On the Stability of Multilinear Dynamical Systems *

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Abstract. In this paper, we investigate the stability properties of discrete-time multilinear dynamical systems. We establish theoretical results on the criteria for determining the internal stability of multilinear dynamical systems via tensor spectral theory. In particular, we show that tensor Z-eigenvalues play a significant role in the stability analysis offering necessary and sufficient conditions if the dynamic tensor is orthogonal decomposable (odeco). Moreover, we build an upper bound for the Z-spectral radii of even-order supersymmetric tensors by tensor unfolding, and apply it to determine the stability efficiently with help of tensor train decomposition. We also discuss the internal stability of multilinear dynamical systems with non-odeco dynamic tensors by exploiting the tensor Frobenius norm and tensor singular values. Furthermore, we explore the Lyapunov stability of multilinear dynamical systems. Finally, we demonstrate our results with numerical examples.

Key words. multilinear dynamical systems, stability, regions of attraction, Z-eigenvalues, tensors, homogeneous polynomials, orthogonal decomposition, tensor train decomposition

AMS subject classifications. 15A18, 15A69, 37N30, 39A30, 65P40, 93D05, 93D20

1. Introduction.

The role of tensor algebra has been explored for modeling and simulation of linear and nonlinear dynamics [8, 9, 10, 14, 21, 24]. The key idea is to tensorize the vector-based dynamical system representation into an equivalent tensor representation, and to exploit tensor algebra. Tensor decomposition techniques such that CANDECOMP/PARAFAC decomposition, higher-order singular value decomposition, and tensor train decomposition, are applied for reducing memory usages and enabling efficient computations in multilinear dynamical systems theory [10, 14]. Gelß [14] applied tensor decompositions for computing numerical solutions of master equations associated with Markov processes on extremely large state spaces, developing tensor representations for operators based on nearest-neighbor interactions, construction of pseudoinverses for dimensionality reduction methods, and the approximation of transfer operators of dynamical systems. In addition, Chen *et al.* [8, 10] developed the tensor algebraic conditions for stability, reachability, and observability for input/output multilinear time-invariant systems, and expressed them in terms of more standard notions of tensor ranks/decompositions to facilitate efficient computations.

Many complex systems such as those arising in biology and engineering can be studied using a network prospective [4, 9, 20, 28, 34, 35, 42]. Most real world data representations are multidimensional, and using graph models to characterize them may cause a loss of higher-order information [7, 9, 43]. Recently, a new tensor-based continuous-time multilinear dynamical system representation with linear control inputs (different from the ones proposed in [8, 10] which can be unfolded to linear dynamical systems via tensor unfolding, an operation that transforms a tensor into a matrix) was proposed by Chen *et al.* [9] for characterizing the

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multidimensional state dynamics of hypergraphs, a generalization of graphs in which edges can contain more than one nodes. The authors derived a Kalman-rank-like condition to determine the minimum number of control nodes needed to achieve controllability/accessibility of uniform hypergraphs. They also proposed minimum number of control nodes as a measure of hypergraph robustness, and found that it is related to the hypergraph degree distribution. The mulitlinear dynamical system evolution, inspired by hypergraphs, is described by the action of tensor vector multiplications between a dynamic tensor and the state vector. As a matter of fact, the multilinear dynamical system belongs to the family of homogeneous polynomial dynamical systems if one expands the tensor vector multiplications.

The stability of homogeneous polynomial dynamical systems is one of the most challenging problems in control theory due to its nature of nonlinearity [1, 2, 18, 37, 39]. In 1983, Samardzija [37] established a necessary and sufficient condition for asymptotic stability in 2-dimensional homogeneous polynomial dynamical systems by formulating a generalized characteristic value problem. In 2019, Ali and Khadir [1] showed that existence of a rational Lyapunov function is necessary and sufficient for asymptotic stability of a homogeneous continuously polynomial dynamical system, and the Lyapunov function can be solved using semidefinite programming. However, the semidefinite programming problem depends on the two degree parameters, and one has to try all possible combinations of the parameters in order to obtain the Lyapunov function. On the other hand, when a homogeneous polynomial dynamical system has degree one, the stability properties can be obtained simply from the locations of the eigenvalues of the dynamic matrix, known as the linear stability. It is therefore conceived that tensor eigenvalues may have the potential to be used for determining the stability properties of homogeneous polynomial dynamical systems.

Tensor eigenvalue problems of real supersymmetric tensors were first explored by Qi [32, 33] and Lim [25] independently in 2005. There are many different notions of tensor eigenvalues such as H-eigenvalues, Z-eigenvalues, M-eigenvalues, and U-eigenvalues [10, 32, 33], which are similar to matrix eigenvalues in different senses. Chen *et al.* [12] showed that the Z-eigenvector associated with the second smallest Z-eigenvalue of a normalized Laplacian tensor can be used for hypergraph partition. In addition, Huang and Qi [17] used M-eigenvalues to prove the strong ellipticity of elasticity tensors in solid mechanics. Furthermore, Chen *et al.* [10] utilized U-eigenvalues to determine the stability of multilinear time-invariant systems (first type of multilienar dynamical systems discussed above). Of particular interest of this paper are Z-eigenvalues. We explore the stability properties of the discrete-time version of the multilinear dynamical system proposed in [9] via tensor spectral theory. The multilinear dynamical systems can be used to capture the discrete-time dynamics of higher-order networks or hypergraphs such as coauthorship networks, film actor/actress networks, brain neural networks, and protein-protein interaction networks [7, 29, 40, 43]. The key contributions of the paper are as follows:

- We establish theoretical results on the criteria for determining the internal stability of multilinear dynamical systems with orthogonal decomposable (odeco) dynamic tensors by exploiting its Z-eigenvalues. Based on the stability conditions, we can obtain the regions of attraction of multilinear dynamical systems.
- We build an upper bound for the Z-spectral radii of even-order supersymmetric tensors by tensor unfolding, and apply it to determine the internal stability of multilinear

dynamical systems without computing orthogonal decomposition. We exploit tensor train decomposition to accelerate the computation of the upper bound.
- We extend the stability analysis to multilinear dynamical systems with general dynamic tensors by using the tensor Frobenius norm and tensor singular values. Similarly, we apply tensor train decomposition to gain computational efficiency.
- We perform preliminary explorations of the Lyapunov stability of multilinear dynamical systems, analogous to the construction of quadratic Lyapunov functions for linear dynamical systems.
- We verify our results on four numerical examples.

Note that all the results are applicable to homogeneous polynomial dynamical systems if one can find the corresponding multilinear dynamical systems.

The paper is organized into eight sections. In section 2, we review tensor preliminaries including tensor products, tensor decompositions, and tensor eigenvalues. Section 3 introduces the notion of odeco tensors and a discrete-time multilinear dynamical system representation with explicit solution. We establish the internal stability criteria for multilinear dynamical systems with odeco and non-odeco dynamic tensors in section 4, and explore the Lyapunov stability conditions in section 5. We verify our results with numerical examples in section 6. We discuss the stabilizability and reachability of multilinear dynamical systems with control in section 7, and conclude in section 8 with future research directions.

2. Tensor preliminaries.

A tensor is a multidimensional array [7, 8, 10, 14, 22, 23]. The order of a tensor is the number of its dimensions, and each dimension is called a mode. A kth order tensor usually is denoted by $\mathsf{T} \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_k}$. It is therefore reasonable to consider scalars $x \in \mathbb{R}$ as zero-order tensors, vectors $\mathbf{v} \in \mathbb{R}^n$ as first-order tensors, and matrices $\mathbf{M} \in \mathbb{R}^{m\times n}$ as second-order tensors. A tensor is called cubical if every mode is the same size, i.e., $\mathsf{T} \in \mathbb{R}^{n\times n\times\cdots\times n}$. A cubical tensor T is called supersymmetric if $\mathsf{T}_{j_1j_2\dots j_k}$ is invariant under any permutation of the indices, and is called diagonal if $\mathsf{T}_{j_1j_2\dots j_k} = 0$ except $j_1 = j_2 = \cdots = j_k$.

