EXPLICIT SOLUTIONS AND STABILITY PROPERTIES OF HOMOGENEOUS POLYNOMIAL DYNAMICAL SYSTEMS VIA TENSOR ORTHOGONAL DECOMPOSITION *

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Abstract. This paper investigates the explicit solutions and stability properties of certain continuous-time homogeneous polynomial dynamical systems via tensor algebra. In particular, if a system of homogeneous polynomial differential equations can be represented by an orthogonally decomposable tensor, we can construct its explicit solution by exploiting tensor Z-eigenvalues and Z-eigenvectors. By utilizing the form of the explicit solution, we are able to discuss the stability properties of the homogeneous polynomial dynamical system. We illustrate that the Z-eigenvalues from the orthogonal decomposition of the corresponding dynamic tensor can be utilized to establish necessary and sufficient stability conditions, similar to these from linear systems theory. Furthermore, we explore the complete solution to the homogeneous polynomial dynamical system with constant inputs. These results are demonstrated via several numerical examples.

Key words. tensor algebra, orthogonal decomposition, Z-eigenvalues, Z-eigenvectors, polynomial dynamical systems, explicit solutions, stability

AMS subject classifications. 15A18, 15A69, 15A72, 34D05, 34D20, 93D20

1. Introduction.

Tensor algebra has been applied to model and simulate nonlinear dynamics [6, 9, 16, 27, 28]. The key idea is to represent nonlinear dynamics using tensor products, and then to exploit tensor decomposition techniques such that CANDECOMP/PARAFAC decomposition, higher-order singular value decomposition, Tucker decomposition, and tensor train decomposition [14, 24, 25, 33]. Kruppa [27, 28] represented a multilinear polynomial dynamical system by a contracted product between a parameter tensor and a monomial tensor, and utilized CANDECOMP/PARAFAC decomposition and Tucker decomposition for efficiently simulating the evolution of the dynamics. Moreover, Chen *et al.* [9] proposed a new tensor-based multilinear dynamical system for characterizing the dynamics of hypergraphs, a generalization of graphs in which edges can contain more than one nodes. The mulitlinear dynamical system evolution is described by the action of tensor vector multiplications between a dynamic tensor and the state vector. In fact, the multilinear dynamical system belongs to the family of homogeneous polynomial dynamical systems if one expands the tensor vector multiplications.

The explicit solution and stability properties of a linear dynamical system can be readily obtained from the eigenvalue decomposition of the dynamic matrix. However, the results can hardly be extended to homogeneous polynomial dynamical systems due to its nonlinear nature [1, 2, 21, 39, 41]. In terms of stability, many methods such as generalized characteristic value problems [39] and optimization-based Lyapunov functions [1] have been proposed to establish stability of some homogeneous polynomial dynamical systems. Similarly, in this paper, we will exploit tensor orthogonal decomposition with Z-eigenvalues and Z-eigenvectors to summarize the explicit

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solutions and stability properties of certain homogeneous polynomial dynamical systems that can be represented by orthogonally decomposable tensors via tensor vector multiplications as defined in [9].

Tensor eigenvalue problems of real supersymmetric tensors were first explored by Qi [34, 35] and Lim [29] independently in 2005. There are many different notions of tensor eigenvalues such as H-eigenvalues, Z-eigenvalues, M-eigenvalues, and U-eigenvalues [10, 34, 35], which have different applications in network theory, machine learning, elasticity theory, and dynamical systems. Surana *et al.* [43] compared the H-eigenvalue spectrum between the two Laplacian tensors for measuring hypergraph distance. Chen *et al.* [13] showed that the Z-eigenvector associated with the second smallest Z-eigenvalue of a normalized Laplacian tensor can be used for hypergraph partition. Moreover, Huang and Qi [19] used M-eigenvalues to prove the strong ellipticity of elasticity tensors in solid mechanics. Furthermore, Chen *et al.* [8, 10] utilized U-eigenvalues to determine the stability of multilinear time-invariant systems, which can be unfolded to linear dynamical systems via tensor unfolding, an operation that transforms a tensor into a matrix. Of particular interest of this paper are Z-eigenvalues.

Recently, Chen [6] investigated the explicit solutions and stability properties of certain discrete-time homogeneous polynomial dynamical systems (also called multilinear dynamical systems in [6]) via tensor orthogonal decomposition. In particular, the author showed that Z-eigenvalues play a significant role in the stability analysis offering necessary and sufficient conditions if the corresponding dynamic tensors are orthogonally decomposable [6]. This paper will focus on continuous-time homogeneous polynomial dynamical systems. Continuous-time polynomial dynamical systems are a popular tool to model various robotic systems [30, 42, 45]. The key contributions of the paper are:

1. We investigate the explicit solutions of certain continuous-time homogeneous polynomial dynamical systems that can be represented by orthogonally decomposable tensors. We derive an explicit solution formula by using the Z-eigenvalues and Z-eigenvectors from the orthogonal decomposition of the corresponding dynamic tensors.
2. According to the form of the explicit solutions, we are able to discuss the stability properties of such homogeneous polynomial dynamical systems. We find that similar to the linear stability, the Z-eigenvalues from the orthogonal decomposition of the corresponding dynamic tensors can offer necessary and sufficient stability conditions. Furthermore, we apply an upper bound of the largest Z-eigenvalue to determine the stability efficiently.
3. We explore the complete solutions of such homogeneous polynomial dynamical systems with constant inputs. We discover that the complete solutions can be solved implicitly by exploiting Gauss hypergeometric functions.
4. We verify our results on four numerical examples. In addition, we discuss the controllability and observability of homogeneous polynomial dynamical systems with linear inputs and outputs.

The paper is organized into six sections. In section 2, we review tensor preliminaries including tensor vector multiplications, tensor eigenvalues, and tensor orthogonal decomposition. We derive an explicit solution formula of the continuous-time homogeneous polynomial dynamical systems that can be represented by orthogonally decomposable tensors in subsection 3.1. In subsection 3.2, we discuss the stability properties of such homogeneous polynomial dynamical systems based on the form of the explicit solutions. We also explore the complete solutions of the homogeneous polynomial dynamical systems with constant inputs in subsection 3.3. Four numerical examples are presented in section 4. We discuss the controllability and observability of the homogeneous polynomial dynamical systems with linear inputs and outputs in section 5. Finally, we conclude in section 6 with future research directions.

2. Tensor preliminaries.

A tensor is a multidimensional array [7, 10, 14, 24, 25, 36]. The order of a tensor is the number of its dimensions, and each dimension is called a mode. A kth order tensor usually is denoted by $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k}$. It is therefore reasonable to consider scalars $x \in \mathbb{R}$ as zero-order tensors, vectors $\mathbf{v} \in \mathbb{R}^n$ as first-order tensors, and matrices $\mathbf{M} \in \mathbb{R}^{m \times n}$ as second-order tensors. A tensor is called cubical if every mode is the same size, i.e., $\mathcal{A} \in \mathbb{R}^{n \times n \times \cdots \times n}$. A cubical tensor $\mathcal{A}$ is called supersymmetric if $\mathcal{A}_{j_1 j_2 \ldots j_k}$ is invariant under any permutation of the indices.

