epsilon_1, \ldots, \epsilon_p$ and $\nu_1, \ldots, \nu_q$ and $J$ an $l \times l$-sized pencil. Furthermore, let $\delta(J)$ denote the number of invariant polynomials containing at least one nonlinear elementary divisor. Then the tensor rank of $P_{(R,S)}$ is given by

$$\sum_{i=1}^{p} (\epsilon_i + 1) + \sum_{j=1}^{q} (\nu_j + 1) + l + \delta(J).$$

In the Appendix, a summary of all the equivalence classes is presented (Table I) as well as the state Kronecker canonical forms representing each class. We see that there are 26 distinct SLOCC classes for $2 \otimes 3 \otimes n$ ($n \geq 6$) systems. This reproduces the findings of Chen et al.17 here obtained in an entirely different way by using matrix pencil analysis.

VI. NONINVERTIBLE TRANSFORMATIONS

A natural question is whether it is possible to transform from one class to another via noninvertible transformations. One obvious constraint is that states with full local ranks cannot preserve their ranks under a noninvertible transformation. Consequently, we cannot nonreversibly convert among inequivalent states with the same local ranks. A possible conjecture might be that unidirectional convertibility is achievable if none of the local ranks increase and at least one decreases; certainly three qubit systems satisfy this hypothesis. This, however, is false, in general, as we will now observe.

Let $|\phi\rangle$ be some state having maximal local ranks of $(2,m,n)$ and suppose $|\phi\rangle$ is a state with ranks $(2,m,n-1)$. If $|\phi\rangle \equiv_{\text{SLOCC}} |\phi\rangle$, Alice and Bob’s matrices inducing the transformation will be full rank while Charlie’s will have rank $n-1$. As for the latter, any such operator can be decomposed into a series of elementary column operations (permutations, column-multiplications, column-additions) on $P_{\phi}$ which results in exactly $n-1$ linearly independent columns. If column $i$
is the resultant linearly dependent column, then immediately after all column-additions of the \( i \)th column are performed, the remaining \( n-1 \) columns must correspond to some state SLOCC equivalent to \( |\phi\rangle \). As a result, we obtain the following criterion.

**Theorem 3:** Let \( |\psi\rangle \) and \( |\phi\rangle \) be states with local ranks \((2,m,n)\) and \((2,m,n-1)\), and let \( c_1, \ldots, c_n \) denote the columns of \( P_{\phi}(\mu,\lambda) \). Then \( |\phi\rangle \ll SLOCC \| |\psi\rangle \) if and only if for some \( 1 \leq i \leq n \), there exists constants \( a_1, \ldots, a_{i-1}, \, a_{i+1}, \ldots, a_n \) and some invertible linear transformation \((\mu,\lambda) \rightarrow (\hat{\mu},\hat{\lambda})\), such that the pencil \( P_{\phi}(\hat{\mu},\hat{\lambda})=c_1+a_1c_i, \ldots, c_n+a_nc_i \) is strictly equivalent to \( P_{\phi}(\mu,\lambda) \).

In general, for transformations in which Charlie’s rank decreases to \( n-k \), one need only modify this theorem by considering subpencils of \( P_{\phi} \) having \( n-k \) columns where to each of the columns is added a linear combination of the nonincluded columns. Likewise, to account for transformations when Bob’s local rank decreases, the above criterion can be applied with the analysis conducted on the rows of \( P_{\phi}(\mu,\lambda) \) instead of its columns.

On the surface, Theorem 3 has limited value since it involves a search for values \( a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \) over the complex numbers. However, in many cases, it is easy to see whether or not such a collection of numbers can be found. For example, for \( 1 \leq i \leq 4 \) in (ABC-19) (see the Appendix), upon any choice of the \( a_i \), and any transformation \( \hat{\mu},\hat{\lambda} \), the resultant pencil \( P_{\phi}(\hat{\mu},\hat{\lambda}) \) will either be rank 2 or it will have an elementary divisor of degree at least 1. However, the state (ABC-18) is rank 3 with no nontrivial elementary divisors. Thus, the transformation (ABC-19) \( \rightarrow \) (ABC-18) is impossible. On the other hand, for the state (ABC-17), when \( i=1 \), we have \( \det P_{\phi}(\mu,\lambda)=\lambda[\lambda^2-\mu(\frac{2}{3}\lambda+\frac{1}{2}a_1)] \) for \( a_3 \neq 0 \). The state (ABC-8) has \( \det P_{\phi}(\mu,\lambda)=\lambda(\mu+\lambda) \times (2\mu+\lambda) \). By choosing \( c_2=\frac{1}{2} \) and \( c_3=-\frac{1}{2} \), these polynomials become equal as well as the elementary divisors, the ranks, and the minimal indices of the pencils. Thus, (ABC-17) \( \rightarrow \) (ABC-8) is achievable by SLOCC.