2.1. Tensor products.

The inner product of two tensors $\mathsf{T}, \mathsf{S} \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_k}$ is defined as

$$\langle \mathsf{T}, \mathsf{S}\rangle = \sum_{j_1=1}^{n_1}\cdots\sum_{j_k=1}^{n_k} \mathsf{T}_{j_1j_2\dots j_k}\mathsf{S}_{j_1j_2\dots j_k}, \tag{2.1}$$

leading to the tensor Frobenius norm $\|\mathsf{T}\|^2 = \langle \mathsf{T}, \mathsf{T}\rangle$. The matrix tensor multiplication $\mathsf{T}\times_p\mathbf{M}$ along mode p for a matrix $\mathbf{M} \in \mathbb{R}^{m\times n_p}$ is defined by

$$(\mathsf{T}\times_p \mathbf{A})_{j_1j_2\dots j_{p-1}ij_{p+1}\dots j_k} = \sum_{j_p=1}^{n_p} \mathsf{T}_{j_1j_2\dots j_p\dots j_k}\mathbf{M}_{ij_p}. \tag{2.2}$$

This product can be generalized to what is known as the Tucker product, for $\mathbf{M}_p \in \mathbb{R}^{m_p\times n_p}$,

$$\mathsf{T}\times_1 \mathbf{M}_1 \times_2 \mathbf{M}_2 \times_3 \cdots \times_k \mathbf{M}_k \in \mathbb{R}^{m_1\times m_2\times\cdots\times m_k}. \tag{2.3}$$

The tensor vector multiplication $\mathsf{T}\times_p \mathbf{v}$ along mode p for a vector $\mathbf{v} \in \mathbb{R}^{n_p}$ is defined by

$$(\mathsf{T}\times_p \mathbf{v})_{j_1j_2\dots j_{p-1}j_{p+1}\dots j_k} = \sum_{j_p=1}^{n_p} \mathsf{T}_{j_1j_2\dots j_p\dots j_k}\mathbf{v}_{j_p}, \tag{2.4}$$

which also can be extended to

$$\mathsf{T} \times_1 \mathbf{v}_1 \times_2 \mathbf{v}_2 \times_3 \cdots \times_k \mathbf{v}_k = \mathsf{T}\mathbf{v}_1\mathbf{v}_2 \dots \mathbf{v}_k \in \mathbb{R} \tag{2.5}$$

for $\mathbf{v}_p \in \mathbb{R}^{n_p}$. If T is supersymmetric and $\mathbf{v}_p = \mathbf{v}$ for all p, the product (2.5) is also known as the homogeneous polynomial associated with T, and we write it as $\mathsf{T}\mathbf{v}^k$ for simplicity.

2.2. Tensor decompositions. There are several definitions of tensor ranks [10, 22, 23], which are related to different notions of tensor decompositions. The multilinear ranks are related to the so-called higher-order singular value decomposition (HOSVD), a multilinear generalization of the matrix singular value decomposition (SVD) [5, 13].

Theorem 2.1 (HOSVD). *A tensor $\mathsf{T} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k}$ can be written as*

$$\mathsf{T} = \mathsf{S} \times_1 \boldsymbol{U}_1 \times_2 \boldsymbol{U}_2 \times_3 \cdots \times_k \boldsymbol{U}_k, \tag{2.6}$$

where $\boldsymbol{U}_p \in \mathbb{R}^{n_p \times n_p}$ are orthogonal matrices, and $\mathsf{S} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k}$ is a tensor of which the subtensors $\mathsf{S}_{j_p=\alpha}$, obtained by fixing the pth index to α, have the properties of:

1. all-orthogonality: two subtensors $\mathsf{S}_{j_p=\alpha}$ and $\mathsf{S}_{j_p=\beta}$ are orthogonal for all possible values of p, α and β subject to $\alpha \neq \beta$;

2. ordering: $\|\mathsf{S}_{j_p=1}\| \geq \cdots \geq \|\mathsf{S}_{j_p=n_p}\| \geq 0$ for all possible values of p.

The Frobenius norms $\|\mathsf{S}_{j_p=j}\|$, denoted by $\gamma_j^{(p)}$, are the p-mode singular values of T.

Analogous to rank-one matrices, a tensor $\mathsf{T} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k}$ is rank-one if it can be written as the outer product of k vectors, i.e., $\mathsf{T} = \mathbf{a}^{(1)} \circ \mathbf{a}^{(2)} \circ \cdots \circ \mathbf{a}^{(k)}$. The CANDECOMP/PARAFAC decomposition (CPD) decomposes a tensor $\mathsf{T} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k}$ into a sum of rank-one tensors as form of outer products. It is often useful to normalize all the vectors and have weights λ_r in descending order in front:

$$\mathsf{T} = \sum_{r=1}^{m} \lambda_r \mathbf{a}_r^{(1)} \circ \mathbf{a}_r^{(2)} \circ \cdots \circ \mathbf{a}_r^{(k)}, \tag{2.7}$$

where $\mathbf{a}_r^{(p)} \in \mathbb{R}^{n_p}$ have unit length, and m is called the CP rank of T if it is the minimum integer that achieves (2.7).

The tensor train decomposition (TTD) of a tensor $\mathsf{T} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k}$ is given by

$$\mathsf{T} = \sum_{j_0=1}^{r_0} \cdots \sum_{j_k=1}^{r_k} \mathsf{T}^{(1)}_{j_0:j_1} \circ \mathsf{T}^{(2)}_{j_1:j_2} \circ \cdots \circ \mathsf{T}^{(k)}_{j_{k-1}:j_k}, \tag{2.8}$$

where $\{r_0, r_1, \dots, r_k\}$ is the set of TT-ranks with $r_0 = r_k = 1$, and $\mathsf{T}^{(p)} \in \mathbb{R}^{r_{p-1} \times n_p \times r_p}$ are the core tensors [31]. Here ":" denotes the colon operation in MATLAB, see details in Appendix C. There exist optimal TT-ranks for the TTD such that

$$r_p = \texttt{rank}\Big(\texttt{reshape}\Big(\mathsf{T}, \prod_{j=1}^{p} n_j, \prod_{j=p+1}^{k} n_j\Big)\Big),$$

for $p = 1, 2, \dots, k-1$. A core tensor $\mathsf{T}^{(p)}$ is called left-orthonormal if $(\bar{\mathbf{T}}^{(p)})^\top \bar{\mathbf{T}}^{(p)} = \mathbf{I} \in \mathbb{R}^{r_p \times r_p}$, and is called right-orthonormal if $\underline{\mathbf{T}}^{(p)}(\underline{\mathbf{T}}^{(p)})^\top = \mathbf{I} \in \mathbb{R}^{r_{p-1} \times r_{p-1}}$ where

$$\bar{\mathbf{T}}^{(p)} = \texttt{reshape}(\mathsf{T}^{(p)}, r_{p-1}n_p, r_p) \text{ and } \underline{\mathbf{T}}^{(p)} = \texttt{reshape}(\mathsf{T}^{(p)}, r_{p-1}, n_p r_p)$$

are the left- and right-unfoldings of the core tensor, respectively [21]. Here $\mathbf{I}$ denotes the identity matrix, and `rank` and `reshape` refer to the rank and reshape operations in MATLAB, respectively, see details in Appendix C. TTD is advantageous in that it provides better compression, and is computationally more robust [31]. Detailed algorithms of computing TTD and left- and right-orthonormalization can be found in Appendix B.

2.3. Tensor eigenvalues. Homogeneous polynomials are closely related to eigenvalue problems. The tensor eigenvalues of real supersymmetric tensors were first explored by Qi [32, 33] and Lim [25] independently. There are many different notions of tensor eigenvalues such as H-eigenvalues, Z-eigenvalues, M-eigenvalues, and U-eigenvalues [10, 32, 33]. Of particular interest of this paper are Z-eigenvalues. Given a kth order supersymmetric tensor $\mathsf{T} \in \mathbb{R}^{n \times n \times \cdots \times n}$, the E-eigenvalues $\lambda \in \mathbb{C}$ and E-eigenvectors $\mathbf{v} \in \mathbb{C}^n$ of T are defined as

$$\begin{cases} \mathsf{T}\mathbf{v}^{k-1} = \lambda \mathbf{v} \\ \mathbf{v}^\top \mathbf{v} = 1 \end{cases}. \tag{2.9}$$

The E-eigenvalues λ could be complex. If λ are real, we call them Z-eigenvalues. Computing the E-eigenvalues and the Z-eigenvalues of a tensor is NP-hard [16]. In 2016, Chen *et al.* [11] proposed numerical methods for computing E-eigenvalues and Z-eigenvalues via homotopy continuation approach, but the methods only work well for small size tensors.