2.1. Tensor vector multiplication.

The tensor vector multiplication $\mathcal{A} \times_p \mathbf{v}$ along mode p for a vector $\mathbf{v} \in \mathbb{R}^{n_p}$ is defined by

$$(\mathcal{A} \times_p \mathbf{v})_{j_1 j_2 \ldots j_{p-1} j_{p+1} \ldots j_k} = \sum_{j_p=1}^{n_p} \mathcal{A}_{j_1 j_2 \ldots j_p \ldots j_k} \mathbf{v}_{j_p}, \tag{2.1}$$

which can be extended to

$$\mathcal{A} \times_1 \mathbf{v}_1 \times_2 \mathbf{v}_2 \times_3 \cdots \times_k \mathbf{v}_k = \mathcal{A}\mathbf{v}_1 \mathbf{v}_2 \ldots \mathbf{v}_k \in \mathbb{R} \tag{2.2}$$

for $\mathbf{v}_p \in \mathbb{R}^{n_p}$. If $\mathcal{A}$ is supersymmetric and $\mathbf{v}_p = \mathbf{v}$ for all $p = 1, 2, \ldots, k$, the product (2.2) is also known as the homogeneous polynomial associated with $\mathcal{A}$, and we write it as $\mathcal{A}\mathbf{v}^k$ for simplicity.

2.2. Tensor eigenvalues.

The tensor eigenvalues of real supersymmetric tensors were first explored by Qi [34, 35] and Lim [29] independently. There are many different notions of tensor eigenvalues including H-eigenvalues, Z-eigenvalues, M-eigenvalues, and U-eigenvalues [10, 34, 35]. Of particular interest of this paper are Z-eigenvalues. Given a kth order supersymmetric tensor $\mathcal{A} \in \mathbb{R}^{n \times n \times \cdots \times n}$, the E-eigenvalues $\lambda \in \mathbb{C}$ and E-eigenvectors $\mathbf{v} \in \mathbb{C}^n$ of $\mathcal{A}$ are defined as

$$\begin{cases} \mathcal{A}\mathbf{v}^{k-1} = \lambda \mathbf{v} \\ \mathbf{v}^\top \mathbf{v} = 1 \end{cases}. \tag{2.3}$$

The E-eigenvalues λ could be complex. If λ are real, we call them Z-eigenvalues. Computing the E-eigenvalues and the Z-eigenvalues of a tensor is NP-hard [18]. However, many numerical algorithms such that homotopy continuation approaches [11, 12] and adaptive shifted power methods [26] are proposed in order to compute the E-eigenvalues or Z-eigenvalues of a tensor.

2.3. Orthogonal decomposition.

There are many types of tensor decompositions including CANDECOMP/PARAFAC decomposition, higher-order singular value decomposition, Tucker decomposition, and tensor train decomposition, which all play important roles in tensor algebra [14, 24, 25, 32, 33]. Tensor orthogonal decomposition is a special case of CANDECOMP/PARAFAC decomposition. A kth order suppersymmetric tensor $\mathcal{A} \in \mathbb{R}^{n \times n \times \cdots \times n}$ is called orthogonally decomposable if it can be written as a sum of vector outer products

$$\mathcal{A} = \sum_{r=1}^{n} \lambda_r \mathbf{v}_r \circ \mathbf{v}_r \circ \overset{k}{\cdots} \circ \mathbf{v}_r, \tag{2.4}$$

where $\lambda_r \in \mathbb{R}$ in the descending order, and $\mathbf{v}_r \in \mathbb{R}^n$ are orthonormal [37]. Here " $\circ$ " denotes the vector outer product.

It is easy to show that λ_r are the Z-eigenvalues of $\mathcal{A}$ with the corresponding Z-eigenvectors $\mathbf{v}_r$. Note that λ_r do not include all the Z-eigenvalues of $\mathcal{A}$, which means that λ_1 may not be the largest Z-eigenvalue of $\mathcal{A}$. Reobeva [37] speculated that orthogonally decomposable tensors satisfy a set of polynomial equations that vanish on the orthogonally decomposable variety, which is the Zariski closure of the set of orthogonally decomposable tensors inside the space of kth order n-dimensional complex supersymmetric tensors. Although the author only proved for the case when $n = 2$, she provided with strong evidence for its overall correctness [37]. Furthermore, a tensor power method was proposed in [37] in order to obtain the orthogonal decomposition of an orthogonally decomposable tensor.

3. Results. In this paper, we are interested in finding the explicit solution to a continuous-time homogeneous polynomial dynamical systems of degree $k-1$ that can be represented by

$$\dot{\mathbf{x}}(t) = \mathcal{A} \times_1 \mathbf{x}(t) \times_2 \mathbf{x}(t) \times_3 \cdots \times_{k-1} \mathbf{x}(t) = \mathcal{A}\mathbf{x}(t)^{k-1}, \tag{3.1}$$

where $\mathcal{A} \in \mathbb{R}^{n\times n\times\cdots\times n}$ is a kth order n-dimensional orthogonally decomposable tensor, and $\mathbf{x}(t) \in \mathbb{R}^n$ is the state variable. Chen [6] investigated the explicit solution and stability properties of the discrete-time version of the system (3.1) by exploiting tensor orthogonal decomposition. In the following subsections, we extend the results to the continuous-time case, which are quite different from the discrete-time case.

3.1. Explicit solutions. Finding an explicit solution of a homogeneous polynomial dynamical system is usually challenging due to its nonlinear nature. However, if a homogeneous polynomial dynamical system can be represented in the form of (3.1) with orthogonally decomposable dynamic tensor $\mathcal{A}$, we can write down its explicit solution in a simple fashion by exploiting its Z-eigenvalues and Z-eigenvectors.

PROPOSITION 3.1. *Suppose that $k \geq 3$ and $\mathcal{A} \in \mathbb{R}^{n\times n\times\cdots\times n}$ is orthogonally decomposable. Let the initial condition $\boldsymbol{x}_0 = \sum_{r=1}^n \alpha_r \boldsymbol{v}_r$. Then the explicit solution to the homogeneous polynomial dynamical system (3.1), given initial condition $\boldsymbol{x}_0$, is given by*

$$\boldsymbol{x}(t) = \sum_{r=1}^n \Big(1 - (k-2)\lambda_r\alpha_r^{k-2}t\Big)^{-\frac{1}{k-2}} \alpha_r \boldsymbol{v}_r, \tag{3.2}$$

where λ_r are the Z-eigenvalues with the corresponding Z-eigenvectors $\boldsymbol{v}_r$ in the orthgogonal decomposition of $\mathcal{A}$. Moreover, if $\lambda_r\alpha_r^{k-2} > 0$ for some r, the solution (3.2) is only defined over the interval

$$t \in \Big[0, \min_S \frac{1}{(k-2)\lambda_r\alpha_r^{k-2}}\Big), \tag{3.3}$$

where $S = \{r = 1, 2, \ldots, n | \lambda_r\alpha_r^{k-2} > 0\}$.

