EECS 562 Nonlinear Control

A Review of

Control System Analysis and Design Via the "Second Method" of Lyapunov: I–Continuous -Time Systems

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Introduction

The celebrated "second method" of Lyapunov can be considered as the most important method in the field of control theory, especially as regards nonlinear control, in which the stability properties can be concluded without solving the nonlinear system. The fundamental idea of the method originates from classical physics in using the energy function $E(x)$ decreasing along the trajectory of x to obtain the stability properties of the equilibrium of the system. Except talking the stability properties without solving the systems, one can rigorously analyze most of nonlinear systems including error criterion, optimization and control design provided that they are globally asymptotically stable whenever explicit Lyapunov functions can be found. The "second method" of Lyapunov is more a unifying principle than a method [1].

In [1], the two famous contributors in modern control theory, Kalman and Bertram, started from dynamical systems governed by vector ordinary differential equations:

$$\dot{x} = f(x, u, t) \text{ for } t > t_0,\ x \in \mathbb{R}^n \text{ and } u \in \mathbb{R}^m \tag{1}$$

Considering the control u as a function of x and t, namely $u = \gamma(x, t)$ for some function γ, we can rewrite (1) into:

$$\dot{x} = f(x, t) \text{ for } t > t_0 \tag{2}$$

which is called an unforced system. We know that Theorem 3.1 in [2] guarantees that there exists a unique solution denoted $\Phi(t; x_0.t_0)$ locally to (2) if f is locally Lipschitz in x and piecewise continuous in t. Although global existence also can be stated, the requirement of a global Lipschitz on f is too restrictive. The equilibrium state x_e of a unforced dynamical system is defined as:

$$f(x_e; t) = 0 \tag{3}$$

After stating the mathemtical background, they introduced the concepts of stability in the sense of Lyapunov. In addition to the usual Lyapunov stability and asymptotical stability that covered in the lecture, there is a third type stability called equiasymptotical stability.

Definition 1. *An equilibrium state x_e of a unforced dynamical system is equiasymptotically stable if*

1. *it is stable and*

2. *every motion starting sufficiently near x_e converges to x_e as $t \to \infty$ uniformly in x_0. In other words, there is some real constant $r(t_0) > 0$ and to every real number $\mu > 0$ there corresponds a real number $T(\mu, r(t_0), t_0)$ such that $\|x_0 - x_e\| < r(t_0)$ implies*

$$\|\Phi(t; x_0, t_0) - x_e\| < \mu \text{ for all } t \geq t_0 + T \tag{4}$$

They also defined global (in the large) and uniform stability afterwards.

Ultimately, Kalman and Bertram outlined six main stability theorems of the "second method" and discussed important applications including control optimization and design. In this review, we are going to summarize the six main theorems in detail from [1] and try to connect it to the materials we have learned in lecture. Furthermore, we will render more straightforward examples from [1] and [2] for the main theorems to readers in order to better understand the "second method" of Lyapunov.

The Main Theorems

The first theorem is the most familiar and classical one of the "second method" that is also discussed carefully in the lecture.

Theorem 1. *Consider the continuous - time, unforced dynamical system (2) where* $f(0, t) =$

0 for all t. Suppose there exists a scalar function $V(x,t)$ with continuous first partial derivatives with respect to x such that $V(0,t) = 0$ and

1. *V(x,t) is positive definite; i.e., there exist a continuous, nondecreasing scalar function α such that $\alpha(0) = 0$ and, for all t and all $x \neq 0$*

$$0 < \alpha(\|x\|) \leq V(x,t) \tag{5}$$

2. *There exists a continuous scalar function γ such that $\gamma(0) = 0$ and the derivative $\dot{V}$ of V along the trajectory starting at t, x satisfies, for all t and all $x \neq 0$,*

$$\dot{V}(x,t) = \frac{\partial V}{\partial t} + \nabla V f(x,t) \leq -\gamma(\|x\|) < 0 \tag{6}$$

3. *There exists a continuous, nondecreasing scalar function β such that $\beta(0) = 0$ and for all t,*

$$V(x,t) \leq \beta(\|x\|) \tag{7}$$

4. *$\alpha(\|x\|) \to \infty$ with $\|x\| \to \infty$*

Then the equilibrium state $x_e = 0$ is uniformly asymptotically stable in the large; $V(x,t)$ is called a Lyapunov function of the system (2).

