AN EQUIVALENT CONDITION FOR THE REDUCIBILTY OF TROPICAL CURVES

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Abstract. In response to a recent algorithm for counting the number of isomorphism classes of irreducible tropical curves through 3d+g-1 general points in the plane satisfying certain psi-class conditions by Hannah Markwig and Johannes Rau in [MR08], a method for determining the reducibility of a tropical cruve in terms of it's dual Newton polytope is required. This paper provides a solution by giving an equivalent condition for reducibility in terms of a mixed subdivision of the dual Newton polytope.

1. Introduction

How many lines pass through 2 general points in the plane? How many quadratics pass through 5 general points? How many cubics pass through 8? Enumerative geometry has been studied since the time of the ancient Greeks and only recently was a complete solution to the problem of counting curves given (by Kontsevich and Manin in [KM94]).

It turns out that in order for the number of curves of degree d, N_d, to be finite (but non-zero), they must be fixed by exactly $3d-1$ general points in the plane. For example, there are infinitely many lines passing through one point, while no lines pass through 3 general points. So, in general one asks this question - how many complex (projective) curves of degree d (genus 0) pass through $3d-1$ fixed general points in the plane. The answers to low degree examples were classically known; $N_1 = N_2 = 1$ and $N_3 = 12$. However, for higher degrees, N_d becomes large and hard to compute. Recently Kontsevich initiated the study of moduli spaces and produced the recursion:

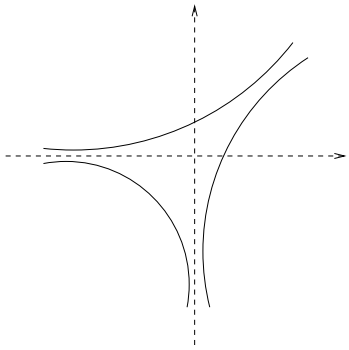
$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)$$

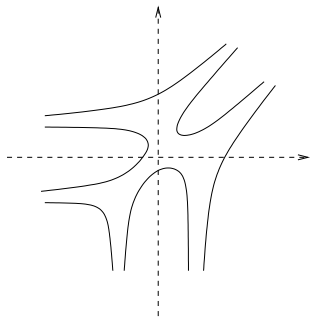
giving a complete solution for the enumeration of complex plane curves of any degree. With amazing elegance, the only information needed to start the recursion is that 1 line passes through 2 distinct points. Euclid might have told us that!

Tropical geometry has given another method of researching enumerative geometry. Let us first give a brief motivating overview of tropical geometry. Let F be a complex plane curve. Consider the map

$$\begin{aligned}\log :&(\mathbf{C}^*)^2 \to \mathbf{R}^2\\ &(z_1, z_2) \mapsto (log|z_1|, log|z_2|)\end{aligned}$$

The image $\log(F \cap (\mathbf{C}^*)^2)$ is called the *amoeba* of F and some examples are included here.

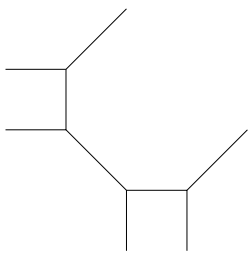


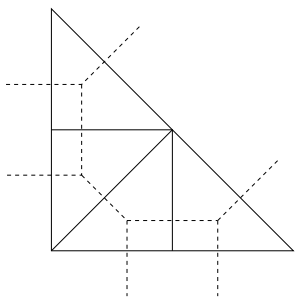


A general curve of degree d will have d "tentacles" in each of the (0,-1), (-1,0), and (1,1) directions. We would like to turn this ameoba into a purely combinitorial construction. To do this consider the map

$$\begin{aligned}\log_t :&(\mathbf{C}^*)^2 \to \mathbf{R}^2\\ &(z_1, z_2) \mapsto \left(-\frac{log|z_1|}{log(t)}, -\frac{log|z_2|}{log(t)}\right)\end{aligned}$$

As $t \to 0$, the image of $F \cap (\mathbf{C}^*)^2$ is a piecewise linear graph in the plane. We call this object a *tropical plane curve*. In fact, tropical curves of degree d have exactly d edges in each of the directions (0,-1), (-1,0), and (1,1), and every vertex is balanced. This makes them rather nice. In particular, a tropical curve is dual to a subdivided convex polytope. Note that this correspondence is not 1:1 - there can exist more than one tropical curve for a subdivided polytope.