Proof. Since $\mathbf{v}_r$ are orthonormal, suppose that

$$\mathbf{x}(t) = \sum_{r=1}^n c_r(t)\mathbf{v}_r = \mathbf{V}\mathbf{c}(t),$$

where $\mathbf{V} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n,\end{bmatrix}$ and $\mathbf{c}(t) = \begin{bmatrix}c_1(t) & c_2(t) & \dots & c_n(t)\end{bmatrix}^\top$. Clearly, $c_r(0) = \alpha_r$ for all $r = 1, 2, \dots, n$. Based on the property of tensor vector multiplications, it can be shown that

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \Big(\sum_{r=1}^{n} \lambda_r \mathbf{v}_r \circ \mathbf{v}_r \circ \cdots \circ \mathbf{v}_r\Big) \times_1 \mathbf{x}(t) \times_2 \mathbf{x}(t) \times_3 \cdots \times_{k-1} \mathbf{x}(t)\\ &= \Big(\sum_{r=1}^{n} \lambda_r \mathbf{v}_r \circ \mathbf{v}_r \circ \cdots \circ \mathbf{v}_r\Big) \times_1 \Big(\sum_{i=1}^{n} c_i(t)\mathbf{v}_i\Big) \times_2 \Big(\sum_{i=1}^{n} c_i(t)\mathbf{v}_i\Big) \times_3 \dots\\ &\times_{k-1} \Big(\sum_{i=1}^{n} c_i(t)\mathbf{v}_i\Big) = \sum_{r=1}^{n} \lambda_r \Big\langle \mathbf{v}_r, \sum_{i=1}^{n} c_i(t)\mathbf{v}_i\Big\rangle^{k-1} \mathbf{v}_r\\ &= \sum_{r=1}^{n} \lambda_r c_r(t)^{k-1}\mathbf{v}_r.\end{aligned}$$

Thus, we have

$$\mathbf{V}\dot{\mathbf{c}}(t) = \mathbf{V}(\boldsymbol{\lambda} * \mathbf{c}(t)^{[k-1]}) \Rightarrow \dot{\mathbf{c}}(t) = \boldsymbol{\lambda} * \mathbf{c}(t)^{[k-1]} \Rightarrow \dot{c}_r(t) = \lambda_r c_r(t)^{k-1},$$

where $\boldsymbol{\lambda} = \begin{bmatrix}\lambda_1 & \lambda_2 & \dots & \lambda_n\end{bmatrix}^\top$, " $*$ " denotes the element-wise multiplication, and the superscript "$[k-1]$" denotes the element-wise $(k-1)$th power. By the method of separation of variables, we can solve for $c_r(t)$, which are given by

$$\begin{aligned}&\int c_r(t)^{-(k-1)} dc_r(t) = \int \lambda_r dt\\ &\Rightarrow c_r(t) = \Big((k-2)(w_r - \lambda_r t)\Big)^{-\frac{1}{k-2}}.\end{aligned}$$

Thus, plugging the initial condition yields

$$c_r(t) = \Big(1 - (k-2)\lambda_r \alpha_r^{k-2} t\Big)^{-\frac{1}{k-2}} \alpha_r,$$

and the result follows immediately. Moreover, if $\lambda_r \alpha_r^{k-2} > 0$ for some r, the corresponding coefficient functions $c_r(t)$ will have singularities at $t = \frac{1}{(k-2)\lambda_r \alpha_r^{k-2}}$. Thus, the domains of $c_r(t)$ are given by $t \in \Big[0, \frac{1}{(k-2)\lambda_r \alpha_r^{k-2}}\Big)$. The other branches of $c_r(t)$ over $t \in \Big(\frac{1}{(k-2)\lambda_r \alpha_r^{k-2}}, \infty\Big)$ do not satisfy the initial conditions, so they are not included in the solutions of $c_r(t)$. Therefore, the domain of the solution (3.2) will be

$$D = \bigcap_{S} \Big[0, \frac{1}{(k-2)\lambda_r \alpha_r^{k-2}}\Big) = \Big[0, \min_{S} \frac{1}{(k-2)\lambda_r \alpha_r^{k-2}}\Big),$$

where $S = \{r = 1, 2, \dots, n | \lambda_r \alpha_r^{k-2} > 0\}$. Note that if $\lambda_r \alpha_r^{k-2} \leq 0$ for all r, the domain of the solution (3.2) will be $D = [0, \infty)$. $\square$

The coefficients α_r can be found from the inner product between $\mathbf{x}_0$ and $\mathbf{v}_r$.

When $k = 2$, the result reduces to the famous linear systems' solutions, i.e.,

$$\lim_{k\to 2} \mathbf{x}(t) = \lim_{k\to 2} \sum_{r=1}^{n} \Big(1 - (k-2)\lambda_r \alpha_r^{k-2} t\Big)^{-\frac{1}{k-2}} \alpha_r \mathbf{v}_r$$
$$= \lim_{p\to\infty} \sum_{r=1}^{n} \Big(1 + \frac{\lambda_r t}{p}\Big)^{p} \alpha_r \mathbf{v}_r = \sum_{r=1}^{n} \exp\{\lambda_r t\} \alpha_r \mathbf{v}_r,$$

where λ_r become the eigenvalues of the dynamic matrix with the corresponding eigenvectors $\mathbf{v}_r$. Furthermore, based on the form of the explicit solution, we can discuss the stability properties of the homogeneous polynomial dynamical system (3.1).

3.2. Stability. In linear control theory, it is conventional to investigate so-called (internal) stability [38]. The stability of a linear dynamical system only relies on the locations of the eigenvalues of the dynamic matrix. Similarly, the equilibrium point $\mathbf{x} = \mathbf{0}$ of the homogeneous polynomial dynamical system (3.1) is called stable if $\|\mathbf{x}(t)\| \leq \gamma \|\mathbf{x}_0\|$ for some initial condition $\mathbf{x}_0$ and $\gamma > 0$, asymptotically stable if $\|\mathbf{x}(t)\| \to 0$ as $t \to \infty$, and unstable if $\|\mathbf{x}(t)\| \to \infty$ as $t \to c$ (c could be positive real numbers or infinity). Here "$\|\cdot\|$" denotes the Frobenius norm. We discover that the stability properties of the homogeneous polynomial dynamical system (3.1) with orthogonally decomposable dynamic tensor are similar to those of linear systems, but depend on both the Z-eigenvalues of $\mathcal{A}$ and initial conditions.

