MATH 494 2018 DISCUSSION 3: AFFINE SPACE AND THE ZARISKI TOPOLOGY

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Contents

1. Introduction: affine space

We will introduce *affine n-space* $\mathbb{A}^n$, the appropriate setting for the geometry of algebraic varieties. The definition of affine space will depend on the choice of a base field k, which we will insist on being algebraically closed. As a set, affine n-space is equal to the k-vector space k^n, however we will give it a special topology, the *Zariski topology*, and describe some of its features. In the next section we will give this topological space a sheaf of rings, the *structure sheaf* on affine space. We will define a *locally ringed space*, and show that affine space with the structure sheaf is a locally ringed space.

2. As a set

As mentioned, we define affine n-space as a set to be the points k^n. We call $\mathbb{A}^1$ the *affine line*, and $\mathbb{A}^2$ the *affine plane*. Note that when $k = \mathbb{C}$, this has the (potentially upsetting) effect of reducing the "dimension" of the space by half. Dimension theory is a bit complicated (and algebraic), and we won't deal with it here.

We note that we have a natural action of the polynomial ring $k[x_1, ..., x_n]$ on $\mathbb{A}^n$, via evaluation. If we stopped thinking about polynomials as functions in 494, we begin again to think of them that way now. Polynomials will be thought of in this document as functions $k^n \to k$. When $n = 1$, the zero sets of these polynomials are exactly the finite sets (can you see how?); when $n > 1$, they are much more exotic. Draw some pictures.

For polynomials $f_1, ..., f_k \in k[x_1, ..., x_n]$, we denote by $V(f_1, ..., f_k)$ the set $\{x \in \mathbb{A}^n \mid f_i(x) = 0 \text{ for all } 1 \leq i \leq k\}$, the *vanishing set* of the f_i. These sets are often called *algebraic* subsets of affine space.

Exercise. Show that if $f_1, ..., f_n \in k[x_1, ..., x_n]$, then $V(f_1, ..., f_k) = V((f_1, ..., f_k))$, where in the second term we replace $f_1, ..., f_k$ by the ideal they generate in $k[x_1, ..., x_k]$. Show that we may further replace $(f_1, ..., f_k)$ by $\sqrt{(f_1, ..., f_k)}$.

3. As a topological space

Now we introduce the Zariski topology. A subset $V \subset \mathbb{A}^n$ is declared to be closed in the Zariski topology if and only if there exists an ideal $I \subset k[x_1, ..., x_n]$ such that $V = V(I)$. As we have shown in the previous discussion section that the polynomial ring $k[x_1, ..., x_n]$ is Noetherian, and after appealing to the above exercise, we see that this is equivalent to the fact that $V = V(f_1, ..., f_k)$ for some polynomials $f_1, ..., f_k \in k[x_1, ..., x_n]$.

Exercise. Verify that this defines a topology on $\mathbb{A}^n$. Describe the Zariski-closed sets in $\mathbb{A}^1$.

If you've seen Hilbert's Nullstellensatz, or if you Google it, you will see that this relationship gives a correspondence between ideals in $k[x_1, ..., x_n]$ and Zariski-closed subsets of affine n-space. We state it for the record, but we will not prove it.

Theorem 3.1 (Hilbert's Nullstellensatz)**.** *There is a one-to-one, inclusion-reversing correspondence*

$$\{\text{closed subsets of } \mathbb{A}^n\} \longleftrightarrow \{\text{radical ideals in } k[x_1, ..., x_n]\}.$$

It follows that the maximal (hence prime, hence radical) ideals of $k[x_1, ..., x_n]$ correspond to the minimal closed subsets of affine space. These are precisely the points, so we obtain the so-called weak Nullstellensatz: the maximal ideals of $\bar{k}[x_1, ..., x_n]$ are the ideals $(x_1 - a_1, ..., x_n - a_n)$.

For any closed subset $V \subset \mathbb{A}^n$, we define $I(V)$ the *ideal associated to* V to be the set $\{f \in k[x_1, ..., x_n] \mid f|_V = 0\}$.

Exercise. Verify that this is a radical ideal. The Nullstellensatz can thus be reformulated as: $I(V(J)) = \sqrt{J}$.

Definition 3.2. A topological space is called *Noetherian* provided that there is no descending chain of closed subsets $V_1 \supsetneq V_2 \supsetneq V_3 \supsetneq \cdots$.

Exercise. Verify that affine space with the Zariski topology is Noetherian, using the Nullstellensatz.

Exercise. Noetherian topological spaces are weird: show that a Noetherian topological space is Hausdorff if and only if it is discrete, with finitely many points. Show that any open subset of a Noetherian topological space is dense.

Definition 3.3. A topological space is *irreducible* provided that it is not the union of two proper closed subsets.

Exercise (Harder). Show that every Noetherian topological space X admits a decomposition $X = X_1 \cup \cdots \cup X_d$ into irreducible closed subsets X_i. (This can be done with contradiction.)

Exercise. Think about what irreducibility means on a ring-theoretic level. Formulate and prove a statement.