With this construction complex (projective) curves are transformed into piecewise linear graphs in $\mathbf{R}^2$. *A priori*, much information about the original structure of the curve F has been lost, but in fact we have not lost so much as to make this degeneration of F unusable. For a detailed discussion of the link between tropical and algebraic geometry, see [Gat06].

How does this relate to the question of counting curves? In [Mik04] Mikhalkin proved his Correspondence Theorem: that the number of isomorphism classes of complex projective curves in $\mathbf{P}^2$ of degree d and genus g, $N(d,g)$, passing through $3d+g-1$ fixed general points is equal to the number of isomorphism classes of tropical curves of degree d and genus g, $N_{trop}(d,g)$ passing through the same number of points. He also gave an algorithm to count $N_{trop}(d,g)$, by counting lattice paths in the dual polytope (with certain multiplicity).

More recently Markwig and Rau (in [MR08]) has developed a similar algorithm to count irreducible tropical curves satisfying certain "psi-class" conditions. These correspond to conditions imposed at the fixed points on the curves we wish to enumerate. The multiplicity of some lattice paths may not count, since they correspond to reducible tropical curves. This is the motivating problem for this paper. In general, we would like a chracterization of irreduciblity of a tropical curve based entirely on the dual Newton polygon. We will solve this problem by proving the following theorem:

Theorem. *The tropical curve T of degree d is reducible into k components if and only if for its dual polytope $\Delta_d = \sum_{i=1}^{k} \Delta_{d_i}$, the subdivision corresponding to T admits a labeling as a mixed subdivision.*

Before coming to this main result, we take some time to cover the defintions and notation used, since tropical geometry is a young subject and standardizations of many of the main ideas have not yet been introduced. A basic review of certain facts from polyhedral geometry is given as well, to familarize the reader with the polyhedral notions which will be used. Though the main motivation for this paper, the algorithms due to Mikhalkin and Markwig are not discussed detail, and any reader who desires to learn more about these is encouraged to read [Mik04] and [MR08].

In proving the main result we will need the following lemma, which is proved in the paper. This in itself is a particularly useful result in relating the geometry of a tropical curve to the geometry of it's dual, and though it perhaps has often been used implicitly, (because it is in some sense "obvious") we do not believe a rigorous proof has yet been given.

Lemma. *Given a reducible tropical curve $T = \bigcup_{i=1}^{n} T_i$, the dual to the intersection of (2,1,or 0 dimensional) cells $C_1, C_2, \ldots, C_n$ (each C_i in T_i) is the Minkowski sum of the duals to each C_i.*

A light introduction to polyhedral geometry will be helpful, but not necessary since we cover the requisite material in section 3.

Finally, many of the basic definitions are, of course, not my own. I used definitions and ideas from [Mar06], [RGST03], [Gat06], and [Mik04], all of which are well worth a read.

2. Basic Notions

As discussed above, a tropical curve is a degeneration of the image of a complex (projective) curve under the log map. Tropical curves also arise naturally from polynomials over the field of Puiseux series under a certain valuation (which corresponds exactly to the aforementioned log map). A treatment of this approach can be found in [RGST03]. In either case, the object one recovers is a piecewise linear graph in $\mathbf{R}^2$ whose vertices are balanced. Thus, tropical curves can be treated entirely combinitorially.

Definition. Let $\bar{\Gamma}$ be a finite graph without 2-valent vertices, and let V_∞ denote the set of all 1-valent vertices. An *abstract tropical curve* is the graph $\Gamma = \bar{\Gamma} \backslash V_\infty$ together with weights $w(e) \in \mathbf{N}$ assigned to each edge $e \in \Gamma$. By removing the 1-valent vertices, Γ may have non-compact edges, called *unbounded edges (or ends)* of Γ.

Let Γ be an abstract tropical plane curve. A *parametrized tropical plane curve* is a weight preserving proper map $h : \Gamma \to \mathbf{R}^2$ such that

(1) For each edge e of Γ, $h(e)$ is a segment (possibly trivial) or ray of an affine line with rational slope.