COROLLARY 3.2. *Suppose that $k \geq 3$ and $\mathcal{A} \in \mathbb{R}^{n\times n\times\cdots\times n}$ is orthogonally decomposable. Let the initial condition $\boldsymbol{x}_0 = \sum_{r=1}^{n} \alpha_r \boldsymbol{v}_r$. For the homogeneous polynomial dynamical system (3.1), the equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ is:*

1. *stable if and only if $\lambda_r \alpha_r^{k-2} \leq 0$ for all $r = 1, 2, \ldots, n$;*
2. *asymptotically stable if and only if $\lambda_r \alpha_r^{k-2} < 0$ for all $r = 1, 2, \ldots, n$;*
3. *unstable if and only if $\lambda_r \alpha_r^{k-2} > 0$ for some $r = 1, 2, \ldots, n$,*

where λ_r are the Z-eigenvalues in the orthogonal decomposition of $\mathcal{A}$.

Proof. First, by the triangle inequality, it can be shown that

$$\|\mathbf{x}(t)\| = \|\sum_{r=1}^{n} c_r(t)\mathbf{v}_r\| \leq \sum_{r=1}^{n} |c_r(t)| \|\mathbf{v}_r\| = \sum_{r=1}^{n} |c_r(t)|.$$

Since $\lambda_r \alpha_r^{k-2} \leq 0$ for all $r = 1, 2, \ldots, n$, the coefficient functions $|c_r(t)|$ are bounded by $|\alpha_r|$ over $t \in [0, \infty)$. Then we have

$$\|\mathbf{x}(t)\| \leq \sum_{r=1}^{n} |\alpha_r| = \|\mathbf{x}_0\|_1 \leq \sqrt{n}\|\mathbf{x}_0\|.$$

Therefore, the equilibrium point $\mathbf{x} = \mathbf{0}$ is stable. On the other hand, since $\mathbf{v}_r$ are orthonormal, $\|\mathbf{x}(t)\| = \|\mathbf{V}\mathbf{c}(t)\| = \|\mathbf{c}(t)\|$ where $\mathbf{V}$ and $\mathbf{c}(t)$ are same as defined in Proposition 3.1. If $\|\mathbf{x}(t)\| = \|\mathbf{c}(t)\| \leq \gamma\|\mathbf{x}_0\|$, all the coefficient functions $c_r(t)$ must be bounded for $t \geq 0$. Thus, $\lambda_r \alpha_r^{k-2}$ must lie in the left-half plane for all $r = 1, 2, \ldots, n$.

Second, if $\lambda_r \alpha_r^{k-2} < 0$ for all $r = 1, 2, \ldots, n$, the coefficient functions $c_r(t)$ follow that $\lim_{t\to\infty} |c_r(t)| = 0$. Therefore,

$$\lim_{t\to\infty} \|\mathbf{x}(t)\| \leq \lim_{t\to\infty} \sum_{r=1}^{n} |c_r(t)| = \sum_{r=1}^{n} \lim_{t\to\infty} |c_r(t)| = 0,$$

and the equilibrium point $\mathbf{x} = \mathbf{0}$ is asymptotically stable. On the other hand, if $\lim_{t\to\infty} \|\mathbf{x}(t)\| = \lim_{t\to\infty} \|\mathbf{c}(t)\| = 0$, all the coefficient functions $c_r(t)$ should satisfy that $\lim_{t\to\infty} |c_r(t)| = 0$. Thus, $\lambda_r \alpha_r^{k-2}$ must lie in the open left-half plane for all $r = 1, 2, \ldots, n$.

Finally, if $\lambda_r \alpha_r^{k-2} > 0$ for some $r = 1, 2, \ldots, n$, the corresponding coefficient functions $c_r(t)$ will have singularities at $t_s = \frac{1}{(k-2)\lambda_r \alpha_r^{k-2}}$ such that $\lim_{t\to t_s} c_r(t) = \infty$. Hence,

$$\lim_{t\to \min t_s} \|\mathbf{x}(t)\| = \lim_{t\to \min t_s} \|\mathbf{c}(t)\| = \infty,$$

and the equilibrium point $\mathbf{x} = \mathbf{0}$ is unstable. On the other hand, if $\lim_{t\to c} \|\mathbf{x}(t)\| = \lim_{t\to c} \|\mathbf{c}(t)\| = \infty$ for some $c > 0$, the coefficient functions $c_r(t)$ should satisfy that $\lim_{t\to c} |c_r(t)| = \infty$ for some r. Since c cannot be infinity, those $c_r(t)$ must have the singularities at $t = c$. Therefore, we have $\lambda_r \alpha_r^{k-2} = \frac{1}{(k-2)c} > 0$ for some r. ◻

When $k = 2$, the above conditions reduce to the famous linearity stability conditions. The inequalities obtained from the asymptotic stability condition can provide us with the region of attraction of the homogeneous polynomial dynamical system (3.1), i.e.,

$$R = \{\mathbf{x} : \lambda_r \alpha_r^{k-2} < 0 \text{ where } \mathbf{x} = \sum_{r=1}^{n} \alpha_r \mathbf{v}_r\}, \tag{3.4}$$

where $\mathbf{v}_r$ are the Z-eigenvectors in the orthogonal decomposition of $\mathcal{A}$ corresponding to the Z-eigenvalues λ_r. Furthermore, when k is even, α_r^{k-2} will be always greater than or equal to zero. Thus, the stability conditions can be simplified for the homogeneous polynomial dynamical system (3.1) of odd degree.

COROLLARY 3.3. *Suppose that $k \geq 4$ is even and $\mathcal{A} \in \mathbb{R}^{n\times n\times\cdots\times n}$ is orthogonally decomposable. For the homogeneous polynomial dynamical system (3.1), the equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ is:*

1. *stable if and only if $\lambda_r \leq 0$ for all $r = 1, 2, \ldots, n$;*
2. *asymptotically stable if and only if $\lambda_r < 0$ for all $r = 1, 2, \ldots, n$;*
3. *unstable if and only if $\lambda_r > 0$ for some $r = 1, 2, \ldots, n$,*

where λ_r are the Z-eigenvalues in the orthogonal decomposition of $\mathcal{A}$.

Proof. The results follow immediately from Corollary 3.2 when k is even. ◻

When k is even, the stability conditions are exactly same as those of linear systems, i.e., the homogeneous polynomial dynamical system (3.1) of odd degree is globally stable if and only if all the Z-eigenvalues λ_r from the orthogonal decomposition of $\mathcal{A}$ lie in the left-half plane. On the other hand, computing the orthogonal decomposition or Z-eigenvalues of a supersymmetric tensor is NP-hard [18, 37]. If we know an upper bound of the largest Z-eigenvalue of a supersymmetric tensor, it will save a great amount of computations for determining the stability of the homogeneous polynomial dynamical systems (3.1). Chen [6] found that the largest Z-eigenvalue of an even-order supersymmetric tensor is upper bounded by the largest eigenvalue of one of its unfolded matrices.