Theorem 1 is very similar to Theorem 4.10 in [2]. Hypotheses 1, 3 and 4 can be interpreted as the Lyapunov function candidate $V(x,t)$ is locally positive definite, decrescent and radially unbounded, while Hypothesis 2 implies that $-\dot{V}(x,t)$ is locally positive definite. Let's look an example from [2].

Example 1. *Consider the scalar system*

$$\dot{x} = -(1 + g(t))x^3 \tag{8}$$

where $g(t)$ is continuous and $g(t) \geq 0$ for all $t \geq 0$. Using the Lyapunov function candidate $V(x,t) = \frac{x^2}{2}$ and choosing $\alpha(x) = \beta(x) = V(x)$ and $\gamma(x) = x^4$, we have

$$\dot{V}(x,t) = -(1+g(t))x^4 \leq -\gamma(\|x\|) < 0 \tag{9}$$

We check that all the assumptions are satisfied, so we can conclude that the equilibrium state $x_e = 0$ is uniformly asymptotically stable in the large; $V(x,t) = \frac{x^2}{2}$ is a Lyapunov function of the system (8).

The next theorem is also mentioned in [2] as Theorem 4.17.

Theorem 2. *Let the function f defined in the differential equation (2) be Lipschitzian. Assume also that $f(0,t) = 0$ and that the equilibrium state $x_e = 0$ is uniformly asymptotically stable in the large. Then there exists a Lyapunov function $V(x,t)$ which is infinitely differentiable with respect to x, t, and satisfies all the hypotheses of Theorem 1.*

Basically, it is just the converse statement of Theorem 1. In other words, the existence of Lyapunov function is a necessary and sufficient condition for uniformly asymptotical stability in the large at the equilibrium state x_e. Although it is hard to come up with an specific example, a special case of Theorem 2 can be used to demonstrate the idea (Theorem 4.6 in [2]).

Example 2. *If a linear system $\dot{x} = Ax$ is globally asymptotically stable, then there exists a quadratic Lyapunov function that satisfies all the hypotheses of Theorem 1.*

Let's consider the Lyapunov function $V(x,t) = x^T P x$ where P is defined as :

$$P = \int_0^\infty \exp{(A^T t)} Q \exp{(At)} dt \text{ for any positive definite matrix } Q \tag{10}$$

Since the matrix Q is positive definite, so are P and $V(x,t)$. It is clear that the hypotheses (1), (3), (4) are satisfied by choosing $\alpha(x) = \beta(x) = V(x,t)$. On the other hand, the *hypotheses (2)*

$$\dot{V}(x,t) = -x^T Q x < 0 \tag{11}$$

is satisfied by choosing $\gamma(x) = x^T Q x$. Thus, $V(x,t) = x^T P x$ is indeed a Lyapunov function of the linear system $\dot{x} = Ax$.

Example 2 actually can be summarized into a more generalized cases by the following theorem.

Theorem 3. *Consider a continuous - time linear dynamical system*

$$\dot{x} = A(t)x + B(t)u(t) \tag{12}$$

subject to the restrictions

1. $\|A(t)\| \leq c_1 < \infty$
2. $0 < c_2 \leq \|B(t)x\| \leq c_3 < \infty$ *for all* $\|x\| = 1$, *all* t

Then the following propositions concerning this system are equivalent:

1. *Any uniform bounded excitation*

$$\|u(t)\| \leq c_4 < \infty \tag{13}$$

   gives rise to a uniformly bounded response for all $t \geq t_0$

$$\|x(t)\| = \|\Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau\| \leq c_5(c_4, \|x_0\|) < \infty \tag{14}$$

2. *For all $t \geq t_0$*

$$\int_{t_0}^{t} \|\Phi(t,\tau)\| d\tau \leq c_6 < \infty \tag{15}$$

3. *The equilibrium state $x_e = 0$ of the unforced system is uniformly asymptotically stable*