3. Multilinear dynamical systems. Before introducing the multilinear dynamical system representation, we first discuss the notion of orthogonal decomposability for supersymmetric tensors, which will play a significant role in the stability analysis.

Definition 3.1. *A suppersymmetric tensor $\mathsf{A} \in \mathbb{R}^{n \times n \times \cdots \times n}$ is called orthogonal decomposable (odeco) if it can be written as a sum of vector outer products*

$$\mathsf{A} = \sum_{r=1}^{n} \lambda_r \boldsymbol{v}_r \circ \boldsymbol{v}_r \circ \cdots \circ \boldsymbol{v}_r, \tag{3.1}$$

where $\lambda_r \in \mathbb{R}$ in the descending order, and $\boldsymbol{v}_r \in \mathbb{R}^n$ are orthonormal.

Clearly, orthogonal decomposition is a very special case of CPD. Reobeva [36] proved that λ_r are the Z-eigenvalues of A with the corresponding Z-eigenvectors $\mathbf{v}_r$. However, λ_r do not include all the Z-eigenvalues of A. Moreover, the author showed that odeco tensors satisfy a set of polynomial equations that vanish on the odeco variety, which is the Zariski closure of the set of odeco tensors inside the space of kth order n-dimensional complex supersymmetric tensors. Note that the author only proved for the case when $n = 2$, but provided with strong evidence for its overall correctness [36]. A tensor power method was also reported in [36] in order to find the orthogonal decomposition of an odeco tensor.

In this paper, we consider the discrete-time version of the multilinear dynamical system proposed in [9], which is given by

$$\mathbf{x}_{t+1} = \mathsf{A} \times_1 \mathbf{x} \times_2 \mathbf{x} \times_3 \cdots \times_{k-1} \mathbf{x} = \mathsf{A}\mathbf{x}_t^{k-1}, \tag{3.2}$$

where $\mathsf{A} \in \mathbb{R}^{n\times n\times\cdots\times n}$ is a supersymmetric odeco/non-odeco dynamic tensor, and $\mathbf{x}_t \in \mathbb{R}^n$ is the state variable (multilinear in the sense of multilinear algebra). In fact, the multilinear dynamical system (3.2) belongs to the family of homogeneous polynomial dynamical systems of degree $k-1$ if one expands the tensor vector multiplications.

For example, suppose that the dynamic tensor $\mathsf{A} \in \mathbb{R}^{3\times 3\times 3}$ with $\mathsf{A}_{123} = \mathsf{A}_{132} = \mathsf{A}_{213} = \mathsf{A}_{231} = \mathsf{A}_{312} = \mathsf{A}_{321} = \frac{1}{2}$ and zero elsewhere. Let

$$\mathbf{x}_t = \begin{bmatrix} x_t^{(1)} & x_t^{(2)} & x_t^{(3)} \end{bmatrix}^\top .$$

Then the multilinear dynamical system $\mathbf{x}_{t+1} = \mathsf{A}\mathbf{x}_t^2$ can be equivalently represented by the following homogeneous polynomial dynamical system

$$\begin{cases} x_{t+1}^{(1)} &= x_t^{(2)} x_t^{(3)} \\ x_{t+1}^{(2)} &= x_t^{(1)} x_t^{(3)} \\ x_{t+1}^{(3)} &= x_t^{(1)} x_t^{(2)} \end{cases}.$$

3.1. Explicit solutions.

We find that if the dynamic tensor A is odeco, we can write down the solution of (3.2) explicitly in a simple fashion.

Proposition 3.2. *Suppose that $k \geq 3$ and $\mathsf{A} \in \mathbb{R}^{n\times n\times\cdots\times n}$ is odeco with orthogonal decomposition (3.1). Let the initial condition $\boldsymbol{x}_0 = \sum_{r=1}^n c_r \boldsymbol{v}_r$. Then the solution of the multilinear dynamical system (3.2) at time q, given initial condition $\boldsymbol{x}_0$, is given by*

$$\boldsymbol{x}_q = \sum_{r=1}^n \lambda_r^\alpha c_r^\beta \boldsymbol{v}_r, \tag{3.3}$$

where $\alpha = \sum_{j=0}^{q-1}(k-1)^j = \frac{(k-1)^q - 1}{k-2}$ and $\beta = (k-1)^q$.

Proof. Based on the property of tensor vector multiplications, we can write down the solution $\mathbf{x}_1$ as follows:

$$\begin{aligned} \mathbf{x}_1 &= \mathsf{A} \times_1 \Big(\sum_{i=1}^n c_i \mathbf{v}_i\Big) \times_2 \Big(\sum_{i=1}^n c_i \mathbf{v}_i\Big) \times_3 \cdots \times_{k-1} \Big(\sum_{i=1}^n c_i \mathbf{v}_i\Big) \\ &= \Big(\sum_{r=1}^n \lambda_r \mathbf{v}_r \circ \mathbf{v}_r \circ \cdots \circ \mathbf{v}_r\Big) \times_1 \Big(\sum_{i=1}^n c_i \mathbf{v}_i\Big) \times_2 \Big(\sum_{i=1}^n c_i \mathbf{v}_i\Big) \times_3 \cdots \times_{k-1} \Big(\sum_{i=1}^n c_i \mathbf{v}_i\Big) \\ &= \sum_{r=1}^n \lambda_r \Big\langle \mathbf{v}_r, \sum_{i=1}^n c_i \mathbf{v}_i \Big\rangle^{k-1} \mathbf{v}_r. \end{aligned}$$

Since all the vectors $\mathbf{v}_r$ are orthonormal, we have

$$\mathbf{x}_1 = \sum_{r=1}^{n} \lambda_r \Big\langle \mathbf{v}_r, \sum_{i=1}^{n} c_i \mathbf{v}_i \Big\rangle^{k-1} \mathbf{v}_r = \sum_{r=1}^{n} \lambda_r \mathbf{c}_r^{k-1} \mathbf{v}_r.$$

Similarly, the solution $\mathbf{x}_2$ can be written as

$$\begin{aligned}\mathbf{x}_2 &= \mathsf{A} \times_1 \Big(\sum_{i=1}^{n} \lambda_i c_i^{k-1}\mathbf{v}_i\Big) \times_2 \Big(\sum_{i=1}^{n} \lambda_i c_i^{k-1}\mathbf{v}_i\Big) \times_3 \cdots \times_{k-1} \Big(\sum_{i=1}^{n} \lambda_i c_i^{k-1}\mathbf{v}_i\Big)\\ &= \sum_{r=1}^{n} \lambda_r \Big\langle \mathbf{v}_r, \sum_{i=1}^{n} \lambda_i c_i^{k-1}\mathbf{v}_i \Big\rangle^{k-1} \mathbf{v}_r = \sum_{r=1}^{n} \lambda_r^{k} \mathbf{c}_r^{(k-1)^2} \mathbf{v}_r.\end{aligned}$$

One can continue to compute $\mathbf{x}_3, \mathbf{x}_4, \dots, \mathbf{x}_q$ in the similar manner. Therefore, the result follows immediately. ∎

Clearly, if A is not odeco, the solution of the system will be extremely complicated. CPD does not possess the orthonormal property, so we cannot write the initial condition in the form of $\mathbf{x}_0 = \sum_{r=1}^{n} c_r \mathbf{v}_r$. In the following section, we will establish stability results on the multilinear dynamical system (3.2) with odeco/non-odeco dynamic tensor A. In particular, when the dynamic tensor A is odeco, we can obtain necessary and sufficient conditions for determining the internal stability of a multilinear dynamical system.

4. Internal stability. In linear control theory, it is conventional to investigate so-called internal stability. The stability of a linear dynamical system solely depends on the locations of the eigenvalues of the dynamic matrix. Similarly to linear dynamical systems, the equilibrium point $\mathbf{x} = \mathbf{0}$ of the multilinear dynamical system (3.2) is called stable if $\|\mathbf{x}_t\| \leq \gamma\|\mathbf{x}_0\|$ for some initial condition $\mathbf{x}_0$ and $\gamma > 0$, asymptotically stable if $\mathbf{x}_t \to \mathbf{0}$ as $t \to \infty$, and unstable if it is not stable.