In a manner similar to that just described, we have used Theorem 3 to analyze all possible transformations among the \( 2 \otimes 3 \otimes n \) equivalence classes. Figure 1 in the Appendix depicts the SLOCC hierarchy among the classes.

**VII. CONCLUSIONS AND FUTURE RESEARCH**

In this article, we have used the theory of matrix pencils to study \( 2 \otimes m \otimes n \) pure quantum states. In doing so, we were able to derive a polynomial-time algorithm for deciding SLOCC equivalence of such states. For all tripartite systems having a finite number of equivalence classes, we have obtained canonical state representatives and determined the partial ordering among these classes based on a criterion for general SLOCC convertibility in \( 2 \otimes m \otimes n \) systems. It is interesting to note that in the hierarchy chart of Table I, there exists certain transformations that are impossible even though the local rank of Charlie decreases by 2. The transformation (ABC-14) to (ABC-7) is such example.

A natural extension of this work is to find efficient algorithms for deciding LOCC equivalence, LOCC convertibility, and SLOCC convertibility in \( 2 \otimes m \otimes n \) systems. We have made progress on those questions. Another natural next line of inquiry might be to consider \( p \otimes m \otimes n \) systems and their corresponding degree \( p \) matrix polynomials. Indeed, much research has been conducted on higher degree elements, especially those having special properties such as being symmetric.23 Unfortunately, there exists no corresponding characterization such as Kronecker’s for strict equivalence of matrix pencils of degree greater than 2. Making the project of generalizing to higher degrees more dubious is the fact that determining SLOCC equivalence for \( p \otimes m \otimes n \) can be reduced from a tensor rank calculation on a set of \( p \) bilinear forms,20 and this problem has no known solution for \( p > 2 \) (the general problem is, in fact, NP-Hard).21,25

As noted in Sec. I, we are not the first to study SLOCC convertibility in multipartite systems, and it would be interesting to try and develop the relationship between our results and the work of others. For example, Miyake’s results involve “hyperdeterminants” and their singularities. It would be valuable to investigate the correspondence between matrix pencils and hyperdeterminants or to introduce the connection to the quantum information community if such a correspondence has already been obtained. In another work, Liang et al. have recently proven a set of
conditions both necessary and sufficient for the convertibility of two qubit mixed bell-diagonal states. As these mixed states can be considered pure with respect to a $2 \otimes 2 \otimes 4$ system, it would be fruitful to study transformations between tripartite “purified” bell diagonal states via our matrix pencil construction and compare it to the convertibility conditions in Ref. 26. Doing so might suggest ways in which purified tripartite pencils can assist in deciding equivalence between general $2 \otimes n$ mixed states.

ACKNOWLEDGMENTS

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APPENDIX

We also direct the reader to the work of Chen et al. for a different derivation of other representatives for the following classes.

1. $2 \otimes 2 \otimes 2$ systems

Here the states are represented as $2 \times 2$ pencils. We first consider the case with no minimal indices. Here there can only be two or one distinct elementary divisors with the latter having possible signatures of $(1,1)$ and $(0,2)$. In matrix and bra-ket form, these correspond to the unnormalized states,

$$\begin{pmatrix} \lambda & \mu + \lambda \\ \frac{\mu}{\lambda} & \mu \end{pmatrix} \text{ (ABC-1) “GHZ-class” } \begin{pmatrix} \lambda & \mu \\ \frac{\mu}{\lambda} & \mu \end{pmatrix} \text{ (A:BC-1) “W-class” }$$

$$(|0\rangle + |1\rangle) |11\rangle + |100\rangle$$

$|100\rangle + |111\rangle$$

$|001\rangle + |100\rangle + |111\rangle$.

By Lemma 5, the tensor rank of (ABC-1) is 2, and indeed by the invertible transformation $|1\rangle(|0\rangle + |1\rangle) + |0\rangle|1\rangle$ on Alice’s part, the state can be brought into the standard form $|000\rangle + |111\rangle$.

On the other hand, (ABC-2) has an elementary divisor of $\lambda^2$ so its tensor rank is 3. Likewise, it can be put in the standard form $|100\rangle + |010\rangle + |001\rangle$ by a SLOCC transformation. By Lemma 2, (A:BC-1) represents a state of pure entanglement shared between Bob and Charlie.