4. There are positive constant c_7, c_8 *such that, whenever* $t \geq t_0$,

$$\|\Phi(t, t_0)\| \leq c_7 \exp -(c_8(t - t_0)) \tag{16}$$

5. Given any positive definite matrix $Q(t)$ *continuous in* t *and satisfying for all* $t \geq t_0$

$$0 \leq c_9 I \leq Q(t) \leq c_{10} I < \infty \tag{17}$$

the scalar function defined by

$$V(x, t) = \int_t^{\infty} \|\Phi(\tau, t)\|^2 Q(\tau) d\tau = \|x\|^2_{P(t)} \tag{18}$$

exists and is a Lyapunov function for the unforced system satisfying the requirement of Theorem 1, with its derivative along the unforced trajectory starting at x, t *being*

$$\dot{V}(x, t) = \|x\|^2_{Q(t)} \tag{19}$$

where $\Phi(t, t_0)$ *is the transition matrix of the system (12) and* $P(t)$ *is the unique solution to the Lyapunov equation:*

$$A(t)^T P(t) + P(t) A(t) = -Q(t) \tag{20}$$

The stability described in Proposition 1 is called uniformly BIBO stability. Namely, every bounded input will yield a bounded output, and the bound of the output is t_0-independent[3]. Proposition 4 states that the system is exponentially stable. Hence, in [3], Brockett concluded that the system (12) is uniformly BIBO stable if and only if the unforced system of (12) is exponentially stable. Moreover, Theorem 3 tells us that both BIBO and exponential stability are equivalent to uniformly asymptotical stability in the sense of Lyapunov in the linear systems. In particular, Proposition 5 is just the time-varying version of Example

2, in which the transition matrix $\Phi(t,0) = \exp(A^T t)$, and it is very useful in proving the Lyapunov indirect method which is widely used in many engineering applications. Let's look at a numeric counterexample from [2].

Example 3. *Consider a second-order linear system with*

$$A(t) = \begin{bmatrix} -1+1.5\cos^2 t & 1-1.5\sin t\cos t \\ -1-1.5\sin t\cos t & -1+1.5\sin^2 t \end{bmatrix} \tag{21}$$

The eigenvalues of $A(t)$ are $-0.25 \pm j0.25\sqrt{7}$, which are lying in the open left-half plane. However, the origin is unstable because the transition matrix

$$\Phi(t,0) = \begin{bmatrix} \exp(0.5t)\cos t & \exp(-t)\sin t \\ -\exp(0.5t)\sin t & \exp(-t)\cos t \end{bmatrix} \tag{22}$$

in which the norm of the transition matrix

$$\|\Phi(t,\tau)\| > \infty \tag{23}$$

Therefore, we can conclude that the origin is not uniformly asymptotically stable by Proposition 3 and 4.

The Lyapunov equation is also useful in driving Routh-Hurwitz conditions[1].

Example 4. *Let's consider Routh-Hurwitz conditions for general second-order cases. Let $Q = I$ and write out (20) in the case when $A(t)$ is a 2×2 matrix independent of t:*

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{24}$$

We can simplify the system to:

$$\begin{bmatrix} 2a_{11} & 2a_{21} & 0 \\ a_{12} & a_{11}+a_{22} & a_{21} \\ 0 & 2a_{12} & 2a_{22} \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{21} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \tag{25}$$

The system has a unique solution if and only if

$$\det A = 4(a_{11}+a_{22})(a_{11}a_{22}-a_{12}a_{21}) = n^2 \operatorname{Tr} A \det A \tag{26}$$

Assuming (26) is achieved, we can compute the matrix P, which is given by

$$P = \frac{-1}{2 \operatorname{Tr} A \det A} \begin{bmatrix} \det A + a_{21}^2 + a_{22}^2 & -(a_{12}a_{22}+a_{12}a_{21}) \\ -(a_{12}a_{22}+a_{12}a_{21}) & \det A + a_{11}^2 + a_{12}^2 \end{bmatrix} \tag{27}$$

In order to make P positive definite, we need to let

$$\det P = \frac{(a_{11}+a_{22})^2 + (a_{12}-a_{21})^2}{2(\operatorname{Tr} A)^2 \det A} \tag{28}$$

$$p_{11} = -\frac{\det A + a_{21}^2 + a_{22}^2}{2 \operatorname{Tr} A \det A} \tag{29}$$

$$\Rightarrow \det A = a_{11}a_{22} - a_{12}a_{21} > 0 \textit{ and } \operatorname{Tr} A = a_{11}+a_{22} < 0 \tag{30}$$

The condition (30) is equivalent to a better-known form:

$$A = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} \tag{31}$$

with $a_1, a_2 > 0$, which are the well-known Routh-Hurwitz inequalities for a constant-coefficient linear differential equation of second order.