LEMMA 3.4. *Let $\mathcal{A} \in \mathbb{R}^{n\times n\times\cdots\times n}$ be an even-order supersymmetric tensor. Then the largest Z-eigenvalue $\lambda_{\max}$ of $\mathcal{A}$ is upper bounded by $\mu_{\max}$ where $\mu_{\max}$ is the largest eigenvalue of $\varphi(\mathcal{A})$ defined by:*

$$\boldsymbol{A} = \varphi(\mathcal{A}) \text{ such that } \mathcal{A}_{j_1 i_1 \ldots j_k i_k} \xrightarrow{\varphi} \boldsymbol{A}_{ji}, \tag{3.5}$$

with $j = j_1 + \sum_{p=2}^{k}(j_p - 1)n^{p-1}$ *and* $i = i_1 + \sum_{p=2}^{k}(i_p - 1)n^{p-1}$.

COROLLARY 3.5. *Suppose that* $k \geq 4$ *is even and* $\mathcal{A} \in \mathbb{R}^{n\times n\times\cdots\times n}$ *is orthogonally decomposable. For the homogeneous polynomial dynamical system (3.1), the equilibrium point* $\boldsymbol{x} = \boldsymbol{0}$ *is:*

1. *stable if* $\mu_{\max} \leq 0$*;*
2. *asymptotically stable if* $\mu_{\max} < 0$,

where $\mu_{\max}$ *is the largest eigenvalue of* $\varphi(\mathcal{A})$ *defined in (3.5).*

Proof. Based on Lemma 3.4, we know that $\lambda_1 \leq \lambda_{\max} \leq \mu_{\max}$. Therefore, the result follows immediately from Corollary 3.3. ∎

Note that λ_1 is the largest Z-eigenvalue in the orthogonal decomposition of $\mathcal{A}$, while $\lambda_{\max}$ is the largest Z-eigenvalue of $\mathcal{A}$. There are many other upper bounds for the largest Z-eigenvalue or Z-spectral radius of a supersymmetric tensor [5, 17, 31, 44]. Given an orthogonally decomposable dynamic tensor, the better upper bound of the largest Z-eigenvalue, the more strong stability conditions we can obtain.

3.3. Constant inputs case. In this subsection, we consider the homogeneous polynomial system (3.1) with constant inputs, i.e.,

$$\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t)^{k-1} + \mathbf{b}, \tag{3.6}$$

where $\mathcal{A} \in \mathbb{R}^{n\times n\times\cdots\times n}$ is a kth order n-dimensional orthgonally decomposable tensor, and $\mathbf{b} \in \mathbb{R}^n$ is a constant input vector. We find that the complete solution to this polynomial dynamical system (3.6) can be solved implicitly by using Gauss hypergeometric functions.

PROPOSITION 3.6. *Suppose that* $k \geq 3$ *and* $\mathcal{A} \in \mathbb{R}^{n\times n\times\cdots\times n}$ *is orthogonally decomposable. Let* $\boldsymbol{x}(t) = \sum_{r=1}^{n} c_r(t)\boldsymbol{v}_r$ *with initial conditions* $c_r(0) = \alpha_r$*. For the polynomial dynamical system (3.6), the coefficient functions* $c_r(t)$ *can be solved implicitly by*

$$t = -\frac{g\left(\frac{k-2}{k-1}, -\frac{\tilde{b}_r}{\lambda_r c_r(t)^{k-1}}\right)}{(k-2)\lambda_r c_r(t)^{k-2}} + \frac{g\left(\frac{k-2}{k-1}, -\frac{\tilde{b}_r}{\lambda_r \alpha_r^{k-1}}\right)}{(k-2)\lambda_r \alpha_r^{k-2}}, \tag{3.7}$$

where λ_r *are the Z-eigenvalues with the corresponding Z-eigenvectors* $\boldsymbol{v}_r$ *in the orthgogonal decomposition of* $\mathcal{A}$*, and* $g(\cdot,\cdot)$ *is the specified Gauss hypergeometric function [20] defined by*

$$g(a, z) = {}_2F_1(1, a; a+1; z) = a\sum_{m=0}^{\infty}\frac{z^m}{a+m}.$$

Proof. Since $\mathbf{x}(t) = \sum_{r=1}^{n} c_r(t)\mathbf{v}_r$, we can rewrite the polynomial dynamical system (3.6) as follows:

$$\mathbf{V}\dot{\mathbf{c}}(t) = \mathbf{V}(\boldsymbol{\lambda} * \mathbf{c}(t)^{[k-1]}) + \mathbf{V}\mathbf{V}^\top\mathbf{b} \Rightarrow \dot{\mathbf{c}}(t) = \boldsymbol{\lambda} * \mathbf{c}(t)^{[k-1]} + \tilde{\mathbf{b}},$$

where $\tilde{\mathbf{b}} = \mathbf{V}^\top\mathbf{b}$. Therefore, for each coefficient function $c_r(t)$, we have

$$\dot{c}_r(t) = \lambda_r c_r(t)^{k-1} + \tilde{b}_r, \tag{3.8}$$

where $\tilde{b}_r$ is the rth entry of $\tilde{\mathbf{b}}$. The differential equation (3.8) is a particular form of the Chini's equation [23], and can be solved implicitly by using Gauss hypergeometric functions. Based on the method of separation of variables, it can be shown that

$$\int \frac{1}{\lambda_r c_r(t)^{k-1} + \tilde{b}_r} dc_r(t) = \int 1 dt \Rightarrow -\frac{g\left(\frac{k-2}{k-1}, -\frac{\tilde{b}_r}{\lambda_r c_r(t)^{k-1}}\right)}{(k-2)\lambda_r c_r(t)^{k-2}} = t + w_r.$$

Plugging the initial conditions yields

$$t = -\frac{g\left(\frac{k-2}{k-1}, -\frac{\tilde{b}_r}{\lambda_r c_r(t)^{k-1}}\right)}{(k-2)\lambda_r c_r(t)^{k-2}} + \frac{g\left(\frac{k-2}{k-1}, -\frac{\tilde{b}_r}{\lambda_r \alpha_r^{k-1}}\right)}{(k-2)\lambda_r \alpha_r^{k-2}},$$

and the proof is complete. □

The solutions of $c_r(t)$ can be further solved by any nonlinear solver given a specific time point t. We then can recover the complete solution of $\mathbf{x}(t)$ based on the values of $c_r(t)$. Moreover, although $g(a, z)$ is defined for $|z| < 1$, it can be analytically continued along any path in the complex plane that avoids the branch points one and infinity [15]. When $k = 3$, the differential equation (3.8) is also known as the Riccati equation, which can be converted to a second-order linear system. Furthermore, we can use the implicit solutions to determine the system properties of the dynamics of $c_r(t)$ at some particular points. Denote the implicit solution (3.7) by $t = f(c_r) + \beta$. For example, if we consider c_r approaches to infinity, it can be shown that

$$\lim_{c_r \to \pm\infty} f(c_r) + \beta = \beta.$$

Thus, if the second terms β in (3.7) are positive for some r, the domains of the coefficient functions $c_r(t)$ will be $[0, \beta)$. We can therefore conclude that the dynamical systems of the coefficient functions are unstable, which can be used for inferring the system properties of the original polynomial dynamical system.

4. Numerical examples. All the numerical examples presented were performed on a Macintosh machine with 16 GB RAM and a 2 GHz Quad-Core Intel Core i5 processor in MATLAB R2020b.

