MATH 296 2018 DISCUSSION 3: GROUP ACTIONS

BEN GOULD

Contents

1. Introduction

On a previous discussion notes I included material on group actions, assuming that you had seen it. I was recently informed that you have not! To rectify this, we define group actions, and give a litany of examples.

Exercise. Define a group. Give ten examples of groups, five abelian and five non-abelian. Recall what it means to conjugate elements in a group. Define the symmetric (or permutation) group S_n on a set of n-elements. When is S_n abelian?

2. Definitions, examples, exercises, etc.

A group G and a set S are fixed. Let $S(G)$ denote the set of the subgroups of G. We do not insist that G is finite.

Definition 2.1. A *group action* of G on S, notated $G \curvearrowright S$, is a function $G \times S \to S$, notated in this document by $(g, s) \mapsto g * s$, which satisfies two criteria: for all $g, g' \in G$, $s \in S$, we have $gg' * s = g * (g' * s)$, and $1_G * s = s$ for each $s \in S$.

Exercise. The *conjugation action* of G on its subgroups is the action $G \times S(G) \to S(G)$ given by $(g, H) \mapsto g * H = gHg^{-1}$. Verify that this is an action.

Exercise. Give some more examples of natural actions of groups on themselves.

Example. Verify that the symmetric group S_n acts by permutations on sets with n elements.

Example. I've heard that you have defined an action of $SL_2(\mathbb{Z})$ on the upper-half plane $\mathbb{H}$. Please recall this, and verify that it is an action.

Example. The special orthogonal group $SO(n)$ acts on $\mathbb{R}^n$ by rotations. Prove this.

Definition 2.2. A group action is *faithful* provided that this map is injective, i.e. $g*s = g'*s$ for all $s \in S$ implies $g = g'$. A group action is *free* if $g * s = g' * s$ for one $s \in S$ implies that $g = g'$. A group action is *transitive* if for each $s, s' \in S$ there exists $g \in G$ such that $g * s = s'$.

Exercise. Show that an action is faithful if and only if $g * s = s$ for all $s \in S$ implies that $g = 1$. Show that an action is free when that statement is true when I replace "all" with "one".

Definition 2.3. For an action of G on S and $s \in S$, the *orbit* of s is the set $\{x \in S \mid \exists g \in G \text{ such that } gs = x\}$. In the same setting, the *stabilizer* of s is the set $\{g \in G \mid gs = s\}$.

Exercise. Show that $G \curvearrowright S$ is transitive if and only if there is a unique orbit. Show that the stabilizer of an element of s forms a subgroup of G.

3. Let's spice things up a bit

We do not care only about sets (or at least, we shouldn't). We care about sets, with some extra structure. "Sets with extra structure" include topological spaces, vector spaces, groups, rings, etc. Sometimes it is convenient (or even necessary) to slightly alter our definitions to accommodate this extra structure.

In the setting of a group, we define a *G-module* to be an abelian group H equipped with an action $G \curvearrowright H$ such that $g * (h_1 + h_2) = g * h_1 + g * h_2$. That is, the group action "knows about" the additive structure on H (and we denote by $+$ the operation on H, as it is abelian).

Exercise. Give an example of a group G and a G-module H. Give an example of a pair of groups G and H with an action of G on H that does not make H into a G-module.

In the setting of a topological space, we add in structure in the following way. Let $G \curvearrowright X$ be an action of G on a topological space X. We define the map $\hat{g} : X \to X$ via $x \mapsto g * x$. We require that this map is continuous.

Exercise. Show that $\hat{g}$ for all $g \in G$ is a homeomorphism.

I encourage you to keep the concept of group actions in mind as you learn linear algebra. (In fact, take a look at the previous discussion section notes.)

Exercise. Ask Noah Luntzlara for more questions about group theory.