Now, let's go back to nonlinear systems and see a so-called "the best" general result on stability of nonlinear systems by Kalman and Bertram.

Theorem 4. *Consider the continuous-time, unforced, stationary dynamical system:*

$$\dot{x} = f(x) \text{ and } f(0) = 0 \tag{32}$$

Assume that f has continuous first partial derivative and that its Jacobian matrix $A(x) = \frac{\partial f}{\partial x}$ satisfies the condition: for any $\epsilon > 0$:

$$\hat{A}(x) = A(x) + A(x)^T \leq -\epsilon I < 0 \tag{33}$$

Then the equilibrium state $x_e = 0$ of the sysetm (32) is asymptotically stable in the large and

$$V(x) = \|f(x)\|^2 \tag{34}$$

is one of its Lyapunov functions.

The proof of Theorem 4 is straightforward, and the key idea is using (33) to show that $\dot{V}$ is negative definite. It is indeed a "best" result because it comes up with a very simple way to find Lyapunov functions for autonomous systems, which is just the norm of f square, and the result is also easy-applicable in many nonlinear systems. Also notice that failure on (33) does not necessarily means the system is unstable. Let's look at an example from [1].

Example 5. *Consider a second order nonlinear system*

$$\begin{cases} \dot{x}_1 & = f_1(x_1) + f_2(x_2) \\ \dot{x}_2 & = x_1 + ax_2 \end{cases} \tag{35}$$

where $f_1(0) = f_2(0) = 0$; f_1 and f_2 are differential functions. It is easy to compute that

$$\hat{A}(x) = \begin{bmatrix} 2f_1'(x_1) & 1 + f_2'(x_2) \\ 1 + f_2'(x_2) & 2a \end{bmatrix} \tag{36}$$

In order to satisfy the condition (33), we must have

$$4af_1'(x_1) - (1 + f_2'(x_2))^2 \geq \epsilon^2 > 0 \textit{ for all } x \tag{37}$$

$$f_1'(x_1) \leq -\epsilon < 0 \textit{ for all } x_1 \tag{38}$$

Before stating Theorem 5, we need some preliminary backgrounds in system optimization.

Definition 2. *Let $\rho(x)$ be a continuous, nonnegative function which serves as the error criterion of a regulator whose purpose is to maintain the system at all times as close as possible to the equilibrium state $x_e = 0$. Thus, $\rho(0) = 0$. For instance, one can take $\rho(x) = x_1^2$. The performance index of the system as the integrated error criterion*

$$V(x) = \int_0^\infty \rho(\Phi(\tau; x, 0))d\tau \tag{39}$$

If the regulator contains some parameters, we can minimize $V(x)$ in terms of those parameters.

Theorem 5. *Consider a unforced, linear, stationary dynamical system with an equilibrium state at the origin and assume*

1. *The error criterion $\rho(x)$ is positive definite.*

2. *The performance index $V(x)$ defined by (39) is finite in some neighborhood of the origin.*

The $x_e = 0$ is asymptotically stable.

Kalman and Bertram provided a very illustrative example in [1].

Example 6. *Consider the second-order linear stationary system*

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - 2\zeta x_2 \end{cases} \tag{40}$$

where ζ is an arbitrary constant called damping ratio. The performance index is given by

$$V(x,\zeta) = \|x\|_P^2 = \int_{t_0}^{\infty} \|\Phi(\tau; x, t_0)\|_Q^2 d\tau \tag{41}$$

where

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}; \mu > 0 \tag{42}$$

Using (20), we can calculate

$$P(\xi) = \begin{bmatrix} \xi + \frac{1+\mu}{4\zeta} & \frac{1}{2} \\ \frac{1}{2} & \frac{1+\mu}{4\zeta} \end{bmatrix} \tag{43}$$

$$V(x,\zeta) = \zeta x_1^2 + \frac{1+\mu}{4\zeta}(x_1^2 + x_2^2) + x_1 x_2 \tag{44}$$

We try to minimize $V(x,\zeta)$ in term of ζ when $x_2 = 0$ and obtain

$$\zeta^*(\mu) = \frac{\sqrt{1+\mu}}{2} \tag{45}$$

Notice that the damping ratio $\zeta^*(1) = 0.707$. From classical control theory, when the damping ratio $\zeta = 0.707$, the systems would have optimal rise time and settling time compared to other choices of ζ. The following graph is taken from [4].

