AN EQUIVALENT CONDITION FOR THE REDUCIBILITY OF TROPICAL CURVES

BRIAN MANN

ABSTRACT. In response to a recent algorithm for counting the number of isomorphism classes of irreducible tropical curves through 3d+g-1 general points in the plane satisfying certain psi-class conditions by Hannah Markwig and Johannes Rau in [MR08], a method for determining the reducibility of a tropical curve in terms of its dual Newton polytope is required. This paper provides a solution by giving an equivalent condition for reducibility in terms of a mixed subdivision of the dual Newton polytope.

1. Introduction

How many lines pass through 2 general points in the plane? How many quadratics pass through 5 general points? How many cubics pass through 8? Enumerative geometry has been studied since the time of the ancient Greeks and only recently was a complete solution to the problem of counting curves given (by Kontsevich and Manin in [KM94]).

It turns out that in order for the number of curves of degree \(d\), \(N_d\), to be finite (but non-zero), they must be fixed by exactly \(3d - 1\) general points in the plane. For example, there are infinitely many lines passing through one point, while no lines pass through 3 general points. So, in general one asks this question - how many complex (projective) curves of degree \(d\) (genus 0) pass through \(3d - 1\) fixed general points in the plane. The answers to low degree examples were classically known; \(N_1 = N_2 = 1\) and \(N_3 = 12\). However, for higher degrees, \(N_d\) becomes large and hard to compute. Recently Kontsevich initiated the study of moduli spaces and produced the recursion:

\[
N_d = \sum_{d_1 + d_2 = d} N_{d_1} N_{d_2} \left( d_1^3 d_2^2 \left( \frac{3d - 4}{3d_1 - 2} \right) - d_1^3 d_2 \left( \frac{3d - 4}{3d_1 - 1} \right) \right)
\]

giving a complete solution for the enumeration of complex plane curves of any degree. With amazing elegance, the only information needed to start the recursion is that 1 line passes through 2 distinct points. Euclid might have told us that!
Tropical geometry has given another method of researching enumerative geometry. Let us first give a brief motivating overview of tropical geometry. Let $F$ be a complex plane curve. Consider the map

$$\log : (\mathbb{C}^*)^2 \to \mathbb{R}^2$$

$$(z_1, z_2) \mapsto (\log |z_1|, \log |z_2|)$$

The image $\log (F \cap (\mathbb{C}^*)^2)$ is called the *amoeba* of $F$ and some examples are included here.

A general curve of degree $d$ will have $d$ "tentacles" in each of the (0,-1), (-1,0), and (1,1) directions. We would like to turn this amoeba into a purely combinatorial construction. To do this consider the map

$$\log_t : (\mathbb{C}^*)^2 \to \mathbb{R}^2$$

$$(z_1, z_2) \mapsto \left(\frac{-\log |z_1|}{\log(t)}, \frac{-\log |z_2|}{\log(t)}\right)$$

As $t \to 0$, the image of $F \cap (\mathbb{C}^*)^2$ is a piecewise linear graph in the plane. We call this object a *tropical plane curve*. In fact, tropical curves of degree $d$ have exactly $d$ edges in each of the directions (0,-1), (-1,0), and (1,1), and every vertex is balanced. This makes them rather nice. In particular, a tropical curve is dual to a subdivided convex polytope. Note that this correspondence is not 1:1 - there can exist more than one tropical curve for a subdivided polytope.
With this construction complex (projective) curves are transformed into piecewise linear graphs in $\mathbb{R}^2$. A priori, much information about the original structure of the curve $F$ has been lost, but in fact we have not lost so much as to make this degeneration of $F$ unusable. For a detailed discussion of the link between tropical and algebraic geometry, see [Gat06].

How does this relate to the question of counting curves? In [Mik04] Mikhalkin proved his Correspondence Theorem: that the number of isomorphism classes of complex projective curves in $\mathbb{P}^2$ of degree $d$ and genus $g$, $N(d,g)$, passing through $3d + g - 1$ fixed general points is equal to the number of isomorphism classes of tropical curves of degree $d$ and genus $g$, $N_{\text{trop}}(d,g)$ passing through the same number of points. He also gave an algorithm to count $N_{\text{trop}}(d,g)$, by counting lattice paths in the dual polytope (with certain multiplicity).