4.1. Explicit solutions. In this example, we try to compute the explicit solution of a homogeneous polynomial dynamical system, and compare it to the trajectory using the MATLAB ODE45 solver. Given a following 3-dimensional homogeneous polynomial dynamical system of degree two

$$\begin{cases} \dot{x}_1 &= 0.0962x_1^2 + 0.0291x_2^2 + 0.0957x_3^2 - 0.0170x_1x_2 - 0.0048x_1x_3 - 0.0322x_2x_3 \\ \dot{x}_2 &= -0.0085x_1^2 + 0.1840x_2^2 + 0.0992x_3^2 + 0.0582x_1x_2 - 0.0322x_1x_3 + 0.0474x_2x_3 \\ \dot{x}_3 &= -0.0024x_1^2 + 0.0237x_2^2 - 0.4400x_3^2 - 0.0322x_1x_2 + 0.1914x_1x_3 + 0.1984x_2x_3 \end{cases},$$

it can be represented in the form of (3.1) with

$$\mathcal{A}_{::1} = \begin{bmatrix} 0.0962 & -0.0085 & -0.0024 \\ -0.0085 & 0.0291 & -0.0161 \\ -0.0024 & -0.0161 & 0.0957 \end{bmatrix}, \ \mathcal{A}_{::2} = \begin{bmatrix} -0.0085 & 0.0291 & -0.0161 \\ 0.0291 & 0.1844 & 0.0237 \\ -0.0161 & 0.0237 & 0.0992 \end{bmatrix},$$
$$\mathcal{A}_{::3} = \begin{bmatrix} -0.0024 & -0.0161 & 0.0957 \\ -0.0161 & 0.0237 & 0.0992 \\ 0.0957 & 0.0992 & -0.4402 \end{bmatrix},$$

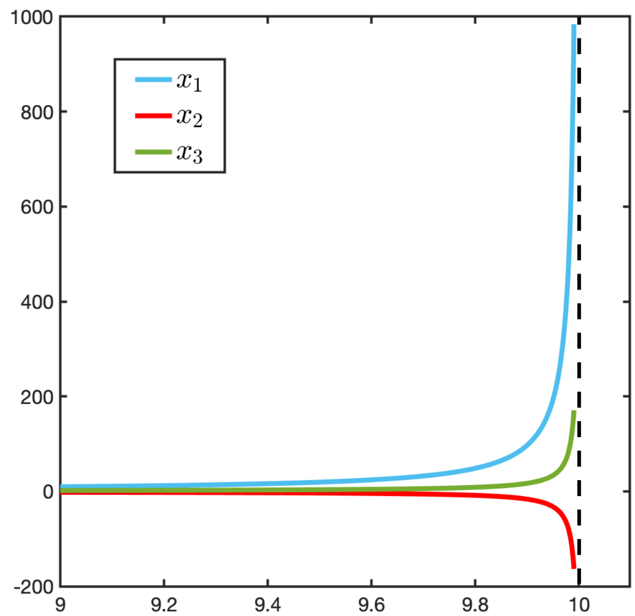

FIG. 1. *Trajectories of the homogeneous polynomial dynamical system with the initial condition* $\boldsymbol{x}_0 = \begin{bmatrix} 0.6516 & -1.3239 & 0.9070 \end{bmatrix}^\top$ *using the MATLAB ODE45 solver.*

such that $\mathcal{A}$ is orthogonally decomposable. Thus, we can write down the explicit solution of the dynamical system according to Proposition 3.1, which is given by

$$\mathbf{x}(t) = \frac{\alpha_1}{1+0.5\alpha_1 t}\begin{bmatrix}-0.1990\\-0.1953\\0.9603\end{bmatrix} + \frac{\alpha_2}{1+0.2\alpha_2 t}\begin{bmatrix}-0.1218\\-0.9674\\-0.2220\end{bmatrix} + \frac{\alpha_3}{1-0.1\alpha_3 t}\begin{bmatrix}0.9724\\-0.1612\\0.1687\end{bmatrix},$$

where α_r can be determined by initial conditions. The results are shown in Table 1, in which we compute the state coordinates for the initial condition $\mathbf{x}_0 = \begin{bmatrix} 0.6516 & -1.3239 & 0.9070 \end{bmatrix}^\top$ with $\alpha_r = 1$ for $r = 1, 2, 3$ at $t = 2, 4, 6, 8$. The domain of the solution is given by $[0, 10)$. It is evident that the state coordinates are very close at each time point between the two approaches. Moreover, since the equilibrium point is unstable given this initial condition, we also see an increasing trend in the Frobenius norm of the state as t approaches to ten, see Table 1 and Figure 1.

4.2. Stability. In this example, we try to verify the stability results discussed in Corollary 3.3. Given a following 2-dimensional homogeneous polynomial dynamical system of degree three

$$\begin{cases} \dot{x}_1 &= -1.2593x_1^3 + 1.6630x_1^2x_2 - 1.5554x_1x_2^2 - 0.1386x_2^3 \\ \dot{x}_2 &= 0.5543x_1^3 - 1.5554x_1^2x_2 - 0.4158x_1x_2^2 - 0.7037x_2^3 \end{cases},$$

TABLE 1

Trajectories of the homogeneous polynomial dynamical system using the explicit solution formula and the MATLAB ODE45 solver. We also report the relative errors between the two trajectories.

Time	$t=0$	$t=2$	$t=4$	$t=6$	$t=8$
	0.6516	1.0290	1.4867	2.3259	4.7753
Explicit Sol.	-1.3239	-0.9901	-0.8712	-0.8915	-1.2170
	0.9070	0.5325	0.4779	0.5609	0.9502
	0.6516	1.0290	1.4866	2.3259	4.7793
ODE45 Solver	-1.3239	-0.9902	-0.8712	-0.8915	-1.2177
	0.9070	0.5325	0.4780	0.5610	0.9509
Relative Error	0	5.1×10^{-5}	2.3×10^{-5}	1.8×10^{-5}	8.1×10^{-4}

TABLE 2

The Frobenius norm of $\mathbf{x}(t)$ for the five initial conditions at $t=0,10,10^2,10^3,10^4,10^5,10^6$.