At the end the paper, Kalman and Bertram talked about control design in stationary linear systems by using the "second method" of Lyapunov. First, let's consider (12) without time-

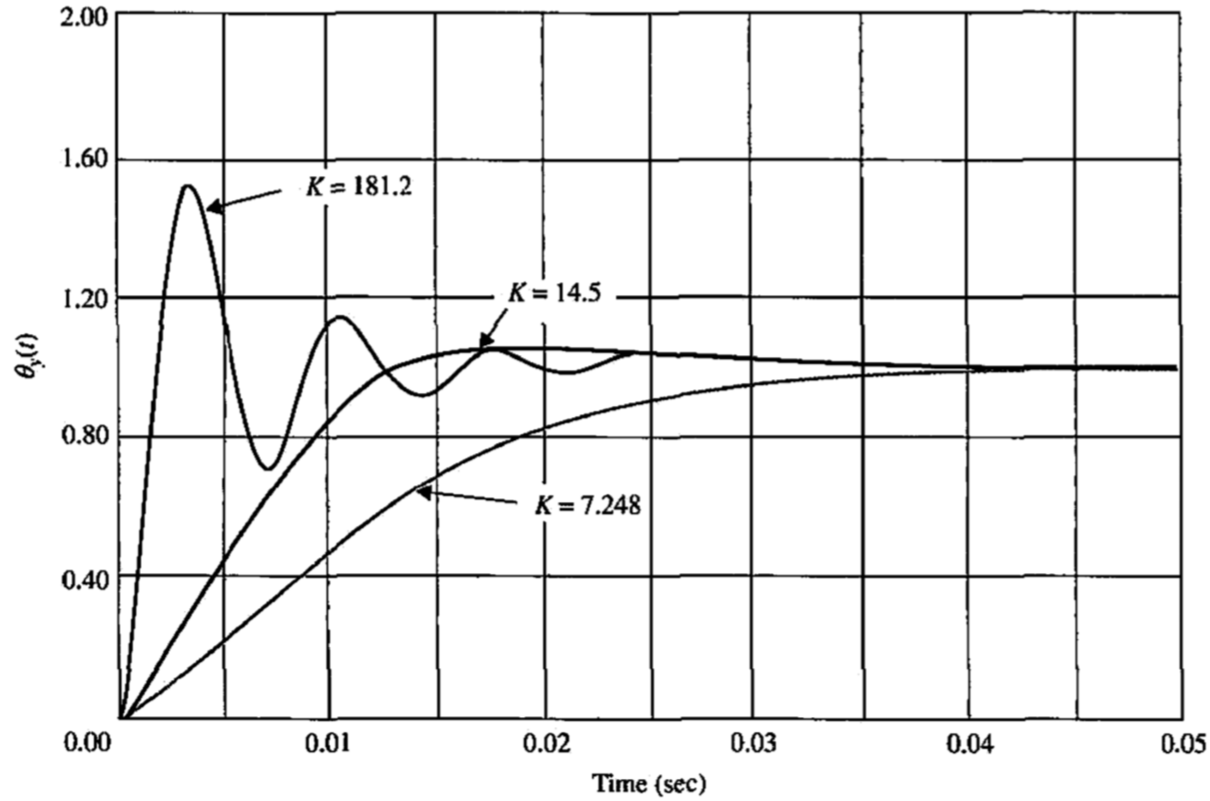

Figure 1: Unit-step responses of the second order system. $K = 181.17, 14.5$ and 7.248 correspond to $\zeta = 0.2, 0.707$ and 1 respectively.

dependence and assume that the pair (A, B) is stabilizable. Taking an arbitrary positive definite matrix Q and applying Theorem 3, we can obtain one of the Lyapunov functions of the system - $V(x) = \|x\|_P^2$ where P is solved from (20). The principle idea of the method is using $V(x)$ as the Lyapunov function of the stationary linear system to make $\dot{V}$ as negative as possible. It is easy to compute that

$$\dot{V}(x,t) = -\|x\|_Q^2 + 2u^T(t)B^TPx(t) \tag{46}$$

Thus, if we take

$$u_i(t) = -a_i \mathrm{sgn}(B^TPx(t))_i \qquad (i = 1, 2, ...m) \tag{47}$$

for arbitrary constants a_i, the time derivative $\dot{V} < 0$ and the system is asymptotically stable at the origin. Let's consider an example in [1].