4.1. Odeco case. The stability properties of the multilinear dynamical system (3.2) with odeco dynamic tensor are similar to these of linear dynamical systems, but it depends on both the Z-eigenvalues of the dynamic tensor A and initial conditions.

Proposition 4.1. *Suppose that $k \geq 3$ and $\mathsf{A} \in \mathbb{R}^{n\times n\times\cdots\times n}$ is odeco with orthogonal decomposition (3.1). Let the initial condition $\boldsymbol{x}_0 = \sum_{r=1}^{n} c_r \boldsymbol{v}_r$. For the multilinear dynamical system (3.2), the equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ is:*

1. *stable if and only if $|c_r\lambda_r^{\frac{1}{k-2}}| \leq 1$ for all $r = 1, 2, \dots, n$;*
2. *asymptotically stable if and only if $|c_r\lambda_r^{\frac{1}{k-2}}| < 1$ for all $r = 1, 2, \dots, n$;*
3. *unstable if and only if $|c_r\lambda_r^{\frac{1}{k-2}}| > 1$ for some $r = 1, 2, \dots, n$.*

Proof. Based on the result from Proposition 3.2, the solution at time q, given initial condition $\mathbf{x}_0$, is $\mathbf{x}_q = \sum_{r=1}^{n} \lambda_r^{\alpha} c_r^{\beta} \mathbf{v}_r$ where $\alpha = \sum_{j=0}^{q-1}(k-1)^j = \frac{(k-1)^q-1}{k-2}$ and $\beta = (k-1)^q$. Therefore, it can be shown that

$$\lambda_r^{\alpha} c_r^{\beta} = \lambda_r^{\frac{(k-1)^q-1}{k-2}} c_r^{(k-1)^q} = \lambda_r^{-\frac{1}{k-2}} (\lambda_r^{\frac{1}{k-2}} c_r)^{(k-1)^q}.$$

Hence, the results follow immediately. ■

When $k = 2$, Proposition 4.1 reduces to the famous linear stability conditions, i.e.,

$$\lim_{k\to 2} \lambda_r^{\frac{(k-1)^q-1}{k-2}} c_r^{(k-1)^q} = \lim_{k\to 2} \lambda_r^{q(k-1)^{q-1}} c_r^{(k-1)^q} = c_r \lambda_r^q.$$

Moreover, the coefficient c_r can be found from the inner product between $\mathbf{x}_0$ and $\mathbf{v}_r$, and thus one may write $|c_r \lambda_r^{\frac{1}{k-2}}|$ as $|\langle \mathbf{x}_0, \lambda_r^{\frac{1}{k-2}} \mathbf{v}_r \rangle|$. Additionally, the inequalities obtained from the asymptotic stability condition can provide us with the region of attraction of the multilinear dynamical system (3.2), i.e.,

$$R = \{\mathbf{x} : |c_r| < |\lambda_r|^{-\frac{1}{k-2}} \text{ where } \mathbf{x} = \sum_{r=1}^{n} c_r \mathbf{v}_r\}. \tag{4.1}$$

If the product between $\max_r |c_r|$ and $\max_r |\lambda_r|^{\frac{1}{k-2}}$ is less than one, the multilinear dynamical system (3.2) will be asymptotically stable.

Definition 4.2. *The Z-spectral radius of a supersymmetric tensor is the maximum of the absolute values of all its Z-eigenvalues.*

Corollary 4.3. *Suppose that $k \geq 3$ and $\mathsf{A} \in \mathbb{R}^{n\times n\times \cdots \times n}$ is odeco. Let $\boldsymbol{x}_0$ be some initial conditions. For the multilinear dynamical system (3.2), the equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable if $\lambda^{\frac{1}{k-2}} \|\boldsymbol{x}_0\| < 1$ where λ is the Z-spectral radius of A.*

Proof. By the Cauchy-Schwarz inequality, $|c_r| \leq \|\mathbf{x}_0\|$ for all r. In addition, $\max_r |\lambda_r| \leq \lambda$. Therefore, the result follows immediately from Proposition 4.1. ■

Based on our numerical experiments, we find that the Z-spectral radius

$$\lambda = \max\{|\lambda_1|, |\lambda_n|\},$$

where λ_1 and λ_n are the largest and the smallest coefficients in the orthogonal decomposition, respectively. This implies that λ_1 is the largest Z-eigenvalue of A, or λ_n is the smallest Z-eigenvalue of A. However, the correctness of this conjecture needs further investigation.

4.2. Upper bounds for Z-spectral radii. Computing the orthogonal decomposition or Z-eigenvalues of a supersymmetric tensor is NP-hard even if we know the tensor is odeco beforehand (e.g., the tensor satisfies a set of polynomial equations that vanish on the odeco variety) [16, 36]. If we can come up with some upper bounds for the Z-spectral radii of the dynamic tensors, it will save a great amount of computations for determining the internal stability of multilinear dynamical systems.

Lemma 4.4. *Suppose that $\mathsf{A} \in \mathbb{R}^{n\times n\times \cdots \times n\times n}$ is an even-order supersymmetric tensor. The Z-spectral radius of A is upper bounded by the spectral radius of the unfolded matrix defined by the following*

$$\boldsymbol{A} = \psi(\mathsf{A}) \text{ s.t. } \mathsf{A}_{j_1 i_1 \dots j_k i_k} \xrightarrow{\psi} \boldsymbol{A}_{ji}, \tag{4.2}$$

where $j = j_1 + \sum_{p=2}^{k} (j_p - 1) n^{p-1}$, and $i = i_1 + \sum_{p=2}^{k} (i_p - 1) n^{p-1}$.

Proof. Based on the results in [8, 10], it can be shown that the tensor unfolding ψ is bijective, and the restriction of ψ^{-1} on the general linear group produces a group isomorphism. Thus, the largest eigenvalue of the unfolded matrix $\mathbf{A}$ can be equivalently solved from the following tensor-based optimization problem

$$\max\{\mathsf{X}^\top * \mathsf{A} * \mathsf{X} : \|\mathsf{X}\| = 1\},$$

where $\mathsf{X} \in \mathbb{R}^{n\times n\times\cdots\times n}$ is a kth order tensor, $*$ denotes the Einstein product, and $\top$ denotes the U-tranpose of a tensor. See Appendix A for the detailed definitions of these notions. On the other hand, it can be shown that the largest Z-eigenvalue of A can be solved from the following tensor-based optimization problem

$$\max\{\mathsf{X}^\top * \mathsf{A} * \mathsf{X} : \|\mathsf{X}\| = 1, \text{ and } \mathsf{X} = \mathbf{v}\circ\mathbf{v}\circ\cdots\circ\mathbf{v}\}.$$

Therefore, the largest Z-eigenvalue of A is always less than or equal to the largest eigenvalue of the unfolded matrix $\mathbf{A}$. Similarly, we can show that the smallest Z-eigenvalue of A is always greater than or equal to the smallest eigenvalue of the unfolded matrix $\mathbf{A}$ by replacing maximization with minimization. Therefore, the result follows immediately. ■

The eigenvalues of the unfoled matrix $\mathbf{A}$ are also called the U-eigenvalues of A [8, 10]. The computational cost for computing spectral radii from the matrix eigenvalue decomposition is much less than that for Z-spectral radii. Once we have an upper bound for the Z-spectral radii of the dynamic tensors, we can determine the internal stability of multilinear dynamical systems without computing orthogonal decomposition or Z-eigenvalues.