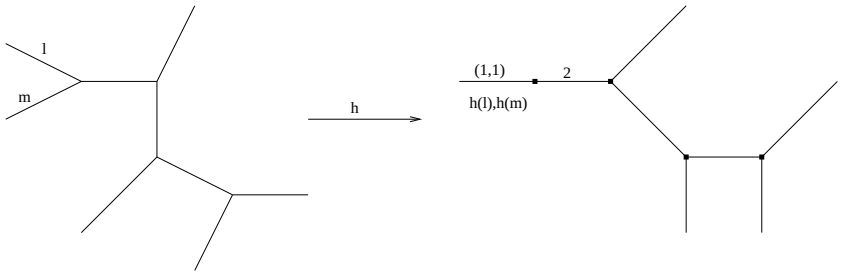
(2) To the image $h(e)$ of each edge $e \in \Gamma$, we assign the weight $w(e)$.

(3) At the image of each vertex, $h(V)$, the following balancing condition holds. If $v_1, v_2, \ldots, v_k$ are primitive vectors in the direction of the non-trivial edges $h(e_1), h(e_2), \ldots, h(e_k)$ incident at V, then

$$\sum_{i=1}^{n} v_i w(e_i) = 0$$

Remark 1. It may happen that the images of the edges in Γ intersect along a segement or ray, in which case we label the intersection of the images with the tuple $(w(e_1), w(e_2), \ldots, w(e_k))$ of the weights of the edges which map to it. The weight of the intersection is the sum of the weights $w(e_i)$. Also, we called the image of a vertex of Γ a *vertex*, and if the image of two edges intersect at a point, we call this point a *crossing*.

Example 1. *A parametrized tropical curve:*



Now that we have a notion of a tropical curve, we should define a notion of equivalence between them. Equivalent tropical curves are not distinguised from one another.

Definition. Two parametrized tropical curves, $h_1 : \Gamma_1 \to \mathbf{R}^2$ and $h_2 : \Gamma_2 \to \mathbf{R}^2$ are said to be *isomorphic* if there exists a homeomorphism $\phi : \Gamma_1 \to \Gamma_2$ which preserves the weights of edges and such that $h_1 = h_2 \circ \phi$.

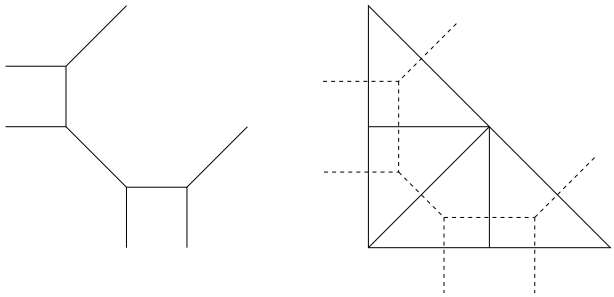
The image $h(\Gamma)$ of a parametrized tropical curve is called an *unparametrized tropical curve* or just a *tropical curve.*

If a tropical curve has exactly d unbounded edges (with edges e whose weight is > 1 counted $w(e)$ times) in each of the directions (-1,0), (0,-1), and (1,1), it is said to be a *projective tropical curve of degree d.* This may sound like a rather absurd classification, but in fact, these are exactly the tropical curves which correspond to complex projective curves in $\mathbf{P}^2$ and are what we are interested in counting.

We mentioned in the introduction that tropical curves are dual to subdivided polygons in the plane. To see how, let us consider how to construct the dual to a tropical curve, say T: pick an edge e of T with weight w. Since e has rational slope, consider a segment in the plane perpendicular to e starting and ending at an integer lattice point and passing through exactly w lattice points. Since each vertex or crossing of T is balanced, doing this for each edge adjacent to e at the vertex or crossing v, we create a polytope. Since, as a topological space, T is connected, repeating this process yields a subdivided polytope.

As projective tropical curves have d edges in the directions (0,-1), (-1,0), and (1,1), they are dual to a subdivision of a triangle in the plane with vertices $(0, d)$, $(d, 0)$, and $(0, 0)$. Denote such a triangle as Δ_d.

Example 2. *Dual to a projective curve:*



An important notion in Markwig's algorithm is that of $n-marked$ tropical curves, and so we will give a defintion, though it is not of extreme importance here. An *n-marked tropical curve* is a tropical curve T together with n special (distinct) points of T called *marked points*. If a point in the intersection of two edges is marked, it is considered a vertex of T.

