MATH 494 2018 DISCUSSION 5: FINITE AND INTEGRAL RING EXTENSIONS

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This document is a partial reproduction of Mircea Mustata's first commutative algebra review for his course 631, Algebraic Geometry I. All rings will be unital and commutative and all homomorphisms will be unital commutative ring homomorphisms.

Let $\varphi : R \to S$ be a ring homomorphism. We say that φ is of *finite type* if S becomes, via φ, a finitely-generated R-algebra. φ is said to be *finite* when S becomes finitely-generated R-module via φ. We say that φ is *integral* when each element $s \in S$ can be expressed as a root of a polynomial $p \in R[x]$, i.e.

$$0 = a_1 s + a_2 s^2 + \cdots + a_n s^n \qquad \text{for some } (a_i)_{i \le n} \in R.$$

Remark. Clearly any finite $\varphi : R \to S$ is also of finite-type; the generators for S over R as a module will also generate it as an algebra. However the converse is not true: the inclusion $R \hookrightarrow R[x]$ is finite-type but not finite. Any R-submodule of $R[x]$ contains polynomials of bounded degree.

Remark. If φ is finite type and integral, then it is finite. For any generating set $s_1, ..., s_m$ of S as an R-algebra, we may write

$$s_i^{d_i} + a_{i,1} s_i^{d_i - 1} + \cdots + a_{i,d_i} = 0.$$

It is clear then that $\{s_1^{a_1} \cdots s_r^{a_r} : 0 \le a_i \le d_i - 1\}$ generate S as an R-module.

Proposition 0.1. *If φ is finite, then it is integral.*

Proof. Let $b_1, ..., b_m$ be generators of S as an R-module. Then it is clear that for all $y \in S$ we can write

$$y b_i = \sum_{j=1}^{m} a_{i,j} y_j, \quad a_{i,j} \in R.$$

Let $A = [a_{i,j}]_{i \le i,j \le m}$ be the matrix of the $a_{i,j}$. Then it is clear that

$$(yI - A) \begin{pmatrix} b_1 \\ \cdots \\ b_m \end{pmatrix} = 0$$

so after multiplying by the adjugate of the matrix $yI - A$, we have if $D = \det(yI - A)$ that $Db_i = 0$ for each i. Then we have that $D \cdot S = 0$, so in particular we have $D \cdot 1_S = 0$. But we can clearly write $D = y^m + c_1 y^{m-1} + \cdots + c_m$ for some $c_i \in R$, so we have that φ is integral.

This is known as the *determinant trick.* □

Remark. We will mostly consider finite type morphisms. From the above, we can conclude that a morphism is finite if and only if it is integral.

Proposition 0.2. *Let* $R \xrightarrow{\varphi} S \xrightarrow{\psi} T$ *be finite (resp. finite type, integral) homomorphisms. Then* $\psi \circ \varphi$ *has the same property.*

Proof. The statement for finite (hence finite-type) are obvious. Now suppose that φ, ψ are integral. Then for any $u \in T$ we may write

$$u^n + b_1 u^{n-1} + \cdots + b_n = 0, \quad b_i \in S.$$

Then since each $b_i \in S$ is integral over R, we have that $R' = R[b_1, ..., b_n]$ is integral. From an early remark above, R' is finite over R. Then since u is integral over R', $R'[u]$ is integral over R, so it is finite. From a proposition above, it is integral. □