Corollary 4.5. *Suppose that $k \geq 4$ is even and $\mathsf{A} \in \mathbb{R}^{n\times n\times\cdots\times n\times n}$ is odeco. Let $\boldsymbol{x}_0$ be some initial conditions. For the multilinear dynamical system (3.2), the equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable if $\mu^{\frac{1}{k-2}}\|\boldsymbol{x}_0\| < 1$ where μ is the spectral radius of $\psi(\mathsf{A})$.*

Proof. The result follows immediately from Lemma 4.4 and Corollary 4.3. ■

The condition offers a conservative region of attraction for the multilinear dynamical system (3.2) without knowing the orthogonal decomposition of A, i.e.,

$$R = \{\mathbf{x} : \|\mathbf{x}\| < \mu^{-\frac{1}{k-2}}\}.$$

However, if a tensor has large dimension n or higher order k, computing the spectral radius of the unfolded matrix is still computationally demanding. Thus, we exploit TTD for finding the spectral radius of the unfolded matrix. In particular, if the dynamic tensor A naturally possesses low TT-ranks structure, the computational time will be significantly reduced.

Corollary 4.6. *Suppose that $k \geq 4$ is even and $\mathsf{A} \in \mathbb{R}^{n\times n\times\cdots\times n\times n}$ is odeco in the tensor train format with first $k-1$ core tensors left-orthonormal, and last k core tensors right-orthonormal. For the multilinear dynamical system (3.2), the equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable if $\mu^{\frac{1}{k-2}}\|\boldsymbol{x}_0\| < 1$ where μ is the largest singular value of the left-unfolding of the kth core tensor $\bar{\mathsf{A}}^{(k)}$.*

Proof. Based on the results of [21], the singular values of $\bar{\mathsf{A}}^{(k)}$ are the singular values of $\psi(\mathsf{A})$. Since $\psi(\mathsf{A})$ is symmetric, the largest singular value of $\psi(\mathsf{A})$ is equal to its spectral radius. Therefore, the result follows immediately from Corollary 4.5. ■

Computing the spectral radius of $\psi(\mathsf{A})$ directly from the eigenvalue decomposition requires an order of $\mathcal{O}(n^{3k})$ number of operations. [1] To the contrary, if the TTD of A is provided, the time complexity of left- and right-orthonormalization is only about $\mathcal{O}(knr^3)$ where r can be viewed as the average of the TT-ranks.

In addition, there are many other upper bounds for Z-spectral radii of supersymmetric tensors [6, 15, 26, 44]. For example, He *et al.* [15] proposed that given a positive kth order supersymmetric tensor A, its Z-spectral radius is upper bounded by

$$\lambda \leq R - l\Big(1 - (\frac{r}{R})^{\frac{1}{k}}\Big), \tag{4.3}$$

where l is the minimum entry of A,

$$r = \min_j \Big(\sum_{j_2=1}^{n} \cdots \sum_{j_k=1}^{n} \mathsf{A}_{jj_2\dots j_k}\Big), \text{ and } R = \max_j \Big(\sum_{j_2=1}^{n} \cdots \sum_{j_k=1}^{n} \mathsf{A}_{jj_2\dots j_k}\Big).$$

Hence, one can also use this upper bound to determine the internal stability of a multilinear dynamical system if the dynamic tensor contains all positive entries. Note that given a dynamic tensor, the better upper bound of the Z-spectral radius, the more strong stability conditions we can obtain. In section 6, we will present an example to show that our upper bound gives more accurate approximation to the Z-spectral radius of an even-order supersymmetric tensor, compared to (4.3).

4.3. Non-odeco case. As mentioned in [36], not all supersymmetric tensors are odeco. Therefore, we offer a more general but relatively weaker stability result for the multilinear dynamical system (3.2) in the following proposition.

Proposition 4.7. *Suppose that $k \geq 3$. Let $\boldsymbol{x}_0$ be some initial conditions. For the multilinear dynamical system (3.2), the equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable if*

$$\|A\|^{\frac{1}{k-2}} \|\boldsymbol{x}_0\| < 1.$$

Proof. Based on Theorem 6 in [19], we have

$$\|\mathbf{x}_{t+1}\| \leq \|\mathsf{A}\| \|\mathbf{x}_t\|^{k-1}.$$

Thus, it can be shown similarly as Proposition 4.1 that at the qth step, we have

$$\|\mathbf{x}_q\| \leq \|\mathsf{A}\|^{\alpha} \|\mathbf{x}_0\|^{\beta},$$

where α and β are the same quantities as defined in Proposition 4.1. Therefore, the result follows immediately. ■

[1] Big O notation: $f(x) = \mathcal{O}(g(x))$ as $x \to \infty$ if and only if there exists a positive real number M and a real number x_0 such that $|f(x)| \leq Mg(x)$ for all $x \geq x_0$.

Proposition 4.7 also holds for non-supersymmetric dynamic tensors A. Similarly, we can obtain a conservative region of attraction, i.e.,

$$R = \{\mathbf{x} : \|\mathbf{x}\| < \|\mathsf{A}\|^{-\frac{1}{k-2}}\}.$$

Moreover, the tensor Frobenius norm is closely related to HOSVD.

Corollary 4.8. *Suppose that $k \geq 3$. Let $\boldsymbol{x}_0$ be some initial conditions. For the multilinear dynamical system (3.2), the equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable if for any p,*

$$\Big(\sum_{j=1}^{n} (\gamma_j^{(p)})^2\Big)^{\frac{1}{k-2}} \|\boldsymbol{x}_0\| < 1,$$

where $\gamma_j^{(p)}$ are the p-mode singular values of A.

Proof. The result follows immediately from Proposition 4.7 and the fact that the Frobenius norm of a tensor is equal to the sum of its p-mode singular values' square for any p. ∎

De Lathauwer *et al.* [13] showed that the p-mode singular values from the HOSVD of a tensor are the singular values of its p-mode unfoldings, see details in [13]. However, computing the tensor Frobenius norm or p-mode singular values is computationally expensive. Similarly, we apply TTD to gain computational efficiency.

Corollary 4.9. *Suppose that $k \geq 3$ and $\mathsf{A} \in \mathbb{R}^{n\times n\times\cdots\times n}$ in the tensor train format with first $k-1$ core tensors left-orthonormal. Let $\boldsymbol{x}_0$ be some initial conditions. For the multilinear dynamical system (3.2), the equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable if*

$$\|\mathsf{A}^{(k)}\|^{\frac{1}{k-2}} \|\boldsymbol{x}_0\| < 1.$$

where $\mathsf{A}^{(k)}$ is the kth core tensor.

Proof. Since the first $k-1$ core tensors are left-orthonormal, contracting them would result in an identify matrix. Therefore, the Frobenius norm of A is equal to the Frobenius norm of its kth core tensor. Hence, the result follows immediately from Proposition 4.7. ∎

A similar result can be obtained if the last $k-1$ core tensors are right-orthonormal. Significantly, computing the Frobenius norm of its kth core tensor which contains only nr_{k-1} entries is much simpler than computing the Frobenius norm of the whole tensor which contains n^k entries. Furthermore, it will be interesting to explore more strong stability conditions as Proposition 4.1 for multilinear dynamical systems with non-odeco dynamic tensors (even non-supersymmetric tensors) in the future.

5. Lyapunov stability. Aside from internal stability, Lyapunov stability also plays an significant role in control systems, particularly in nonlinear polynomial dynamical systems [1, 2, 18, 37, 39]. We here extend the Lyapunov stability conditions to multilinear dynamical systems, analogous to the construction of quadratic Lyapunov functions in linear dynamical systems. First, we introduce the notion of positive definiteness for supersymmetric tensors and the p-mode tensor tensor multiplication.

Definition 5.1. *A kth order n-dimensional supersymmetric tensor $\mathsf{T} \in \mathbb{R}^{n\times n\times\cdots\times n}$ is called positive definite if for any nonzero vector $\boldsymbol{x} \in \mathbb{R}^n$, its corresponding homogeneous polynomial $\mathsf{T}\boldsymbol{x}^k > 0$.*

Definition 5.2. *Given kth order n-dimensioanl supersymmetric tensors $\mathsf{P}, \mathsf{A} \in \mathbb{R}^{n\times n\times\cdots\times n}$, the p-mode tensor tensor multiplication with respect to P is defined by*

$$(\mathsf{P} \bullet_p \mathsf{A})_{i_1 i_2 \dots i_{p-1} i_{p+1} \dots i_k j_2 j_3 \dots j_k} = \sum_{i_p=1}^{n} \mathsf{P}_{i_1 i_2 \dots i_p \dots i_k} \mathsf{A}_{i_p j_2 \dots j_k},$$

which can be extended to

$$\mathsf{P} \bullet_1 \mathsf{A} \bullet_2 \mathsf{A} \bullet_3 \cdots \bullet_k \mathsf{A} = \mathsf{P} \bullet \{\mathsf{A}, \mathsf{A}, \dots, \mathsf{A}\}.$$

We propose a Lyapunov function candidate for the multilinear dynamical system (3.2) of a form $V(\mathbf{x}) = \mathsf{P}\mathbf{x}^k$. When $k = 2$, it reduces to the famous quadratic Lyapunov functions for linear dynamical systems, i.e., $V(\mathbf{x}) = \mathbf{x}^\top \mathbf{P}\mathbf{x}$.