Example 7. *Consider the following stationary linear system with*

$$A = \begin{bmatrix} -0.01 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2.26 & -0.2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 2.5 \end{bmatrix}$$

Let's take $Q = I$. By (20), we can solve for P:

$$P = \begin{bmatrix} 50.00 & 4.64 & 22.10 \\ 4.64 & 9.71 & 2.28 \\ 22.10 & 2.28 & 13.88 \end{bmatrix}$$

Therefore, the optimal control is given by

$$u_1^*(t) = -sgn(B^T P x) = -sgn(22.10x_1 + 2.28x_2 + 13.88x_3) \text{ for } a_1 = 0.4 \tag{48}$$

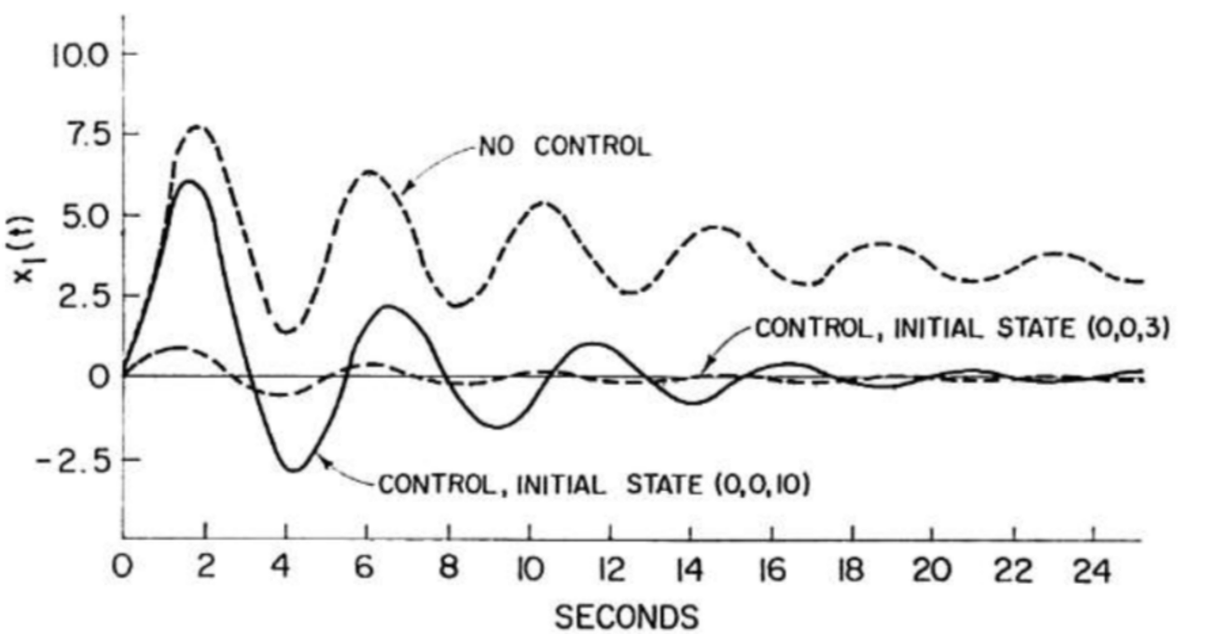

Figure 2: Response of relay control system.