More recently Markwig and Rau (in [MR08]) has developed a similar algorithm to count irreducible tropical curves satisfying certain "psi-class" conditions. These correspond to conditions imposed at the fixed points on the curves we wish to enumerate. The multiplicity of some lattice paths may not count, since they correspond to reducible tropical curves. This is the motivating problem for this paper. In general, we would like a characterization of irreducibility of a tropical curve based entirely on the dual Newton polygon. We will solve this problem by proving the following theorem:

**Theorem.** The tropical curve $T$ of degree $d$ is reducible into $k$ components if and only if for its dual polytope $\Delta_d = \sum_{i=1}^{k} \Delta_{d_i}$, the subdivision corresponding to $T$ admits a labeling as a mixed subdivision.

Before coming to this main result, we take some time to cover the definitions and notation used, since tropical geometry is a young subject and standardizations of many of the main ideas have not yet been introduced. A basic review of certain facts from polyhedral geometry is given as well, to familiarize the reader with the polyhedral notions which will be used. Though the main motivation for this paper, the algorithms due to Mikhalkin and Markwig are not discussed detail, and any reader who desires to learn more about these is encouraged to read [Mik04] and [MR08].

In proving the main result we will need the following lemma, which is proved in the paper. This in itself is a particularly useful result in relating the geometry of a tropical curve to the geometry of it’s dual, and though it perhaps has often been used implicitly, (because it is in some sense "obvious") we do not believe a rigorous proof has yet been given.

**Lemma.** Given a reducible tropical curve $T = \bigcup_{i=1}^{n} T_i$, the dual to the intersection of (2,1,or 0 dimensional) cells $C_1, C_2, \ldots, C_n$ (each $C_i$ in $T_i$) is the Minkowski sum of the duals to each $C_i$. 
A light introduction to polyhedral geometry will be helpful, but not necessary since we cover the requisite material in section 3.

Finally, many of the basic definitions are, of course, not my own. I used definitions and ideas from [Mar06], [RGST03], [Gat06], and [Mik04], all of which are well worth a read.

2. Basic Notions

As discussed above, a tropical curve is a degeneration of the image of a complex (projective) curve under the log map. Tropical curves also arise naturally from polynomials over the field of Puiseux series under a certain valuation (which corresponds exactly to the aforementioned log map). A treatment of this approach can be found in [RGST03]. In either case, the object one recovers is a piecewise linear graph in $\mathbb{R}^2$ whose vertices are balanced. Thus, tropical curves can be treated entirely combinatorially.

Definition. Let $\bar{\Gamma}$ be a finite graph without 2-valent vertices, and let $V_\infty$ denote the set of all 1-valent vertices. An abstract tropical curve is the graph $\Gamma = \bar{\Gamma}\backslash V_\infty$ together with weights $w(e) \in \mathbb{N}$ assigned to each edge $e \in \Gamma$. By removing the 1-valent vertices, $\Gamma$ may have non-compact edges, called unbounded edges (or ends) of $\Gamma$.

Let $\Gamma$ be an abstract tropical plane curve. A parametrized tropical plane curve is a weight preserving proper map $h : \Gamma \to \mathbb{R}^2$ such that

1. For each edge $e$ of $\Gamma$, $h(e)$ is a segment (possibly trivial) or ray of an affine line with rational slope.

2. To the image $h(e)$ of each edge $e \in \Gamma$, we assign the weight $w(e)$.

3. At the image of each vertex, $h(V)$, the following balancing condition holds. If $v_1, v_2, \ldots, v_k$ are primitive vectors in the direction of the non-trivial edges $h(e_1), h(e_2), \ldots, h(e_k)$ incident at $V$, then

$$\sum_{i=1}^{n} v_i w(e_i) = 0$$

Remark 1. It may happen that the images of the edges in $\Gamma$ intersect along a segment or ray, in which case we label the intersection of the images with the tuple $(w(e_1), w(e_2), \ldots, w(e_k))$ of the weights of the edges which map to it. The weight of the intersection is the sum of the weights $w(e_i)$. Also, we called the image of a vertex of $\Gamma$ a vertex, and if the image of two edges intersect at a point, we call this point a crossing.

Example 1. A parametrized tropical curve:
Now that we have a notion of a tropical curve, we should define a notion of equivalence between them. Equivalent tropical curves are not distinguished from one another.

**Definition.** Two parametrized tropical curves, $h_1 : \Gamma_1 \to \mathbb{R}^2$ and $h_2 : \Gamma_2 \to \mathbb{R}^2$ are said to be isomorphic if there exists a homeomorphism $\phi : \Gamma_1 \to \Gamma_2$ which preserves the weights of edges and such that $h_1 = h_2 \circ \phi$.