Proposition 5.3. *Suppose that $V(\boldsymbol{x}) = \mathsf{P}\boldsymbol{x}^k$ is a Lyapunov function candidate where P is a kth order n-dimensional supersymmetric positive definite tensor. For the multilinear dynamical system (3.2), the equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable in the sense of Lyapunov if the following polynomial*

$$p(\boldsymbol{x}) = \mathsf{B}\boldsymbol{x}^{k(k-1)} - \mathsf{P}\boldsymbol{x}^k < 0, \tag{5.1}$$

where $\mathsf{B} = \mathsf{P} \bullet \{\mathsf{A}, \mathsf{A}, \dots, \mathsf{A}\}$ is a $k(k-1)$th order n-dimensional tensor.

Proof. First, since P is positive definite, $V(\mathbf{x}) > 0$ for any nonzero $\mathbf{x}$. Second, based on the property of tensor tensor multiplications, it can be shown that

$$V(\mathbf{x}_{t+1}) - V(\mathbf{x}_t) = \mathsf{P}(\mathsf{A}\mathbf{x}_t^{k-1})^k - \mathsf{P}\mathbf{x}_t^k = (\mathsf{P} \bullet \{\mathsf{A}, \mathsf{A}, \dots, \mathsf{A}\})\mathbf{x}_t^{k(k-1)} - \mathsf{P}\mathbf{x}_t^k.$$

Therefore, the result follows immediately. ∎

When $k = 2$, the polynomial $p(\mathbf{x})$ reduces to the classical Lyapunov equations for linear dynamical systems, i.e.,

$$p(\mathbf{x}) = (\mathbf{P} \bullet \{\mathbf{A}, \mathbf{A}\})\mathbf{x}^2 - \mathbf{P}\mathbf{x}^2 = \mathbf{x}^\top(\mathbf{A}^\top\mathbf{P}\mathbf{A} - \mathbf{P})\mathbf{x} \Rightarrow \mathbf{A}^\top\mathbf{P}\mathbf{A} - \mathbf{P} = -\mathbf{Q}.$$

However, for $k \geq 3$, determining the negative definiteness of $p(\mathbf{x})$, formed by the difference between two homogeneous polynomials, usually is hard. It will be an important direction for future research. Nevertheless, we will try to demonstrate Proposition 5.3 using a simple example in section 6.

6. Numerical examples.

All the numerical examples presented were performed on a Macintosh machine with 16 GB RAM and a 2 GHz Quad-Core Intel Core i5 processor in MATLAB R2020b, and used the MATLAB tensor toolbox [3, 38] and MATLAB TT toolbox [30, 31].

Table 1
Stability results for different initial conditions.

IC	$\max\{\lvert c_r\lambda_r\rvert\}$	$\lVert\mathsf{A}\rVert\lVert\mathbf{x}_0\rVert$	Stability
a	0.9735	28.7712	Asym. Stable
b	0.6032	0.9413	Asym. Stable
c	1	53.9410	Stable
d	1.0053	1.5688	Unstable

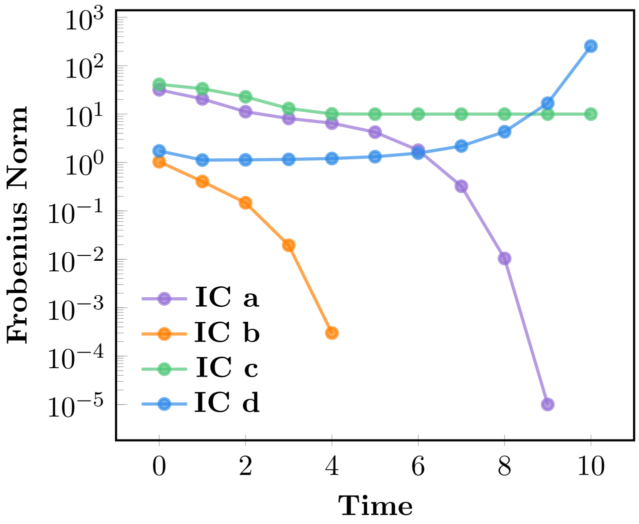

Figure 1. *Stability results for different initial conditions corresponding to Table 1. When the norm of $\boldsymbol{x}_t$ is less than 10^{-5}, we omit the point.*

6.1. Stability with odeco dynamic tensors. In this example, we try to verify the internal stability results discussed in section 4. Given an odeco tensor $\mathsf{A} \in \mathbb{R}^{3\times3\times3}$ with orthogonal decomposition (columns of $\mathbf{V}$ are $\mathbf{v}_r$ in (3.1))

$$\mathbf{V} = \begin{bmatrix} -0.8482 & -0.5212 & 0.0947 \\ -0.4840 & 0.6899 & -0.5382 \\ 0.2152 & -0.5024 & -0.8374 \end{bmatrix} \text{ and } \boldsymbol{\lambda} = \begin{bmatrix} 0.9 \\ 0.1 \\ 0.02 \end{bmatrix},$$

we compute the trajectories $\mathbf{x}_t$ for four initial conditions, which are given by

$$\mathbf{x_a} = \begin{bmatrix} 3 \\ 10 \\ 30 \end{bmatrix}, \ \mathbf{x_b} = \begin{bmatrix} 0.6 \\ 0.6 \\ 0.6 \end{bmatrix}, \ \mathbf{x_c} = \begin{bmatrix} -2.2720 \\ -15.1148 \\ -38.3064 \end{bmatrix}, \ \mathbf{x_d} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The results are shown in Table 1 and Figure 1. For each initial condition, we calculate the quantities $\max\{\lvert c_r\lambda_r\rvert\}$ and $\lVert\mathsf{A}\rVert\lVert\mathbf{x}_0\rVert$, and compare them to one. Clearly, the locations of $c_r\lambda_r$ determine the stability of the multilinear dynamical system. The region of attraction R of the multilinear dynamical system can be obtained by

$$R = \left\{ \mathbf{x} : \begin{array}{l} |-0.8482x_1 - 0.4840x_2 + 0.2152x_3| < \frac{10}{9} \\ |-0.5212x_1 + 0.6899x_2 - 0.5024x_3| < 10 \\ |0.0947x_1 - 0.5382x_2 - 0.8374x_3| < 50 \end{array} \right\},$$

where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$. In addition, the stability condition stated in Proposition 4.7 is weaker than that in Proposition 4.1 as seen in Figure 1 IC a and b .