Evidently, by examining Lemma 1, the only possible classes included in three qubit systems are those with Bob and Charlie having nonmaximal local ranks. When $h=1$, $g=0$, the only possibility is $\epsilon_1=1$, while for $h=0$, $g=1$, it must be $\epsilon_1=1$. The case of $h=1$, $g=1$, there are no nonzero minimal indices. These three states are given by

$$\begin{pmatrix} \lambda & \mu \\ \frac{\mu}{\lambda} & \mu \end{pmatrix} \text{ (AC:B) } \begin{pmatrix} \lambda & \mu \\ \frac{\mu}{\lambda} & \mu \end{pmatrix} \text{ (AB:C) } \begin{pmatrix} \lambda & \mu \\ \frac{\mu}{\lambda} & \mu \end{pmatrix} \text{ (A:B:C) }$$

$|011\rangle + |101\rangle$ $|011\rangle + |101\rangle$ $|011\rangle$.

We see that (A:B:C) represents the product states while (AC:B) and (AB:C) are the bipartite pure entanglement with respect to the specified partitioning.

2. $2 \otimes 2 \otimes 3$ systems

Since we are only concerned with the states of maximal local ranks for Bob and Charlie, we only consider pencils having $h=g=0$. The only possible minimal indices are $\epsilon_1=1$ and $\epsilon_1=2$ which correspond to the states
TABLE I. Summary equivalence classes in $2 \otimes 3 \otimes 6$ systems.

<table>
<thead>
<tr>
<th>Representative</th>
<th>Local ranks</th>
<th>Tensor rank</th>
<th>Representative</th>
<th>Local ranks</th>
<th>Tensor rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A:B:C)</td>
<td>(1,1,1)</td>
<td>1</td>
<td>(AB:C)</td>
<td>(2,2,1)</td>
<td>2</td>
</tr>
<tr>
<td>(AC:B)</td>
<td>(2,1,2)</td>
<td>2</td>
<td>(A:BC-1)</td>
<td>(1,2,2)</td>
<td>2</td>
</tr>
<tr>
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<tr>
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<td>(ABC-4)</td>
<td>(2,2,3)</td>
<td>3</td>
</tr>
<tr>
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<td>(ABC-6)</td>
<td>(2,3,2)</td>
<td>3</td>
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<td>3</td>
</tr>
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<td>(ABC-9)</td>
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<td>2</td>
</tr>
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<td>(ABC-11)</td>
<td>(2,3,3)</td>
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<td>(ABC-13)</td>
<td>(2,3,3)</td>
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<td>(ABC-19)</td>
<td>(2,3,5)</td>
<td>5</td>
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<tr>
<td>(ABC-20)</td>
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<td>5</td>
<td>(ABC-21)</td>
<td>(2,3,5)</td>
<td>6</td>
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</tbody>
</table>

FIG. 1. Complete hierarchy of SLOCC equivalence classes; arrows indicate a noninvertible transformation.
Lemma 5, the tensor rank of both these states is 3. In fact, an explicit three-term expansion of classes of states with maximal local ranks, are

$$072205-14 \text{ Chitambar, Miller, and Shi J. Math. Phys. 51, 072205 (2010)}$$

$$\begin{bmatrix}
\lambda & \mu & \lambda \\
\cdot & \cdot & \cdot \\
\cdot & \mu + \lambda & \cdot
\end{bmatrix} (\text{ABC-3}) \begin{bmatrix}
\lambda & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{bmatrix} (\text{ABC-4})
$$

|001⟩ + |100⟩ + |112⟩ |001⟩ + |012⟩ + |100⟩ + |111⟩.

The state (ABC-3) has a single elementary divisor of $\lambda$ while (ABC-4) has none. According to Lemma 5, the tensor rank of both these states is 3. In fact, an explicit three-term expansion of (ABC-3) is given by $\frac{1}{3} |00⟩|01⟩|+⟩ + \frac{1}{3} |00⟩|−⟩|−⟩ + \frac{1}{3} |+⟩|00⟩|02⟩$, where $|±⟩_{ij} = |i⟩ ± |j⟩$.

3. 2⊗2⊗n systems for $n ≥ 4$

As noted in the discussion above, it is enough to consider $2⊗2⊗4$ systems. For states with Bob and Charlie having full local ranks, the only possible minimal indices are $\epsilon_1 = \epsilon_2 = 1$ which corresponds to the state

$$\begin{bmatrix}
\lambda & \mu & \lambda \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \mu
\end{bmatrix} (\text{ABC-5})
$$

|001⟩ + |013⟩ + |100⟩ + |112⟩.

4. 2⊗3⊗2 systems

These pencils are simply the transpose of $2⊗3$ pencils and thus contribute two equivalence classes of states with maximal local ranks,

$$\begin{bmatrix}
\lambda & \cdot & \cdot \\
\mu & \cdot & \cdot \\
\cdot & \cdot & \lambda
\end{bmatrix} (\text{ABC-6}) = (\text{ABC-3})^T \begin{bmatrix}
\lambda & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \mu
\end{bmatrix} (\text{ABC-7}) = (\text{ABC-4})^T
$$

|010⟩ + |100⟩ + |121⟩ |010⟩ + |021⟩ + |100⟩ + |111⟩.