The image $h(\Gamma)$ of a parametrized tropical curve is called an unparametrized tropical curve or just a tropical curve.

If a tropical curve has exactly $d$ unbounded edges (with edges $e$ whose weight is $> 1$ counted $w(e)$ times) in each of the directions (-1,0), (0,-1), and (1,1), it is said to be a projective tropical curve of degree $d$. This may sound like a rather absurd classification, but in fact, these are exactly the tropical curves which correspond to complex projective curves in $\mathbb{P}^2$ and are what we are interested in counting.

We mentioned in the introduction that tropical curves are dual to subdivided polygons in the plane. To see how, let us consider how to construct the dual to a tropical curve, say $T$: pick an edge $e$ of $T$ with weight $w$. Since $e$ has rational slope, consider a segment in the plane perpendicular to $e$ starting and ending at an integer lattice point and passing through exactly $w$ lattice points. Since each vertex or crossing of $T$ is balanced, doing this for each edge adjacent to $e$ at the vertex or crossing $v$, we create a polytope. Since, as a topological space, $T$ is connected, repeating this process yields a subdivided polytope.

As projective tropical curves have $d$ edges in the directions (0,-1), (-1,0), and (1,1), they are dual to a subdivision of a triangle in the plane with vertices $(0,d)$, $(d,0)$, and $(0,0)$. Denote such a triangle as $\Delta_d$.

**Example 2.** Dual to a projective curve:
An important notion in Markwig’s algorithm is that of \( n \)-marked tropical curves, and so we will give a definition, though it is not of extreme importance here. An \( n \)-marked tropical curve is a tropical curve \( T \) together with \( n \) special (distinct) points of \( T \) called marked points. If a point in the intersection of two edges is marked, it is considered a vertex of \( T \).

The idea here is that the marked points correspond to the psi-class conditions which define the classes of curves which we are enumerating. In particular they come from edges of abstract tropical curves which are shrunk to a point by the parametrization. For more detail see [MR08].

Remark 2. For the remainder of the paper, tropical curve of degree \( d \) will be a projective \( n \)-marked tropical curve of degree \( d \) for some \( n \in \mathbb{N} \). Since all projective tropical curves of degree \( d \) are dual to subdivisions of a triangle \( \Delta_d \), for a tropical curve \( T \) of degree \( d \) we will write \( \Delta_T \) for the triangle \( \Delta_d \) together with the subdivision given by \( T \).

The remaining notion we need is one of reducible tropical curves, which one might think of as a curve which is the union of two strictly smaller curves. However, we cannot define reducible curves in terms of the image of a parametrization, since in this setting we cannot distinguish among crossings and vertices. However, if we retain the information of the graph \( \Gamma \), a precise definition follows:

Definition. A tropical curve \( T \) is reducible if it can be parametrized by a disconnected abstract tropical curve. The images of these disjoint abstract curves, say \( \{T_i\}_{i=1}^n \) are said to be components of the tropical curve, and we write \( T = \bigcup_{i=1}^n T_i \).

Example 3. Reducible tropical curves:
3. Polyhedral Geometry

Now we make a short diversion into polyhedral geometry, in order to familiarize the reader with some basic notions which the proof will require. The definitions and results in this sections are from [San04].

Definition. If $P_1, \ldots, P_n$ are convex polytopes, then their Minkowski sum, $P_1 + \cdots + P_n$ is defined as the set $\{p_1 + \cdots + p_n | p_i \in P_i \}$.

Example 4. The Minkowski sum of the following lattice polytopes in $\mathbb{R}^2$, is
**Definition.** Let \( P = \sum_{i=1}^{n} P_i \). A **Minkowski cell** is a full dimensional polytope \( B \), where \( B = \sum_{i=1}^{n} B_i \) with \( B_i \) a (perhaps not full-dimensional) polytope having vertices in \( P_i \).

A **mixed subdivision** of \( P = \sum_{i=1}^{n} P_i \) is a family \( S \) of Minkowski cells covering \( P \) with the property that for any two Minkowski cells, \( B = \sum_{i=1}^{n} B_i \) and \( C = \sum_{i=1}^{n} C_i \) in \( S \), and for all \( i \) in the index set, the intersection of \( B_i \) and \( C_i \) is a face of both. We say \( B \) and \( C \) **intersect properly** in the Minkowski sense.