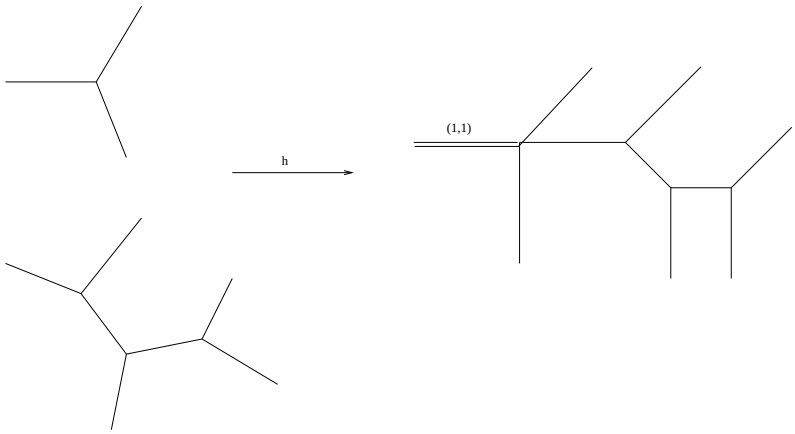
The idea here is that the marked points correspond to the psi-class conditions which define the classes of curves which we are ennumerating. In particular they come from edges of abstract tropical curves which are shrunk to a point by the parametrization. For more detail see [MR08].

Remark 2. For the remainder of the paper, *tropical curve of degree d* will be a projective n-marked tropical curve of degree d for some $n \in \mathbf{N}$. Since all projective tropical curves of degree d are dual to subdivisions of a triangle Δ_d, for a tropical curve T of degree d we will write Δ_T for the triangle Δ_d together with the subdivision given by T.

The remaining notion we need is one of *reducible* tropical curves, which one might think of as a curve which is the union of two strictly smaller curves. However, we cannot define reducible curves in terms of the image of a parametrization, since in this setting we cannot distinguish among crossings and vertices. However, if we retain the information of the graph Γ, a precise definition follows:

Definition. A tropical curve T is *reducible* if it can be parametrized by a disconnected abstract tropical curve. The images of these disjoint abstract curves, say $\{T_i\}_{i=1}^n$ are said to be *components* of the tropical curve, and we write $T = \bigcup_{i=1}^{n} T_i$.

Example 3. *Reducible tropical curves:*

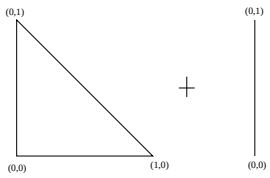


3. Polyhedral Geometry

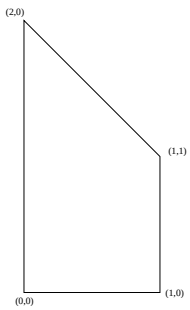
Now we make a short diversion into polyhedral geometry, in order to familiarize the reader with some basic notions which the proof will require. The definitions and results in this sections are from [San04].

Definition. If $P_1, \ldots, P_n$ are convex polytopes, then their *Minkowski sum*, $P_1 + \cdots + P_n$ is defined as the set $\{p_1 + \cdots + p_n | p_i \in P_i\}$.

Example 4. *The Minkowski sum of the following lattice polytopes in* $\mathbf{R}^2$,



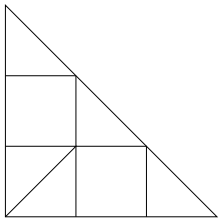
is



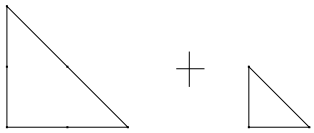
Definition. Let $P = \sum_{i=1}^{n} P_i$. A *Minkowski cell* is a full dimensional polytope B, where $B = \sum_{i=1}^{n} B_i$ with B_i a (perhaps not full-dimensional) polytope having vertices in P_i.

A *mixed subdivision* of $P = \sum_{i=1}^{n} P_i$ is a family S of Minkowski cells covering P with the property that for any two Minkowski cells, $B = \sum_{i=1}^{n} B_i$ and $C = \sum_{i=1}^{n} C_i$ in S, and for all i in the index set, the intersection of B_i and C_i is a face of both. We say B and C *intersect properly* in the Minkowski sense.

