The following is a problem found on Problem Set 4 (and probably in Artin's *Algebra* as well).

Problem. *Let x, y, z, w be variables. Prove that $xy - zw$ is an irreducible element of the polynomial ring $\mathbf{C}[x, y, z, w]$.*

We give a slick (although not very short) proof using some of the algebraic geometry introduced in the discussion sections. Notice first that

$$xy - zw = \det \begin{bmatrix} x & z \\ w & y \end{bmatrix} \in M_2(\mathbf{C}).$$

Thus it suffices to show that the determinant is irreducible as a polynomial over $\mathbf{C}$. It will be critical that we work in the complex numbers, as all of the algebro-geometric constructions in the discussion sections depend on the fact that the base field is algebraically closed. We induct on n, the size of the matrices in question; in the 1-dimensional case this is trivial. We consider the inductive case.

Recall from the discussion sections the notions of irreducible closed sets and the Zariski topology on affine space (we only consider complex affine space $\mathbf{A}^n_{\mathbf{C}}$ here). Prove the following lemmas.

Lemma 1. Let X be a topological space and $U \subset X$ be a subset. Suppose we have an open cover $U = U_1 \cup \cdots \cup U_r$ of U. Show that U is irreducible if and only if each U_i is irreducible and for each i, j, we have $U_i \cap U_j \neq \emptyset$.

Lemma 2. Show that the set

$$M_{n-1} = \{A \in M_n(\mathbf{C}) \mid \operatorname{rank}(A) \leq n - 1\}$$

is closed (in $\mathbf{A}^n_{\mathbf{C}}$) and irreducible. *Hint:* use the fact that a $k \times k$ matrix has rank $\leq d$ if and only if each of its $d \times d$ minors vanish, and apply Lemma 1. Here, use the inductive hypothesis to assume that each minor is given by an irreducible polynomial.

Now identify $V(\det)$ with M_{n-1}. As the polynomial ring is a UFD, we have that $V(I(\det)) = \sqrt{(p)}$ for $p \in \mathbf{C}[x_{i,j}]$ (this uses a definition and result from a discussion section). Thus we have $\det = p^k$.

We finish the proof with a fact from differential topology. We have

$$d(\det)_X(B) = \operatorname{tr}(\operatorname{Adj}(X) \cdot B).$$

Although this fact is non-trivial to prove, a more-or-less quick proof is not hard to come by in textbooks or the on the internet. When the rank of X is $n - 1$, its adjugate has rank 1 (this is essentially equivalent to the fact stated in the hint from Lemma 2). This forces that $d(\det)_X(B) \neq 0$, which when $k > 1$ is a contradiction (as p^k and p^{k-1} have the same zeroes; if this is not clear, write out the derivatives in question). It follows that $\det = p$ is irreducible, as desired.