6.2. Stability using upper bounds of Z-spectral radii. In this example, we try to apply the upper bound of the Z-spectral radii defined in (4.2) to obtain a conservative region of attraction for a multilinear dynamical system. Suppose that the dynamic tensor $\mathsf{A} \in \mathbb{R}^{2\times2\times2\times2}$ is odeco and is given by

$$\mathsf{A}_{::11} = \begin{bmatrix} 0.2285 & 0.0376 \\ 0.0376 & 0.2243 \end{bmatrix}, \quad \mathsf{A}_{::12} = \begin{bmatrix} 0.0376 & 0.2243 \\ 0.2243 & 0.0124 \end{bmatrix},$$
$$\mathsf{A}_{::21} = \begin{bmatrix} 0.0376 & 0.2243 \\ 0.2243 & 0.0124 \end{bmatrix}, \quad \mathsf{A}_{::22} = \begin{bmatrix} 0.2243 & 0.0124 \\ 0.0124 & 0.2229 \end{bmatrix}.$$

The unfolded matrix $\psi(\mathsf{A})$ therefore is given by

$$\psi(\mathsf{A}) = \begin{bmatrix} 0.2285 & 0.0376 & 0.0376 & 0.2243 \\ 0.0376 & 0.2243 & 0.2243 & 0.0124 \\ 0.0376 & 0.2243 & 0.2243 & 0.0124 \\ 0.2243 & 0.0124 & 0.0124 & 0.2229 \end{bmatrix}.$$

The spectral radius of $\psi(\mathsf{A})$ is $\mu = \frac{1}{2}$, and thus the conservative region of attraction of the multilinear dynamical system is an open disk with radius $\sqrt{2}$ centered at the origin (note that the second upper bound (4.3) produces $\lambda \leq 1.0263$, which will give an even more conservative region of attraction). We test four initial conditions to verify the region of attraction, which are given by

$$\mathbf{x_a} = \begin{bmatrix} -1.4 \\ 0 \end{bmatrix}, \quad \mathbf{x_b} = \begin{bmatrix} 0.9 \\ -0.9 \end{bmatrix}, \quad \mathbf{x_c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{x_d} = \begin{bmatrix} 1.2 \\ 1.2 \end{bmatrix}.$$

The results are shown in Figure 2. It is clear to see that the trajectories of the multilinear dynamical system with initial conditions started within the open disk converge to the origin, see IC a and b. Moreover, since the region of attraction is conservative, we see a trajectory started on the circle also converge to the origin, see IC c.

6.3. Lyapunov stability. In this example, we try to establish the Lyapunov stability of a simple multilinear dynamical system. Suppose that the dynamic tensor $\mathsf{A} \in \mathbb{R}^{2\times2\times2\times2}$ is a diagonal tensor with diagonal entries $\mathsf{A}_{1111} = 0.5$ and $\mathsf{A}_{2222} = 0.1$. Clearly, A is odeco. Let $V(\mathbf{x}) = \mathsf{P}\mathbf{x}^4$ where $\mathsf{P} \in \mathbb{R}^{2\times2\times2\times2}$ is the identity tensor (i.e., all diagonal entries are equal to one). Let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top$. According to Proposition 5.3, we have

$$p(\mathbf{x}) = 0.0625x_1^{12} + 0.0001x_2^{12} - (x_1^4 + x_2^4) = x_1^4(0.0625x_1^8 - 1) + x_2^4(0.0001x_2^8 - 1).$$

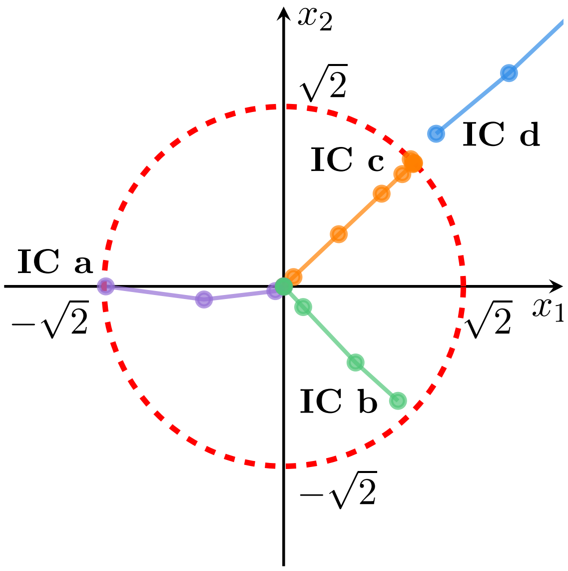

Figure 2. *A conservative region of attraction (red dashed line) of the multilinear dynamical system, and four initial conditions with their trajectories.*

Table 2

Run time comparison between the TTD-based and direct methods in finding conservative regions attraction of the multilinear dynamical systems. When $k = 32$, the computer does not have enough memory to execute the operation for the direct method.

k	TTD(s)	Direct(s)	$\lVert\mathsf{A}\rVert^{-\frac{1}{k-2}}$	Relative error
15	5.40×10^{-4}	6.77×10^{-4}	0.72	5.62×10^{-16}
20	6.20×10^{-4}	7.00×10^{-3}	0.73	3.50×10^{-16}
25	9.83×10^{-4}	0.20	0.74	5.79×10^{-16}
30	6.65×10^{-4}	34.29	0.74	3.84×10^{-16}
31	5.80×10^{-3}	201.30	0.69	1.75×10^{-15}
32	1.20×10^{-3}	-	0.78	-

Therefore, $p(\mathbf{x}) < 0$ if $|x_1| < \sqrt{2}$ and $|x_2| < \sqrt{10}$, which offers a region of attraction for the multilinear dynamical system in the sense of Lyapunov. Significantly, this region of attraction is exactly same as the one identified using the results from Proposition 4.7.

6.4. Computation time comparison. In this example, we consider multilinear dynamical systems with random kth order tensors $\mathsf{A} \in \mathbb{R}^{2\times 2\times\cdots\times 2}$ in the tensor train format with low TT-ranks structure. We compare the run time of Corollary 4.9 with Proposition 5.3 for finding conservative regions of attraction of the multilinear dynamical systems. The results are shown in Table 2. When $k \geq 25$, the TTD-based method for obtaining the regions of attraction exhibits a significant time advantage compared to the direct method for which the time increases exponentially.

7. Discussion. The first two numerical examples reported here highlight that tensor spectral theory plays a significant role in the stability analysis of the discrete-time multilinear dynamical system (3.2). In particular, when the dynamic tensor A is odeco, its Z-eigenvalues together with initial conditions can provide necessary and sufficient criteria for determining the internal stability of a multilinear dynamical system. However, more theoretical and numerical investigations are required to evaluate the stability properties of multilinear dynamical systems with non-odeco dynamic tensors. Moreover, stabilizability of the multilinear dynamical system (3.2) needs to be considered for control/feedback designs. Stabilizability is also associated to another important concept called reachability in systems theory.

We are interested in exploring the reachability of the multilinear dynamical system (3.2) with linear control inputs, i.e.,

$$\mathbf{x}_{k+1} = \mathsf{A}\mathbf{x}_k^{k-1} + \mathbf{B}\mathbf{u}_k, \tag{7.1}$$

where $\mathbf{B} \in \mathbb{R}^{n\times m}$ is the control matrix, and $\mathbf{u}_k \in \mathbb{R}^m$ is a control input. Chen *et al.* [9] proved that for the continuous-time case, the multilinear dynamical system is controllable if and only if k is even, and the span of the smallest Lie algebra of vector fields $\{\mathsf{A}, \mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m\}$ is $\mathbb{R}^n$ at the origin where $\mathbf{B} = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_m\end{bmatrix}$. We believe that a similar result should hold for the discrete-time case, analogous to that in linear systems theory.

Definition 7.1. *Suppose that $\boldsymbol{B} = \begin{bmatrix}\boldsymbol{b}_1 & \boldsymbol{b}_2 & \dots & \boldsymbol{b}_m\end{bmatrix}$. Let $\mathcal{R}_0$ be the linear span of the vectors $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_m\}$ and $\mathsf{A} \in \mathbb{R}^{n\times n\times\cdots\times n}$ be a supersymmetric tensor. For each integer $q \geq 1$, define $\mathcal{R}_q$ inductively as the linear span of*

$$\mathcal{R}_{q-1} \cup \{\mathsf{A}\boldsymbol{v}_1\boldsymbol{v}_2\dots\boldsymbol{v}_{k-1} | \boldsymbol{v}_l \in \mathcal{R}_{q-1}\}. \tag{7.2}$$

Denote the subspace $\mathcal{R} = \cup_{q\geq 0}\mathcal{R}_q$.

Conjecture 7.2. *Suppose that k is even. The multilinear dynamical system (7.1) is reachable if and only if the subspace $\mathcal{R}$ spans $\mathbb{R}^n$, or equivalently, the matrix $\boldsymbol{R}$, including all the column vectors from $\mathcal{R}$, has rank n.*

Although we are not able to fully prove Conjecture 7.2, one can validate its correctness with help of Gröbner basis for specific values of n and k. In addition, the reason that k has to be even is because the roots of polynomial systems of even degree might all be complex.

