MATH 494 2018 DISCUSSION 1: SHEAVES ON EUCLIDEAN SPACE AND MANIFOLDS

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1. Introduction

This discussion will build on your quiz from last week. I've reproduced the quiz here:

Consider the set of equivalence classes of pairs defined as the following.

$$\mathcal{A} = \big\{(U, f) \mid U \in \mathbb{R}^2 \text{ open set containing } (0,0)\ f : U \to \mathbb{R} \text{ continuous function}\big\} / \sim$$

where $(U_1, f_1) \sim (U_2, f_2)$ if and only if there exists an open set $U \subset U_1 \cap U_2$ containing $(0,0)$, such that the restrictions $f_1|_U = f_2|_U$. Addition and multiplication are defined as

$$\overline{(U_1, f_1)} + \overline{(U_2, f_2)} := \overline{(U_1 \cap U_2, f_1|_{U_1 \cap U_2} + f_2|_{U_1 \cap U_2})}$$

and

$$\overline{(U_1, f_1)} \cdot \overline{(U_2, f_2)} := \overline{(U_1 \cap U_2), f_1|_{U_1 \cap U_2} \cdot f_2|_{U_1 \cap U_2}}.$$

You were asked to prove that $\mathcal{A}$ is a ring, to determine the units of $\mathcal{A}$, and to exhibit a nontrivial ideal of $\mathcal{A}$.

We are going to be considering a slight generalization of the above setting. We will not in this section be concerned with the behavior of continuous functions $\mathbb{R}^2 \to \mathbb{R}$ at 0, but on some general open subset $V \subset \mathbb{R}^2$. Thus in what follows we will let $\mathcal{A}(V)$ be the set of pairs (U, f) up to equivalence as above, where each $U \subset V$. We will return to the notion of function behavior at a point later.

Definition 1.1. An element of $\mathcal{A}$ we call a *germ (of a function)* over V.

The concept of germs over an open set in $\mathbb{R}^2$ can of course be lifted to the setting of an abstract manifold, via coordinate charts. If you have experience with manifolds, I encourage you to keep this setting in mind. If not, Euclidean n-space is the setting we will be in.

Germs allow us to consider the behavior of functions *locally*, on small open sets. As such, we may describe functions relatively, i.e. give a description of them which depends on a choice of neighborhood. Depending on which neighborhood of $\mathbb{R}^n$ we choose, a given pair of functions may be equal or distinct. Thus we see that with the definitions given, the rings $\mathcal{A}(V)$ as V ranges over all open subset of $\mathbb{R}^2$, give a total description of the continuous functions $\mathbb{R}^2 \to \mathbb{R}$, which is also local.

2. Pre-sheaves

This motivates the definition of a pre-sheaf:

Definition 2.1. For a topological space X, a *pre-sheaf of sets* (resp. abelian groups, rings, etc.) $\mathcal{F}$ *over* X is an assignment $U \to \mathcal{F}(U)$ for every open set $U \subset X$ of a set (resp. ring, abelian group) to U, where for inclusions $U \subset V$ of open sets, we have *restriction maps* $\mathrm{res}_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$, which obeys the following compatibility condition: when $U \subset V \subset W$ is a tower of open sets in X, the following triangle commutes:

$$\begin{array}{ccc} \mathcal{F}(W) & \xrightarrow{\mathrm{res}_V^W} & \mathcal{F}(V) \\ & {\scriptstyle \mathrm{res}_U^W} \searrow \quad \swarrow {\scriptstyle \mathrm{res}_U^V} & \\ & \mathcal{F}(U) & \end{array}$$

That is, $\mathrm{res}_U^V \circ \mathrm{res}_V^W = \mathrm{res}_U^W$. The elements of $\mathcal{A}(U)$ we call the *sections of* $\mathcal{A}$ *over* U.

For those of you who have seen category theory, a pre-sheaf could alternatively be defined as a contravariant functor from the category whose objects are the open subsets of X and whose maps are (strict) inclusions, which for now we denote $\mathrm{Cat}(X)$, to the category of sets, rings, abelian groups, etc.

In our setting, $X = \mathbb{R}^2$, and $\mathcal{F}(U) = \mathcal{A}(U)$. We claim that this defines a pre-sheaf of rings over X, as we now show.

Example. We need only describe the restriction maps, and check the compatibility condition. So let $U \subset V \subset W$ be a tower of open subsets of $\mathbb{R}^2$. Then the rings $\mathcal{A}(U) \leftarrow \mathcal{A}(V) \leftarrow \mathcal{A}(W)$ should be a tower of rings. Indeed, we define restriction maps (as you might expect) by restriction of functions. That is, for $\overline{(U, f)} \in \mathcal{A}(V)$, we define $\mathrm{res}_U^V(\overline{(V, f)})$ to be $\overline{(U, f|_V)}$. This is obviously independent of the choice of representative (U, f) and well-defined as a function. As function restriction is compatible in the sense of the above definition, it follows that $\mathcal{A}$ is indeed a presheaf over $\mathbb{R}^2$.

In general the restriction maps might be much more exotic maps, and do not in any way have to depend on the topological structure of X.