Time	$t=0$	$t=10$	$t=10^2$	$t=10^3$	$t=10^4$	$t=10^5$	$t=10^6$
IC 1	1.4142	0.2655	0.0864	0.0274	0.0087	0.0027	8.7×10^{-4}
IC 2	50.9902	0.2740	0.0867	0.0274	0.0087	0.0027	8.7×10^{-4}
IC 3	104.4031	0.2740	0.0867	0.0274	0.0087	0.0027	8.7×10^{-4}
IC 4	203.9608	0.2740	0.0867	0.0274	0.0087	0.0027	8.7×10^{-4}
IC 5	1280.6248	0.2740	0.0867	0.0274	0.0087	0.0027	8.7×10^{-4}

it can be represented in the form of (3.1) with

$$\mathcal{A}_{::11}=\begin{bmatrix}-1.2593 & 0.5543\\ 0.5543 & -0.5185\end{bmatrix},\ \mathcal{A}_{::12}=\begin{bmatrix}0.5543 & -0.5185\\ -0.5185 & -0.1386\end{bmatrix},$$
$$\mathcal{A}_{::21}=\begin{bmatrix}0.5543 & -0.5185\\ -0.5185 & -0.1386\end{bmatrix},\ \mathcal{A}_{::22}=\begin{bmatrix}-0.5185 & -0.1386\\ -0.1386 & -0.7037\end{bmatrix},$$

such that $\mathcal{A}$ is orthogonally decomposable. The two Z-eigenvalues in the orthogonal decomposition of $\mathcal{A}$ are $\lambda_1=-1$ and $\lambda_2=-2$. Therefore, according to Corollary 3.3, the homogeneous polynomial dynamical system is asymptotically stable for arbitrary initial conditions. The results are shown in Table 2, in which we compute the Frobenius norm of $\mathbf{x}(t)$ for five random initial conditions at $t=0,10,10^2,10^3,10^4,10^5,10^6$. The five initial conditions are given by

$$\textbf{IC 1}=\begin{bmatrix}1\\1\end{bmatrix},\ \textbf{IC 2}=\begin{bmatrix}10\\50\end{bmatrix},\ \textbf{IC 3}=\begin{bmatrix}100\\30\end{bmatrix},$$
$$\textbf{IC 4}=\begin{bmatrix}-40\\-200\end{bmatrix},\ \text{and } \textbf{IC 5}=\begin{bmatrix}-1000\\800\end{bmatrix}.$$

It is clear to see that all the trajectories of the homogeneous polynomial dynamical system with the five initial conditions converge to the origin. In addition, we plot the vector field of the dynamical system, which also indicates that the equilibrium point $\mathbf{x}=\mathbf{0}$ is asymptotically stable, see Figure 2.

4.3. Stability using the upper bound. In this example, we try to apply the upper bound of the largest Z-eigenvalue defined in (3.5) to determine the stability of the homogeneous polynomial dynamical system defined in the last example. The

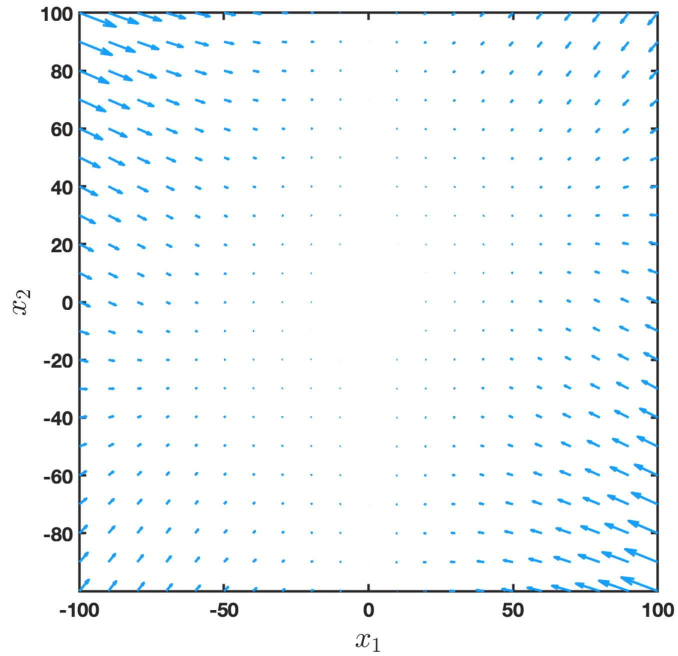

FIG. 2. *Vector field plot of the homogeneous polynomial dynamical system.*

unfolded matrix $\varphi(\mathcal{A})$ is given by

$$\varphi(\mathcal{A}) = \begin{bmatrix} -1.2593 & 0.5543 & 0.5543 & -0.5185 \\ 0.5543 & -0.5185 & -0.5185 & -0.1386 \\ 0.5543 & -0.5185 & -0.5185 & -0.1386 \\ -0.5185 & -0.1386 & -0.1386 & -0.7037 \end{bmatrix}.$$

The maximum eigenvalue of the unfolded matrix is $\mu_{\max} = 0$. Therefore, according to Corollary 3.5, the homogeneous polynomial dynamical system is stable at the equilibrium point $\mathbf{x} = \mathbf{0}$ (although we know that the system is actually asymptotically stable from the last example).

4.4. Constant inputs case. In this example, we try to solve the complete solution of a homogeneous polynomial dynamical system with constant inputs which is given by

$$\begin{cases} \dot{x}_1 & = 1.2593x_1^3 - 1.6630x_1^2x_2 + 1.5554x_1x_2^2 + 0.1386x_2^3 - 0.4105 \\ \dot{x}_2 & = -0.5543x_1^3 + 1.5554x_1^2x_2 + 0.4158x_1x_2^2 + 0.7037x_2^3 + 1.3533 \end{cases}.$$

The above dynamics can be represented in the form of (3.6) such that $\mathcal{A} \in \mathbb{R}^{2\times2\times2\times2}$ is orthogonally decomposable, and $\mathbf{b} = \begin{bmatrix} -0.4105 & 1.3533 \end{bmatrix}^\top$. The Z-eigenvalues in the orthogonal decompostion of $\mathcal{A}$ are $\lambda_1 = 2$ and $\lambda_2 = 1$ with the corresponding Z-eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -0.8819 & 0.4717 \end{bmatrix}^\top$ and $\mathbf{v}_2 = \begin{bmatrix} 0.4714 & 0.8819 \end{bmatrix}^\top$. Suppose that the initial condition is $\mathbf{x}_0 = \begin{bmatrix} -0.4105 & 1.3533 \end{bmatrix}^\top$. Then the differential equations for

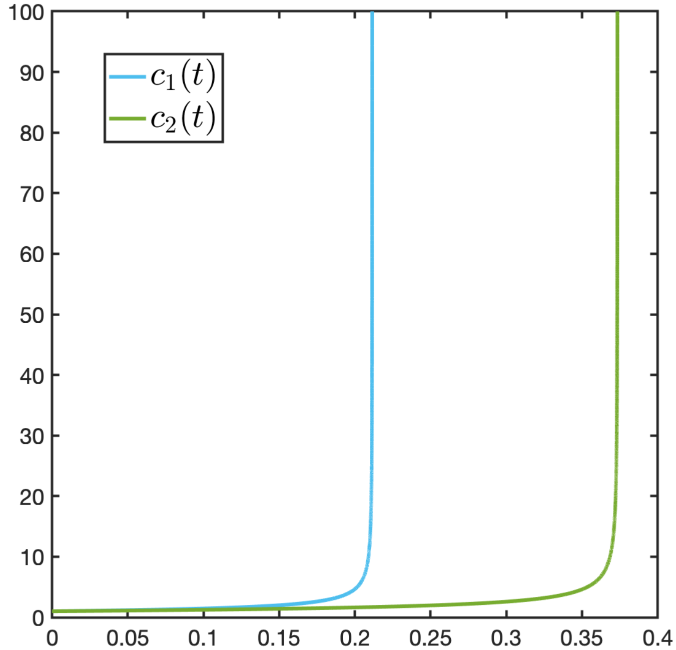

FIG. 3. *Trajectories of the coefficient functions $c_1(t)$ and $c_2(t)$ with initial conditions $c_1(0) = c_2(0) = 1$. Here we only show the branches that satisfy the initial conditions.*