5. 2⊗3⊗3 systems

Here we have $3⊗3$ pencils and for those having no minimal indices, the representative states are

$$\begin{bmatrix}
\lambda & \cdot & \cdot \\
\cdot & \mu + \lambda & \cdot \\
\cdot & \cdot & \mu
\end{bmatrix} (\text{ABC-8}) \begin{bmatrix}
\lambda & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \mu + \lambda
\end{bmatrix} (\text{ABC-9}) \begin{bmatrix}
\lambda & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \mu
\end{bmatrix} (\text{A;BC}-2)
$$

|100⟩ + (|0⟩ + |1⟩)|11⟩ + |022⟩ |100⟩ + |111⟩ + |022⟩ |100⟩ + |111⟩ + |122⟩,

$$\begin{bmatrix}
\lambda & \mu & \lambda \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \lambda
\end{bmatrix} (\text{ABC-10}) \begin{bmatrix}
\lambda & \mu & \lambda \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \mu + \lambda
\end{bmatrix} (\text{ABC-11}) \begin{bmatrix}
\lambda & \mu & \lambda \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \mu
\end{bmatrix} (\text{ABC-12})
$$

|001⟩ + |100⟩ + |111⟩ + |122⟩ |001⟩ + |100⟩ + |111⟩ |001⟩ + |012⟩ + |100⟩ + |111⟩ + |122⟩.

For $3⊗3$ pencils, the only possible minimal indices are $\epsilon_1 = \nu_1 = 1$ corresponding to the representative state

$$\begin{bmatrix}
\lambda & \mu & \lambda \\
\cdot & \cdot & \mu \\
\cdot & \cdot & \lambda
\end{bmatrix} (\text{ABC-13})$$

|001⟩ + |012⟩ + |100⟩ + |122⟩.
6. $2 \otimes 3 \otimes 4$ systems

For a minimal index of $\epsilon_1 = 1$, we have the classes represented by

\[
\begin{pmatrix}
\lambda & \mu & \cdot & \cdot \\
\cdot & \cdot & \lambda & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{pmatrix} \quad (\text{ABC-14})
\]

\[
\begin{pmatrix}
\lambda & \mu & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{pmatrix} \quad (\text{ABC-19})
\]

\[
\begin{pmatrix}
\lambda & \mu & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{pmatrix} \quad (\text{ABC-21})
\]

\[
|001\rangle + |100\rangle + |112\rangle + |123\rangle \\
+ |012\rangle + |100\rangle + |111\rangle + |123\rangle
\]

We also have the states with $\epsilon_1 = 2$ and $\epsilon_1 = 3$, respectively,

\[
\begin{pmatrix}
\lambda & \mu & \cdot & \cdot \\
\cdot & \cdot & \lambda & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{pmatrix} \quad (\text{ABC-15})
\]

\[
\begin{pmatrix}
\lambda & \mu & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{pmatrix} \quad (\text{ABC-18})
\]

\[
\begin{pmatrix}
\lambda & \mu & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{pmatrix} \quad (\text{ABC-20})
\]

\[
|001\rangle + |013\rangle + |100\rangle \\
+ |012\rangle + |023\rangle + |100\rangle + |111\rangle + |122\rangle \\
\]

7. $2 \otimes 3 \otimes 5$ systems

The possibilities are $\epsilon_1 = 1, \epsilon_2 = 1$ and $\epsilon_1 = 1, \epsilon_2 = 2$ corresponding to

\[
\begin{pmatrix}
\lambda & \mu & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\end{pmatrix} \quad (\text{ABC-16})
\]

\[
|001\rangle + |010\rangle + |112\rangle + |123\rangle \\
+ |012\rangle + |023\rangle + |100\rangle + |111\rangle + |122\rangle
\]

8. $2 \otimes 3 \otimes n$ systems for $n \geq 6$

We must only consider $n = 6$ which allows for $\epsilon_1 = 1, \epsilon_2 = 1, \epsilon_3 = 1$ with representative

\[
\begin{pmatrix}
\lambda & \mu & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{pmatrix} \quad (\text{ABC-24})
\]

\[
|001\rangle + |013\rangle + |025\rangle + |100\rangle + |112\rangle + |124\rangle
\]

---

2. For two matrices $R$ and $S$, a vector $x$ and a constant $\lambda$, if $Rx = \lambda Sx$, $\lambda$ is a generalized eigenvalue of $(R, S)$, and $x$ the associated eigenvector. The set of generalized eigenvalues are precisely \{\nu : \det(\mu R + \lambda S) = 0\} (see, e.g., Sec. 7.7 of Ref. [27].