**Example 5.** For an example of such a subdivision let \( P \) be the lattice polytope (with subdivision as in the picture).

If \( P = \)

then, calling the left polytope \( A \) and the right \( B \), a mixed subdivision of \( P = A + B \) is (with obvious notation):

\[
\begin{align*}
P(0,3)P(1,2)P(0,2) &= a_{(0,1)}a_{(0,2)}a_{(1,1)} + b_{(0,1)} \\
P(0,2)P(0,1)P(1,2) &= a_{(0,1)}a_{(1,1)} + b_{(0,0)}b_{(0,1)} \\
P(0,0)P(0,1)P(1,1) &= a_{(0,0)}a_{(0,1)}a_{(1,1)} + b_{(0,0)} \\
P(0,0)P(1,1)P(1,0) &= a_{(0,0)}a_{(1,1)}a_{(1,0)} + b_{(0,0)} \\
P(1,1)P(2,1)P(1,2) &= a_{(1,1)} + B \\
P(1,0)P(1,1)P(2,1)P(2,0) &= a_{(1,1)}a_{(1,0)} + b_{(0,0)}b_{(1,0)} \\
P(2,0)P(2,1)P(3,0) &= a_{(1,0)}a_{(1,1)}a_{(2,0)} + b_{(1,0)}
\end{align*}
\]
The reader can either believe that all the cells intersect properly, or check for himself. It is not true that every subdivision of \( Q = \sum Q_i \) is a mixed subdivision. For a non-example consider \( P = A + B \) from above. The subdivision of \( P \):

\[
\begin{array}{c}
\text{does not admit a non-trivial labeling as a mixed subdivision. Also note that when we say that a subdivision of a polytope \( P \) is mixed, we must refer also to the particular Minkowski sum decomposition we are given. Indeed, if } P = \sum_{i=1}^{k} P_i \text{ with } P_i = P, \text{ then so long as all the Minkowski cells of } P \\
\text{intersect properly and cover } P, \text{ we have found a (trivial) mixed subdivision for } P.
\end{array}
\]

As noted in the definition, tropical curves themselves can be viewed as completely combinatorial objects. Any tropical curve is also a cell complex, where the vertices are 0-dimensional cells, the edges 1-dimensional cells, and and regions of \( \mathbb{R}^2 \) which these edges bound (and which do not contain any other edges) are 2-dimensional cells. It is clear that these 2-dimensional cells are dual to vertices of the dual Newton polytope. The next section uses this view of a tropical curve almost exclusively.

Finally, we will make use of one more result from [San04], the following lemma:

**Lemma 1** ([San04] Lemma 2.1). Let \( I \subset \{1, \ldots, k\} \). Suppose \( S \) is a mixed subdivision of \( P = \sum_{i=1}^{k} P_i \). Then \( \{ \sum_{i \in I} B_i : \sum_{i=1}^{k} B_i \in S \} \) is a mixed subdivision of \( \sum_{i \in I} P_i \).

4. **A Reducibility Condition for Tropical Curves**

We are finally ready to prove the main result of the paper, relating the concept of mixed subdivisions of the polytopes to reducibility of tropical curves. But first, a small definition to make the proof of the main technical lemma somewhat less notationally cumbersome:
Definition. Two 2-dimensional cells are adjacent if they meet along a 1-dimensional cell. A (finite) cell path $\gamma$ of a tropical curve $T$ is a finite sequence of 2-dimensional cells, $\{A_i\}_{i=0}^n$, such that for each $0 \leq i < n$, $A_i$ and $A_{i+1}$ are adjacent.

The original cell, denoted $O_T$, of an unparametrized tropical curve $T$ is the 2-dimensional cell of $T$ dual to $(0,0)$ in the Newton polytope.

Lemma 2. Let $T$ be a reducible tropical curve so that $T = T_1 \cup T_2$. For any two cells (2, 1, or 0 dimensional) $C_1$ and $C_2$ of $T_1$ and $T_2$ resp. the dual (in $\Delta_T$) to the intersection of $C_1$ and $C_2$ is the Minkowski sum of the duals $C_1'$ and $C_2'$ (in $\Delta_{T_i}$).