Example 5. *For an example of such a subdivision let P be the lattice polytope*



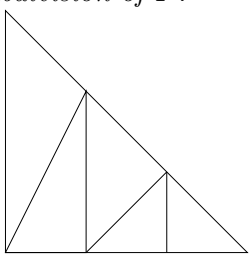
(with subdivison as in the picture).
If $P =$



then, calling the left polytope A and the right B, a mixed subdivision of $P = A + B$ is (with obvious notation):

$$
\begin{aligned}
p_{(0,3)}p_{(1,2)}p_{(0,2)} &= a_{(0,1)}a_{(0,2)}a_{(1,1)} + b_{(0,1)} \\
p_{(0,2)}p_{(0,1)}p_{(1,1)}p_{(1,2)} &= a_{(0,1)}a_{(1,1)} + b_{(0,0)}b_{(0,1)} \\
p_{(0,0)}p_{(0,1)}p_{(1,1)} &= a_{(0,0)}a_{(0,1)}a_{(1,1)} + b_{(0,0)} \\
p_{(0,0)}p_{(1,1)}p_{(1,0)} &= a_{(0,0)}a_{(1,1)}a_{(1,0)} + b_{(0,0)} \\
p_{(1,1)}p_{(2,1)}p_{(1,2)} &= a_{(1,1)} + B \\
p_{(1,0)}p_{(1,1)}p_{(2,1)}p_{(2,0)} &= a_{(1,1)}a_{(1,0)} + b_{(0,0)}b_{(1,0)} \\
p_{(2,0)}p_{(2,1)}p_{(3,0)} &= a_{(1,0)}a_{(1,1)}a_{(2,0)} + b_{(1,0)}
\end{aligned}
$$

The reader can either believe that all the cells intersect properly, or check for himself. It is not *true that every subdivision of* $Q = \sum Q_i$ *is a mixed subdivision. For a non-example consider* $P = A + B$ *from above. The subdivision of* P*:*



does not admit a non-trivial labeling as a mixed subdivision. Also note that when we say that a subdivision of a polytope P *is mixed, we must refer also to the particular Minkowski sum decomposition we are given. Indeed, if* $P = \sum_{i=1}^{1} P_i$ *with* $P_i = P$*, then so long as all the Minkowski cells of* P *intersect properly and cover* P*, we have found a (trivial) mixed subdivision for* P*.*

As noted in the defintion, tropical curves themselves can be viewed as completely combinitorial objects. Any tropical curve is also a cell complex, where the vertices are 0-dimensional cells, the edges 1-dimensional cells, and and regions of $\mathbf{R}^2$ which these edges bound (and which do not contain any other edges) are 2-dimensional cells. It is clear that these 2-dimensional cells are dual to vertices of the dual Newton polytope. The next section uses this view of a tropical curve almost exclusively.

Finally, we will make use of one more result from [San04], the following lemma:

Lemma 1 ([San04] Lemma 2.1)**.** *Let* $I \subset \{1, \ldots, k\}$*. Suppose* S *is a mixed subdivision of* $P = \sum_{i=1}^{k} P_i$*. Then* $\{\sum_{i \in I} B_i : \sum_{i=1}^{k} B_i \in S\}$ *is a mixed subdivision of* $\sum_{i \in I} P_i$*.*

4. A Reducibility Condition for Tropical Curves

We are finally ready to prove the main result of the paper, relating the concept of mixed subdivisions of the polytopes to reducibilty of tropical curves. But first, a small definition to make the proof of the main technical lemma somewhat less notationally cumbersome:

Definition. Two 2-dimensional cells are *adjacent* if they meet along a 1-dimensional cell. A *(finite) cell path* γ of a tropical curve T is a finite sequence of 2-dimensional cells, $\{A_i\}_{i=0}^n$, such that for each $0 \leq i < n$, A_i and A_{i+1} are adjacent.

The *original cell*, denoted $\mathbf{O}_T$, of an unparamentrized tropical curve T is the 2-dimensional cell of T dual to $(0,0)$ in the Newton polytope.