Example. We describe the *constant pre-sheaf* over an arbitrary topological space X. Fix your favorite category $\mathcal{C}$, say sets or commutative rings or abelian groups, etc., and pick an object $c \in \mathrm{Ob}(\mathcal{C})$. Then the assignment $U \to \mathcal{F}(U) = c$ is a $\mathcal{C}$-valued pre-sheaf over X.

We would like the pre-sheaves over a topological space X to form a category; that is, we need to define what it means to give a *morphism* of pre-sheaves over X.

Definition 2.2. Let $\mathcal{F}$ and $\mathcal{G}$ be pre-sheaves over X that take values in the same category $\mathcal{C}$, and let $U \subset V \subset X$ be an inclusion of open subsets. A *morphism of pre-sheaves* ϕ is defined to be a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ {\scriptstyle \mathrm{res}(\mathcal{F})_U^V}\downarrow & & \downarrow{\scriptstyle \mathrm{res}(\mathcal{G})_U^V} \\ \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) \end{array}$$

That is, giving a morphism $\mathcal{F} \to \mathcal{G}$ is equivalent to giving a morphism in $\mathcal{C}$ between all $\mathcal{F}(U)$ and $\mathcal{G}(U)$ which are compatible with restrictions.

In the categorical sense as given above, a morphism of pre-sheaves could alternatively be defined as a natural transformation between the functors $\mathcal{F} : \mathrm{Cat}(X) \to \mathcal{C}$ and $\mathcal{G} : \mathrm{Cat}(X) \to \mathcal{C}$.

3. Stalks and sheaves

We now return to the notions of the behavior of a pre-sheaf over a points, and other local behavior. If, as on the quiz, we would like to use pre-sheaves to describe the behavior of a pre-sheaf $\mathcal{F}$ "around a point" $x \in X$, we have the tools to do so with presheaves. This is (perhaps unfortunately) a categorical construction, but in the case of germs of functions the result is not very abstract.

For a point $x \in X$, the open subsets of X containing x form a poset, with $U \leq V$ if and only if $V \subset U$. Call this poset $(\mathcal{P}, \leq)$. This poset is also *filtering*, i.e. every pair of elements has an upper bound, given by the intersection. Thus we have the following construction.

Definition 3.1. For $\mathcal{F}$ a presheaf over X, *the stalk of $\mathcal{F}$ over x* is defined as

$$\varinjlim_{U \in (\mathcal{P}, \leq)} \mathcal{F}(U).$$

As all of the categories we are dealing with (rings, abelian groups, sets, etc.) are co-complete, i.e. have all of the colimits of their diagrams, we have all of the stalks that we want for any presheaf $\mathcal{F}$.

Stalks are very nice to have, however when we think locally but not necessarily around a point, we run into the obvious problem that infinite intersections of open sets are not necessarily open; there is not a canonical choice of open set $U \ni x$ in X to use to describe the behavior of $\mathcal{F}$ in a neighborhood of x. We might want to choose a sequence of open sets containing x and consider, somehow, the behavior of all of the sets in the sequence over x (as you could do, say, with a direct limit), but there is no reason why such an object would be independent of the choice of such a sequence.

This motivates the definition of a sheaf. Sheaves are and should be thought of as presheaves which can be described locally:

Definition 3.2. A *sheaf* $\mathcal{F}$ over a topological space X is a presheaf over X which satisfies two additional criteria: let $U \subset X$ be open and $\{U_i\}_{i \in I}$ be an open cover.

(1) (Uniqueness axiom) If there exist $s, t \in \mathcal{F}(U)$ such that $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.

(2) (Existence axiom) If there exist $s_i \in \mathcal{F}(U_i)$ for all $i \in I$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there is $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

The uniqueness axiom guarantees that s as in the existence axiom is always unique.

We claim that presheaves of germs form sheaves over Euclidean space or manifolds, and we leave the process of checking the two axioms given in the definition as an exercise.

Sheaves, as we have noted, give a local description of the functions over a topological space. They also give a *relative* understanding of the topological space. To clarify a bit, it is a longstanding philosophy in algebraic geometry, and other fields such as topology and number theory, that objects (i.e. spaces) should not only be considered, but more so maps between them. When we focus on settings of the form $f : X \to Y$, we recover the absolute (i.e. non-relative) setting when Y is a point. For an example, we consider sheaves over manifolds again:

Example. Fact: for a map of rank r manifolds $F : M \to N$, alternatively a continuous map $\mathbb{R}^m \to \mathbb{R}^n$, F is differentiable if and only if for all $U \subset N$ and $f \in C^r(U)$, $f \circ F$ is differentiable. In the language of sheaves, where the sheaves over M and N are as we have discussed, this is expressed in the formula "*manifold sections pull back under differentiable maps*".

Remark. There is a notion of making a "best" sheaf out of a presheaf, called *sheafification*, which we will not cover here. It is unique up to unique isomorphism, and preserves stalks. For a presheaf $\mathcal{F}$ over X, we denote by $\mathcal{F}^+$ the sheafification of $\mathcal{F}$.

END