Proof. We first prove the result for 2-dimensional cells of $T$, and then extend to 1 and 0-dimensional cells. Let $O = O_{T_1} \cap O_{T_2}$. Clearly $O = O_T$, the orginal cell of $T$. Let $K$ be any 2-dimensional cell of $T$. $K = K_1 \cap K_2$, where $K_1$ and $K_2$ are 2-dimensional cells of $T_1$ and $T_2$ resp. Let $\gamma = \{A_i\}_{i=0}^n$ be a path through adjacent 2-dimensional cells, with $A_0 = O$ and $A_n = K$.

$\gamma$ starts at $A_0 = O$, which is dual to $(0,0)$. Suppose $O$ and $A_1$ are adjacent at the edge $e_1$, and $A_1$ is dual to the point $(a_1, b_1)$ with $a_1, b_1 \in \mathbb{Z}$. So $e_1$ is dual to the segment from $(0,0)$ to $(a_1, b_1)$. Note $e_1$ is either an edge of $T_1$ or $T_2$ or the intersection of an edge from each. In particular, moving from $O$ to $A_1$ across the edge $e_1$ is the same as adding a vector, say $v_1 = (a_1, b_1)$, to the point $(0,0)$ dual to $O$.

Suppose $A_2$ is dual to the point $(a_2, b_2)$ and is adjacent to $A_1$ on the edge $e_2$. As above, this edge either lies in $h_1$ or $h_2$ or is an intersection. Again, moving from $A_1$ to $A_2$ along $\gamma$ adds the vector $v_2 = (a_2 - a_1, b_2 - b_1)$ to $(a_1, b_1)$.

Continuing this process to $A_n = K$, we see that going from $A_{i-1}$ to $A_i$ is equivalent to adding the vector $v_i = (a_i - a_{i-1}, b_i - b_{i-1})$ to the point $(a_{i-1}, b_{i-1})$, dual to $A_{i-1}$ (with $(a_i, b_i)$ dual to $A_i$). So for the dual point $(a_n, b_n)$ to $A_n = K = K_1 \cap K_2$ we have:

\begin{equation}
(a_n, b_n) = \sum_{i=1}^n v_i
\end{equation}

But each $v_i$ came from the intersection of some set of edges, which were in either $T_1$ or $T_2$. In particular, if $I_1$ is the index set for edges $e_i$ in $T_1$ and similarly for $I_2$, if the edges $\{e_j\}_{j \in I_j}$ are viewed in $T_{j}$, then clearly the dual to $K_j$ is $\sum_{i \in I_j} v_j$.

By the associativity of vector sums:

\begin{equation}
\sum_{i=1}^n v_i = \sum_{j \in I_1} v_j + \sum_{k \in I_2} v_k
\end{equation}
So the dual to $K = K_1 \cap K_2$ is the Minkowski sum of the duals to $K_1$ and $K_2$ (since for points, Minkowski summation is just componentwise addition).

We need to check that the result is independent of the choice of cell path $\gamma$. Suppose we have some other path $\gamma'$ from $O$ to $K$. $\gamma'$ gives rise to a set of vectors $u_1, u_2, \ldots, u_m$ such that $(a_n, b_n) = K^\vee = \sum_{j=1}^m u_j = \sum_{i=1}^n v_i$. But as above $K = K_1 \cap K_2$, so there exist subsets of the vectors $u_i$ which come from edges in $T_1$ or edges in $T_2$. Let the index sets for these subsets be $R_1$ and $R_2$. So $\sum_{i \in R_1} u_i$ is dual to $K_1$ and $\sum_{j \in R_2} u_j$ is dual to $K_2$. In particular,

\[
\sum_{i \in I_2} v_i = \sum_{k \in R_2} u_k\]

so

\[
\sum_{i=1}^n v_i = \sum_{j \in I_1} v_j + \sum_{k \in I_2} v_k = \sum_{r \in R_1} u_r + \sum_{s \in R_2} u_s = \sum_{l=1}^m u_l
\]

as desired. Thus $\gamma$ and $\gamma'$ are equivalent choices of path.