Lemma 2. *Let T be a reducible tropical curve so that $T = T_1 \cup T_2$. For any two cells (2,1, or 0 dimensional) C_1 and C_2 of T_1 and T_2 resp. the dual (in Δ_T) to the intersection of C_1 and C_2 is the Minkowski sum of the duals $C_1^\vee$ and $C_2^\vee$ (in Δ_{T_i}).*

Proof. We first prove the result for 2-dimensional cells of T, and then extend to 1 and 0-dimensional cells. Let $\mathbf{O} = \mathbf{O}_{T_1} \cap \mathbf{O}_{T_2}$. Clearly $\mathbf{O} = \mathbf{O}_T$, the orginal cell of T. Let K be any 2-dimensional cell of T. $K = K_1 \cap K_2$, where K_1 and K_2 are 2-dimensional cells of T_1 and T_2 resp. Let $\gamma = \{A_i\}_{i=0}^n$ be a path through adjacent 2-dimensional cells, with $A_0 = \mathbf{O}$ and $A_n = K$.

γ starts at $A_0 = \mathbf{O}$, which is dual to $(0,0)$. Suppose $\mathbf{O}$ and A_1 are adjacent at the edge e_1, and A_1 is dual to the point (a_1, b_1) with $a_1, b_1 \in \mathbf{Z}$. So e_1 is dual to the segment from $(0,0)$ to (a_1, b_1). Note e_1 is either an edge of T_1 or T_2 or the intersection of an edge from each. In particular, moving from $\mathbf{O}$ to A_1 across the edge e_1 is the same as adding a vector, say $v_1 = (a_1, b_1)$, to the point $(0,0)$ dual to $\mathbf{O}$.

Suppose A_2 is dual to the point (a_2, b_2) and is adjacent to A_1 on the edge e_2. As above, this edge either lies in h_1 or h_2 or is an intersection. Again, moving from A_1 to A_2 along γ adds the vector $v_2 = (a_2 - a_1, b_2 - b_1)$ to (a_1, b_1).

Continuing this process to $A_n = K$, we see that going from A_{i-1} to A_i is equivalent to adding the vector $v_i = (a_i - a_{i-1}, b_i - b_{i-1})$ to the point (a_{i-1}, b_{i-1}), dual to A_{i-1} (with (a_i, b_i) dual to A_i). So for the dual point (a_n, b_n) to $A_n = K = K_1 \cap K_2$ we have:

$$(a_n, b_n) = \sum_{i=1}^n v_i \tag{1}$$

But each v_i came from the intersection of some set of edges, which were in either T_1 or T_2. In particular, if I_1 is the index set for edges e_i in T_1 and similarly for I_2, if the edges $\{e_j\}_{j \in I_j}$ are viewed in T_j, then clearly the dual to K_j is $\sum_{i \in I_j} v_j$.

By the associativity of vector sums:

$$\sum_{i=1}^n v_i = \sum_{j \in I_1} v_j + \sum_{k \in I_2} v_k \tag{2}$$

So the dual to $K = K_1 \cap K_2$ is the Minkowski sum of the duals to K_1 and K_2 (since for points, Minkowski summation is just componentwise addition).

We need to check that the result is independant of the choice of cell path γ. Suppose we have some other path γ' from $\mathbf{O}$ to K. γ' gives rise to a set of vectors $u_1, u_2, \ldots, u_m$ such that $(a_n, b_n) = K^\vee = \sum_{j=1}^{m} u_j = \sum_{i=1}^{n} v_i$. But as above $K = K_1 \cap K_2$, so there exist subsets of the vectors u_i which come from edges in T_1 or edges in T_2. Let the index sets for these subsets be R_1 and R_2. So $\sum_{i \in R_1} u_i$ is dual to K_1 and $\sum_{j \in R_2} u_j$ is dual to K_2. In particular,

$$\sum_{i \in I_j} v_i = \sum_{k \in R_j} u_k \tag{3}$$

so

$$\sum_{i=1}^{n} v_i = \sum_{j \in I_1} v_j + \sum_{k \in I_2} v_k = \sum_{r \in R_1} u_r + \sum_{s \in R_2} u_s = \sum_{l=1}^{m} u_l \tag{4}$$

as desired. Thus γ and γ' are equivalent choices of path.