8. Conclusion. In this paper, we investigated the stability properties of discrete-time multilinear dynamical systems, inspired by hypergraphs. In contrast to linear dynamical systems, the internal stability of the multilinear dynamical system (3.2) depends on both the spectrum of the dynamic tensor A and initial conditions. In particular, when the dynamic tensor A is odeco, we can obtain necessary and sufficient conditions by exploiting tensor Z-eigenvalues. We also provided an upper bound for the Z-spectral radii of even-order supersymmetric tensors, which can be used to determine the internal stability of multilinear dynamical systems efficiently with help of tensor train decomposition. In addition, we discussed the internal stability of multilinear dynamical systems with non-odeco dynamic tensors by using the tensor Frobenius norm and p-mode singular values. Finally, we performed preliminary explorations of the Lyapunov stability of multilinear dynamical systems. All the results are applicable to homogeneous polynomial dynamical systems if one can find the corresponding multilinear dynamical systems.

It will be worthwhile to explore more strong stability conditions regarding multilinear dynamical systems with non-odeco dynamic tensors, and more work is required to fully realize the Lyapunov theory in multilinear dynamical systems. For example, how does one derive the Lyapunov equations for the multilinear dynamical systems (3.2)? It will also be interesting to explore the stability properties of continuous-time multilinear dynamical systems. As mentioned in section 7, we also intend to analyze the stabilizability and reachability of multilinear control systems in future work. One particular application we plan to investigate is that of higher-order genomic networks. Recent advances in genomics technology, such as multiway chromosomal conformation capture (Pore-C), have inspired us to consider the human genome as a hypergraph [41]. Stability or stabilizability will be important in analyzing such higher-order networked systems.

Appendix A. Definitions for Lemma 4.4.

Definition A.1. *Given two* $2k$*th order tensors* $\mathsf{A}, \mathsf{B} \in \mathbb{R}^{n\times n\times\cdots\times n\times n}$*, the Einstein product between* A *and* B *is defined by*

$$(\mathsf{A} * \mathsf{B})_{j_1 i_1 \dots j_k i_k} = \sum_{l_1=1}^{n} \cdots \sum_{l_k=1}^{n} \mathsf{A}_{j_1 l_1 \dots j_k l_k} \mathsf{B}_{l_1 i_1 \dots l_k i_k}.$$

Definition A.2. *Given a* $2k$*th order tensor* $\mathsf{A} \in \mathbb{R}^{n\times n\times\cdots\times n\times n}$*, the U-transpose of* A*, denoted by* $\mathsf{A}^\top$*, is defined by*

$$\mathsf{A}_{j_1 i_1 \dots j_k i_k} = (\mathsf{A}^\top)_{i_1 j_1 \dots i_k j_k}.$$

Note that the kth order tensor $\mathsf{X} \in \mathbb{R}^{n\times n\times\cdots\times n}$ shown in the optimization problem in the proof of Lemma 4.4 can be viewed as a $2k$th order tensor, i.e., $\mathsf{X} \in \mathbb{R}^{n\times 1\times n\times 1\times\cdots\times n\times 1}$ in order to fit in the Einstein product.

Appendix B. Tensor train decomposition algorithms.

Algorithm B.1 Computing the TTD of a tensor [31]

1: **Input:** Given a kth order tensor $\mathsf{T} \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_k}$ in the full format, a truncation threshold ϵ
2: **for** $j = 1, 2, \dots, k-1$ **do**
3: Set $\mathbf{T} = \texttt{reshape}(\mathsf{T}, r_{j-1}n_j, n_{j+1}n_{j+2}\dots n_k)$ with $r_0 = r_k = 1$
4: Compute the economy-size matrix SVD of $\mathbf{T}$, i.e., $\mathbf{T} = \mathbf{U}\mathbf{S}\mathbf{V}^\top$ with truncation threshold ϵ where $\mathbf{S} \in \mathbb{R}^{s\times s}$
5: Set $r_j = s$, and $\mathsf{T}^{(j)} = \texttt{reshape}(\mathbf{U}, r_{j-1}, n_j, r_j)$
6: Define the remainder $\mathsf{T} = \texttt{reshape}(\mathbf{S}\mathbf{V}^\top, r_j, n_{j+1}, n_{j+1}, \dots, n_k)$
7: **end for**
8: Set $\mathsf{T}^{(k)} = \mathsf{T}$
9: **Return:** The core tensors $\mathsf{T}^{(p)}$ with TT-ranks $\{r_0, r_1, \dots, r_k\}$.

Algorithm B.2 Left-orthonormalization [21]

1: **Input:** Given core tensors $\mathsf{T}^{(1)}, \mathsf{T}^{(1)}, \ldots, \mathsf{T}^{(k)}$ with TT-ranks $\{r_0, r_1, \ldots, r_k\}$, a parameter d with $1 \leq d \leq k-1$
2: **for** $j = 1, 2, \ldots, d$ **do**
3: Set $\mathbf{T}^{(j)} = \texttt{reshape}(\mathsf{T}^{(j)}, r_{j-1}n_j, r_j)$
4: Compute the QR factorization of $\mathbf{T}^{(j)}$, i.e., $\mathbf{T}^{(j)} = \mathbf{QR}$ with $\mathbf{Q} \in \mathbb{R}^{r_{j-1}n_j \times s}$
5: Set $r_j = s$ and $\mathsf{T}^{(j)} = \texttt{reshape}(\mathbf{Q}, r_{j-1}, n_j, r_j)$
6: Compute $\mathbf{T}^{(j+1)} = \mathbf{RT}^{(j+1)}$ where $\mathbf{T}^{(j+1)} = \texttt{reshape}(\mathsf{T}^{(j+1)}, r_j n_{j+1}, r_{j+1})$
7: Set $\mathsf{T}^{(j+1)} = \texttt{reshape}(\mathbf{T}^{(j+1)}, r_j, n_{j+1}, r_{j+1})$
8: **end for**
9: **Return:** The first d core tensors that are left-orthonormal.

Algorithm B.3 Right-orthonormalization [21]

1: **Input:** Given core tensors $\mathsf{T}^{(1)}, \mathsf{T}^{(1)}, \ldots, \mathsf{T}^{(k)}$ with TT-ranks $\{r_0, r_1, \ldots, r_k\}$, a parameter d with $2 \leq d \leq k$
2: **for** $j = k, k-1, \ldots, d$ **do**
3: Set $\mathbf{T}^{(j)} = \texttt{reshape}(\mathsf{T}^{(j)}, r_{j-1}, n_j r_j)$
4: Compute the QR factorization of $(\mathbf{T}^{(j)})^\top$, i.e., $\mathbf{T}^{(j)} = \mathbf{R}^\top\mathbf{Q}^\top$ with $\mathbf{Q}^\top \in \mathbb{R}^{s \times n_j r_j}$
5: Set $r_{j-1} = s$ and $\mathsf{T}^{(j)} = \texttt{reshape}(\mathbf{Q}^\top, r_{j-1}, n_j, r_j)$
6: Compute $\mathbf{T}^{(j-1)} = \mathbf{T}^{(j-1)}\mathbf{R}^\top$ where $\mathbf{T}^{(j-1)} = \texttt{reshape}(\mathsf{T}^{(j-1)}, r_{j-2}, n_{j-1}r_{j-1})$
7: Set $\mathsf{T}^{(j-1)} = \texttt{reshape}(\mathbf{T}^{(j-1)}, r_{j-2}, n_{j-1}, r_{j-1})$
8: **end for**
9: **Return:** The last d core tensors that are right-orthonormal.

Appendix C. MATLAB functions.

C.1. The colon operator. The colon : is one of the most useful operators in MATLAB, which can create vectors, subscript arrays and specify for iterations. For our purpose, it acts as shorthand to include all subscripts in a particular array dimension [27]. For example, $\mathbf{A}_{:i}$ is equivalent to $\mathbf{A}_{ji}$ for all j.

C.2. The `reshape` operator. The command $\mathsf{B} = \texttt{reshape}(\mathsf{A}, n_1, n_2, \ldots, n_k)$ reshapes a tensor A into a $n_1 \times n_2 \times \cdots \times n_k$ order tensor such that the number of elements in B matches the number of elements in A [27].

C.3. The `rank` operator. The command $r = \texttt{rank}(\mathbf{A})$ computes the rank of $\mathbf{A}$ [27].

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