However, the above design has two major mathematical difficulties. First, since $u^*(t)$ is discontinuous, it is not guaranteed that the existence of solution to the system. Second, the state variables are not measurable. and the matrices A and B are not fully known. The first difficulty can be tackled by using $\mathrm{sat}(x)$ function instead of $\mathrm{sgn}(x)$ function:

$$u_i^*(t) = -a_i \mathrm{sat}(kB^T Px)_i \tag{49}$$

for large constant k. To solve the second difficulty, let y denote the measurable states and z denote the unmeasurable states. The idea is similar to design an observer in linear feedback control system. Then we can rewrite the stationary linear system into:

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \tag{50}$$

Assume that $\hat{z}$ is the estimate states of z, and $\hat{A_{ij}}$ and $\hat{B}_i$ are estimated matrices to A_{ij} and B_i respectively. We also use tilde to denote the errors of these estimates. Then the overall system can be written as:

$$\begin{cases} \dot{x} = \hat{A}x + \hat{B}u(t) + \tilde{A}x + \tilde{B}u(t) \\ \dot{\tilde{z}} = \hat{A}_{22}\tilde{z} + \tilde{A}_2 x \end{cases} \tag{51}$$

with optimal control

$$u_i^*(t) = -a_i \mathrm{sat}(k\hat{B}^T \hat{P}x)_i \tag{52}$$

where $\hat{A}_2$ is the second column vector of $\hat{A}$. In order to achieve asymptotical stability, we require the following theorem.

Theorem 6. *The unforced, stationary dynamical system (51) is asymptotically stable in the large, for arbitrary large $k \geq 0$, if the following condition hold:*

1. *$Re(\hat{A}_{22}) < 0$ for all i.*

2. $\tilde{A}$ *and* $\tilde{B}$ *are sufficiently small.*

Discussion

We are going to discuss the meaning of Corollary 1.1, 1.2 and 1.3 followed by the most classical "second method" result. Basically, they are the relaxations of Theorem 1. Corollary 1.1 relaxes some of the hypotheses in Theorem 1 to achieve different types of weaker stability. We know that Hypothesis 4 stands for radially unboundness of V, so it is clear that the result of Theorem 1 will not be global (uniformly asymptotically stable) if Hypothesis 4 is removed. More interestingly, Hypothesis 3 which stands for "decrescent" determines whether the system is asymptotically stable or equiasymptotically stable. Furthermore, if we modify Hypothesis 2 that $\dot{V}$ is negative semidefinite, then the system only can achieve stability (or uniform stability).

Corollary 1.2 relaxes the system to be time-independent, and it is similar to Theorem 4.2 in [2] except that Khalil disregarded the difference between asymptotical stability and equiasymptotical stability.

Theorem 7. *Let* $x = 0$ *be an equilibrium point for* $\dot{x} = f(x)$*. Let* $V\colon \mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable function such that* $V(0) = 0$ *and* $V(x) > 0$ *for all* $x \neq 0$*,* $\|x\| \to \infty \Rightarrow V(x) \to \infty, and \dot{V}(x) < 0$ *for all* $x \neq 0$*. Then* $x = 0$ *is globally asymptotically stable.*

Moreover, Corollary 1.3 continues to relax the second condition further in Corollary 1.2 that $\dot{V}$ can be negative semidefinite with $\dot{V}(\Phi(t; x_0, t_0))$ not vanishing identically for $x_0 \neq 0$. It is referred to LaSalle's invariant principle in [2].

Theorem 8. *Let* $x = 0$ *be an equilibrium point for* $\dot{x} = f(x)$*. Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable, radially unbounded, positive definite function such that* $\dot{V}(x) \leq 0$ *for all* $x \in \mathbb{R}$*. Let* $S = \{x \in \mathbb{R}^n | \dot{V}(x) = 0\}$ *and suppose that no solution can stay identically in* S*, other than the trivial solution* $x(t) = 0$*. Then, the origin is globally asymptotically* *stable.*

Conclusion

The "second method" of Lyapunov provides us a more insight view towards nonlinear systems. Not only do we talk about stability properties of the equilibrium state without solving the solution explicitly, but also we can use it to estimate the transient responses, optimize nonlinear systems and design an optimal control. In this review, we summarize the content of [1] and discuss the six main theorems with couple simple examples from both [1] and [2] to help readers understand the "second method" of Lyapunov.

References

[1] Bertram J. E. Kalman, R. E. Control system analysis and design via the second method of lyapunov: I–continuous -time systems. *Journal of Basic Engineering*, 1960/06/01.

[2] H.K. Khalil. *Nonlinear Systems.* Pearson Education. Prentice Hall, 2002.

[3] R. Brockett. *Finite Dimensional Linear Systems.* Society for Industrial and Applied Mathematics, Philadelphia, PA, 2015.

[4] B.C. Kuo. *Automatic control systems.* Prentice-Hall, 1982.