To finish the proof, given any 2 cells C_1 and C_2 of T_1 and T_2 resp. of dimension 2, 1, or 0, their duals are a point or the convex hull of the duals to the 2-dimensional cells which they bound. If $C_1 \cap C_2 \neq \emptyset$, then these 2-dimensional cells intersect eachother as well. Thus the dual to the intersection $C_1 \cap C_2$ is the convex hull of the points dual to intersection of these 2-dimensional cells. But the above result gives that this is the convex hull of the set $\{p_1 + p_2 \mid p_i \text{ vertex of dual to } C_i\}$. Since the Minkowski sum of 2 convex polytopes P_1 and P_2 is the convex hull of the set $\{p_1 + p_2 \mid p_i \text{ vertex of } P_i\}$, replacing P_i with C_i dual, we have equality as desired.

□

Corollary 1. *The lemma above holds for* $T = \bigcup_{i=1}^{n} T_i$ *for any* $n \in \mathbf{Z}_{>0}$.

Proof. This result follows by an easy induction. Using Lemma 1 as our base case, suppose the result holds for $n-1$ curves. Then, given some $T = \bigcup_{i=1}^{n} T_i$, let $T' = \bigcup_{i=1}^{n-1} T_i$. By induction, the dual to each cell $C = \bigcap_{i=1}^{n-1} C_i$ of T' is the Minkowski sum of the duals to each C_i. Since $T = T' \cup T_n$, for any cell K of T, $K = C \cap C_n$, for some $C \subset T'$ and $C_n \subset T_n$. By the lemma, the dual to $C \cap C_n$ is the Minkowski sum of the duals, $C^\vee + C_n^\vee$. But $C^\vee = \sum_{i=1}^{n-1} C_i^\vee$, so

$$K^\vee = \sum_{i=1}^{n} C_i^\vee \tag{5}$$

as desired.

□

Theorem 1. *The unparametrized tropical curve T of degree d is reducible into k components if and only if for $\Delta_d = \sum_{i=1}^{k} \Delta_{d_i}$, Δ_T admits a labeling as a mixed subdivision.*

Proof. ($\Rightarrow$) We suppose T is reducible into k components $T_1, \ldots, T_k$ of degrees $d_1, \ldots, d_k$ resp. Then clearly, as non-subdivided polytopes, $\Delta_d = \sum_{i=1}^{k} \Delta_{d_i}$. Let the set of all interior polytopes of Δ_T dual to vertices and crossings of T be our family S of Minkowski cells. We must show they intersect properly in the sense of Minkowski subdivisions. Any $P \in S$ is dual to the intersection in T of cells $P_1, \ldots, P_k$ of $T_1, \ldots, T_k$, so by the lemma, $P = \sum_{i=1}^{k} P_i$. So, given any two $Q = \sum_{i=1}^{k} Q_i$ and $Q' = \sum_{i=1}^{k} Q'_i$ in S, Q_i and Q'_i are members of the subdivision of Δ_{T_i} and so intersect in a face of each. Thus $\mathrm{sub}(\Delta_T)$ is mixed.

($\Leftarrow$) Suppose $\Delta_d = \sum_{i=1}^{k} \Delta_{d_i}$ and Δ_T is mixed. Let S be the family of polytopes in Δ_T. By the Santos's Lemma 1 in section 3, the mixed subdivision on Δ_T induces a mixed subdivision on each Δ_{d_m} by restricting each $P = \sum_{i=1}^{k} P_i$ in S to P_m, some $1 \leq m \leq k$. Since the induced subdivisions on the Δ_{d_i} are mixed, in particular the set of P_m for a fixed m cover Δ_{d_m} and intersect in faces. So we have a set S_m of convex polygons which cover Δ_{d_m} and intersect in faces, so it is dual to a graph in $\mathbf{R}^2$ whose vertices are balanced and which has unbounded edges only in the directions (0,-1), (-1,0), and (1,1). Also, each edge of the graph has an associated weight in $\mathbf{Z}_{>0}$ corresponding to the number of integer lattice points which its dual edge intersects. Hence, each Δ_{d_i} is dual to a tropical curve.

Finally, since we are given the mixed subdivision on $\Delta_d = \sum_{i=1}^{n} \Delta_{d_i}$, we know where each edge and vertex of the component curves T_i sit in the larger curve T. Thus, we can parametrize T by k disconnected graphs, one for each curve T_i, and so T is reducible.

□

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