To finish the proof, given any 2 cells $C_1$ and $C_2$ of $T_1$ and $T_2$ resp. of dimension 2, 1, or 0, their duals are a point or the convex hull of the duals to the 2-dimensional cells which they bound. If $C_1 \cap C_2 \neq \emptyset$, then these 2-dimensional cells intersect each other as well. Thus the dual to the intersection $C_1 \cap C_2$ is the convex hull of the points dual to intersection of these 2-dimensional cells. But the above result gives that this is the convex hull of the set $\{p_1 + p_2 \mid p_i \text{ vertex of } P_i\}$. Since the Minkowski sum of 2 convex polytopes $P_1$ and $P_2$ is the convex hull of the set $\{p_1 + p_2 \mid p_i \text{ vertex of } P_i\}$, replacing $P_i$ with $C_i$ dual, we have equality as desired.

\[\square\]

**Corollary 1.** The lemma above holds for $T = \bigcup_{i=1}^n T_i$ for any $n \in \mathbb{Z}_{>0}$.

**Proof.** This result follows by an easy induction. Using Lemma 1 as our base case, suppose the result holds for $n-1$ curves. Then, given some $T = \bigcup_{i=1}^n T_i$,

let $T' = \bigcup_{i=1}^{n-1} T_i$. By induction, the dual to each cell $C = \bigcap_{i=1}^n C_i$ of $T'$ is the Minkowski sum of the duals to each $C_i$. Since $T = T' \cup T_n$, for any cell $K$ of $T$, $K = C \cap C_n$, for some $C \subset T'$ and $C_n \subset T_n$. By the lemma, the dual to $C \cap C_n$ is the Minkowski sum of the duals, $C^\vee + C_n^\vee$. But $C^\vee = \sum_{i=1}^{n-1} C_i^\vee$, so
\[
K' = \sum_{i=1}^{n} C_i'
\]

as desired.

\[\blacksquare\]

**Theorem 1.** The unparametrized tropical curve \( T \) of degree \( d \) is reducible into \( k \) components if and only if for \( \Delta_d = \sum_{i=1}^{k} \Delta_{d_i} \), \( \Delta_T \) admits a labeling as a mixed subdivision.

**Proof.** (\( \Rightarrow \)) We suppose \( T \) is reducible into \( k \) components \( T_1, \ldots, T_k \) of degrees \( d_1, \ldots, d_k \) resp. Then clearly, as non-subdivided polytopes, \( \Delta_d = \sum_{i=1}^{k} \Delta_{d_i} \). Let the set of all interior polytopes of \( \Delta_T \) dual to vertices and crossings of \( T \) be our family \( S \) of Minkowski cells. We must show they intersect properly in the sense of Minkowski subdivisions. Any \( P \in S \) is dual to the intersection in \( T \) of cells \( P_1, \ldots, P_k \) of \( T_1, \ldots, T_k \), so by the lemma,

\[ P = \sum_{i=1}^{k} P_i. \]

So, given any two \( Q = \sum_{i=1}^{k} Q_i \) and \( Q' = \sum_{i=1}^{k} Q'_i \) in \( S \), \( Q_i \) and \( Q'_i \) are members of the subdivision of \( \Delta_{T_i} \) and so intersect in a face of each. Thus \( \text{sub}(\Delta_T) \) is mixed.

(\( \Leftarrow \)) Suppose \( \Delta_d = \sum_{i=1}^{k} \Delta_{d_i} \) and \( \Delta_T \) is mixed. Let \( S \) be the family of polytopes in \( \Delta_T \). By the Santos’s Lemma 1 in section 3, the mixed subdivision on \( \Delta_T \) induces a mixed subdivision on each \( \Delta_{d_m} \) by restricting each \( P = \sum_{i=1}^{k} P_i \) in \( S \) to \( P_m \), some \( 1 \leq m \leq k \). Since the induced subdivisions on the \( \Delta_{d_i} \) are mixed, in particular the set of \( P_m \) for a fixed \( m \) cover \( \Delta_{d_m} \) and intersect in faces. So we have a set \( S_m \) of convex polygons which cover \( \Delta_{d_m} \) and intersect in faces, so it is dual to a graph in \( \mathbb{R}^2 \) whose vertices are balanced and which has unbounded edges only in the directions (0,-1), (-1,0), and (1,1). Also, each edge of the graph has an associated weight in \( \mathbb{Z}_{>0} \) corresponding to the number of integer lattice points which its dual edge intersects. Hence, each \( \Delta_{d_i} \) is dual to a tropical curve.

Finally, since we are given the mixed subdivision on \( \Delta_d = \sum_{i=1}^{n} \Delta_{d_i} \), we know where each edge and vertex of the component curves \( T_i \) sit in the larger curve \( T \). Thus, we can parametrize \( T \) by \( k \) disconnected graphs, one for each curve \( T_i \), and so \( T \) is reducible.
References


