MATH 494 2018 DISCUSSION 1: NOETHERIAN RINGS

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1. Introduction

Noetherian rings are a particularly well-behaved class of commutative rings. They are, fortunately, the generic family of rings that occur in commutative algebra and algebraic geometry. Indeed, so says Ravi Vakil:

"In general, I like having as few hypotheses as possible. Certainly a hypothesis that isnt necessary to the proof is a red herring. But if a reasonable hypothesis can make the proof cleaner and more memorable, I am willing to include it.

In particular, Noetherian hypotheses are handy when necessary, but are oth- erwise misleading. Even Noetherian-minded readers (normal human beings) are better off having the right hypotheses, as they will make clearer why things are true."

So in a sense, in this discussion you will learn how to be normal human beings.

2. Some recollections

We will fortunately only require a few basic definitions from ring theory. We will enforce that all of our rings will be commutative with multiplicative identity. We recall that an *ideal* I of a ring R is a subset of R which forms an abelian group under addition and which absorbs multiplication from R; that is, $rI \subset I$ for all $r \in R$.

(Aside: consider an arbitrary surjection of commutative rings $f : R \to S$; via the first isomorphism we may recover S as a quotient of R via $R/\ker(f)$. We know that $\ker(f)$ forms a prime ideal of R, and indeed every prime ideal of R arises as such a kernel (consider the projection map $R \to R/\mathfrak{p}$). Thus by studying ideals we also study all rings over R, that is, rings S equipped with homomorphisms $R \to S$, such that the structure morphism is surjective. This has the benefit of greatly enlarging the set of rings that we understand after only considering a small class of rings; we may recover all of these rings by looking at the ideal structures of a single ring R.)

An ideal $J \subset R$ is *finitely-generated* when there are $j_1, ..., j_n \in J$, for $n \geq 0$, such that $J = (j_1, ..., j_n)$. The j_i are the *generators* of J. In particular, we note that not all ideals are finitely generated (one can concont ideals which have minimal generating sets of arbitrary cardinality, can you think of some examples?).

For those of us who are not so set-theoretically minded, this might seem a bit worrying. Noetherian rings will solve this problem, letting us deal only with the finitely-generated case. But before we introduce the definitions, let's look at some of our favorite rings.

Example. Let's look at: $\mathbb{Z}$. What do we know already about ideals in $\mathbb{Z}$? What about prime ideals? Maximal ideals? What do the quotients of $\mathbb{Z}$ look like? What kind of initialisms do we know that apply to $\mathbb{Z}$?

Example. Let's look at: $S = R[x]$ for an arbitrary ring R. Is S a PID? When is S a PID? What properties does S inherit from R? If R is a PID, is S as well?

3. Noetherian rings

Now we introduce our main object of study, Noetherian rings. These rings are named after the esteemed mathematician Emmy Noether, and if you have time I greatly recommend looking into her work, which was remarkable and changed the landscape of modern mathematics and physics. That said:

Proposition 3.1. *Let a ring R be given (recall our assumptions). The following are equivalent:*

(1) Every ideal of R is finitely generated.
(2) There is no strictly increasing sequence of ideals in R

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$$

(3) Any nonempty family of ideals $\{J_i\}_{i \in I}$ in R has a maximal element (by inclusion).

Definition 3.2. If any (hence all) of the equivalent properties above are satisfied, then R is called *Noetherian.*

We note now that when we have the language of modules, we will be able to state this result and this definition in slightly greater generality. The interested reader can find such statements in any standard text on the subject. We proceed now with the proof.

Proof. Suppose that (1) holds. If there is an infinite increasing sequence of ideals of R as in (2), then consider $I = \bigcup_{j \geq 1} I_j$; this is an ideal of R, so it is finitely generated by (1). If $r_1, ..., r_m$ generate I, then we can find k such that $r_i \in I_k$ for each i. In this case we have $I = I_k$, contradicting that the sequence is increasing.

Suppose that (2) holds. If a nonempty family $\mathcal{F}$ of ideals of R has no maximal element, choose $I_1 \in \mathcal{F}$. Since this is not maximal, there is $I_2 \in \mathcal{F}$ such that $I_1 \subsetneq I_2$. Continuing this way we obtain a sequence as in (2) which is strictly increasing.

Now suppose finally that (3) holds, and let I be an ideal of R. Consider the family $\mathcal{F}$ of all finitely generated ideals contained in I; it is nonempty as it contains the zero ideal. By (3), $\mathcal{F}$ has a maximal element I'. If $I' \neq I$ thene there is $u \in I \setminus I'$, so that the ideal $I' + uI'$ is finitely generated sub-ideal of I strictly containing I', a contradiction. Thus $I' = I$, so I is finitely generated. □

Do we know any examples already of Noetherian rings? A moment's consideration will suggest that all fields are Noetherian, and that $\mathbb{Z}$ is Noetherian. It is not a waste of time to try to think of more that we have already encountered. In particular we have the following result:

Proposition 3.3 (Hilbert's basis theorem)**.** *When R is Noetherian, so is $R[x]$. Inductively, we have $R[x_1, ..., x_n]$ is Noetherian as well.*

Proof. Let I be an ideal in $R[x]$. We consider the following recursive program: if $I \neq 0$, let $f_1 \in I$ be of minimal degree. If $I \neq (f_1)$, then let $f_2 \in I \setminus (f_1)$ be of minimal degree. Suppose now that $f_1, ..., f_n$ have been chosen; if $I \neq (f_1, ..., f_n)$, choose $f_{n+1} \in I \setminus (f_1, ..., f_n)$ be of minimal degree.

If this process terminates, then I is finitely generated. Towards a contradiction, suppose that this never occurs. We write

$$f_i = a_i x_i^d + \text{lower degree terms} \quad a_i \neq 0.$$

By our standing assumptions of minimality, we have

$$d_1 \leq d_2 \leq \cdots$$

Let J be the ideal of R generated by the a_i for all $i \geq 1$. Since R is Noetherian, J is finitely generated, so that there is m such that $a_1, ..., a_m$ generate J. In particular, there are $u_1, ..., u_m$ such that

$$a_{m+1} = \sum_{i=1}^{m} a_i u_i.$$

In this case we have

$$h := f_{m+1} - \sum_{i=1}^{m} u_i x_i^{d_{m+1} - d_i} f_i \in I \setminus (f_1, ..., f_m)$$

with $\deg(h) < d_{m+1}$, a contradiction. □