the coefficient functions $c_r(t)$ are given by

$$\begin{cases} \dot{c}_1(t) & = 2c_1(t)^3 + 1 \\ \dot{c}_2(t) & = c_2(t)^3 + 1 \end{cases}$$

with $c_1(0) = c_2(0) = 1$. Thus, the coefficient functions $c_1(t)$ and $c_2(t)$ can be solved implicitly by

$$t = -\frac{g\Big(\frac{2}{3}, -\frac{1}{2c_1(t)^3}\Big)}{4c_1(t)^2} + \frac{g\Big(\frac{2}{3}, -\frac{1}{2}\Big)}{4} \text{ and } t = -\frac{g\Big(\frac{2}{3}, -\frac{1}{c_2(t)^3}\Big)}{2c_2(t)^2} + \frac{g\Big(\frac{2}{3}, -1\Big)}{2},$$

respectively. The trajectories of $c_1(t)$ and $c_2(t)$ are shown in Figure 3, both of which increase rapidly as t approaches to $g\Big(\frac{2}{3}, -\frac{1}{2}\Big)/4$ and $g\Big(\frac{2}{3}, -1\Big)/2$, respectively. This implies that the original polynomial dynamical system is unstable. In addition, we compare the complete solution solved from the implicit equations of $c_r(t)$ to the trajectory using the MATLAB ODE45 solver, in which the solution is defined over $t \in \Big[0, g\Big(\frac{2}{3}, -\frac{1}{2}\Big)/4\Big)$. It is clear that the state coordinates are very close at each time point between the two approaches, see in Table 3.

5. Discussion. In linear systems theory, it is important to investigate the controllability and observability of a dynamical system. The controllability of homogeneous polynomial dynamical systems was studied extensively back in 1970s and 80s

Table 3
Trajectories of the polynomial dynamical system using the implicit solution equations of $c_r(t)$ and the MATLAB ODE45 Solver. We also report the relative errors between the two trajectories.

Time	$t = 0$	$t = 0.05$	$t = 0.1$	$t = 0.15$	$t = 0.2$
Complete Sol.	-0.4105 1.3533	-0.5165 1.5330	-0.6965 1.7773	-1.0941 2.1768	-3.3331 3.6261
ODE45 Solver	-0.4105 1.3533	-0.5165 1.5330	-0.6965 1.7773	-1.0941 2.1768	-3.3360 3.6277
Relative Error	0	4.1×10^{-7}	6.0×10^{-7}	1.2×10^{-5}	6.7×10^{-4}

[3, 4, 22, 40]. In particular, Jurdjevic and Kupka [22] obtained strong results in terms of the controllability of homogeneous polynomial dynamical systems with constant input multipliers. In addition, Chen *et al.* [9] showed that the Lie algebra-based rank condition from [22] can be represented using tensor vector multiplications for determining the controllability of the system (3.1). We are interested in exploring the homogeneous polynomial dynamical system with linear inputs and outputs

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathcal{A}\mathbf{x}(t)^{k-1} + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{cases}, \tag{5.1}$$

where $\mathcal{A} \in \mathbb{R}^{n\times n\times\cdots\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, and $\mathbf{C} \in \mathbb{R}^{p\times n}$. $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, and $\mathbf{y}(t) \in \mathbb{R}^p$ are the state, input, and output variables, respectively. Since the result of controllability has been provided in [9], we here simply discuss the observability of the dynamical system (5.1) by the duality principal.

Definition 5.1. *Let $\mathcal{O}_0$ be the linear span of $\{\boldsymbol{c}_1^\top, \boldsymbol{c}_2^\top, \ldots, \boldsymbol{c}_p^\top\}$ ($\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_p \in \mathbb{R}^n$ are the row vectors of $\boldsymbol{C}$) and $\mathcal{A} \in \mathbb{R}^{n\times n\times\cdots\times n}$ be a supersymmetric tensor. For each integer $q \geq 1$, define $\mathcal{O}_q$ inductively as the linear span of*

$$\mathcal{O}_{q-1} \cup \{\mathcal{A}\boldsymbol{v}_1\boldsymbol{v}_2\ldots\boldsymbol{v}_{k-1} | \boldsymbol{v}_l \in \mathcal{O}_{q-1}\}. \tag{5.2}$$

Denote the subspace $\mathcal{O}(\mathcal{A}, \boldsymbol{C}) = \cup_{q\geq 0}\mathcal{O}_q$.

Proposition 5.2. *Suppose that k is even. The dynamical system (5.1) is observable if and only if the subspace $\mathcal{O}(\mathcal{A}, \boldsymbol{C})$ spans $\mathbb{R}^n$, or equivalently, the matrix $\boldsymbol{O}$, including all the column vectors from $\mathcal{O}(\mathcal{A}, \boldsymbol{C})$, has rank n.*

Proof. According to the duality principal, the proof can be formulated similarly to Corollary 1 in [9]. ∎

Since $\mathcal{O}(\mathcal{A}, \mathbf{C})$ is a finite-dimensional vector space, there exists an integer $q \leq n$ such that $\mathcal{O}(\mathcal{A}, \mathbf{C}) = \mathcal{O}_q$. When $k = 2$ and $q = n - 1$, it reduces to the famous Kalman rank condition for observability. Controllability and observability are related to the stability of an input-output dynamical system with the corresponding notions - stabilizability and detectability. It will be interesting to develop the state feedback and observer design frameworks for the polynomial dynamical system (5.1) according to the controllability and observability conditions.

6. Conclusion. This paper investigated the explicit solutions and stability properties of certain continuous-time homogeneous polynomial dynamical systems that can be represented by orthogonally decomposable tensors. We derived an explicit solution formula using the Z-eigenvalues and Z-eigenvectors from the orthogonal decomposition of the corresponding dynamic tensors. By utilizing the form of the explicit solutions, the stability properties of such homogeneous polynomial dynamical systems could be formalized. In particular, the Z-eigenvalues can offer can offer necessary and sufficient stability conditions. Furthermore, we explored the complete solutions of such homogeneous polynomial dynamical systems with constant inputs. Future work will explore stability conditions for homogeneous polynomial dynamical systems with non-orthogonally decomposable dynamic tensors or even non-supersymmetric dynamic tensors. We will also explore the potential of tensor algebra-based computations for Lyapunov equations and Lyapunov stability. This will be particularly important for applications in the robotic context [30, 42, 45]. As mentioned in section 5, stabilizability and detectability are also important for future research.

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