MATH 296 2018 DISCUSSIONS 1 & 2: IMPOSSIBILITY THEOREMS FOR ELEMENTARY INTEGRATION

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1. Introduction

It is a well-known result in calculus generally that one cannot find an antiderivative for the real-valued function e^{-x^2} "in terms of elementary functions". Of course this is of great use in probability theory, as the function e^{-x^2} is the meaningful part of the definition of a Gaussian distribution, and the (im)possibility of computations with this integrand poses serious problems for the theory. In number theory, a well-known (and very deep) result is the *prime number theorem*, which states that asymptotically, we have $\phi(x) \sim x/\log(x)$, where ϕ is the totient function. It turns out that there is a slightly better approximation for $\phi(x)$ given by the "logarithmic integral" $\mathrm{Li}(x) = \int_2^x dt/\log(t)$. Here is an exercise: show that $x/\log(x) \sim \mathrm{Li}(x)$ in the limit $x \to \infty$. It is another popular result that there is no antiderivative for $\mathrm{Li}(x)$, again "in terms of elementary functions". Of course such a result has wide-reaching implications for analytic number theory. Applying the change of variables $u = \log(t)$ we obtain the integrand e^u/u; this is the integrand we will consider in the ensuing proofs, where we will show that this also cannote be integrated in elementary terms.

In this discussion and the next we will precisely define what we mean by "integration in terms of elementary functions", and prove the two impossibility statements above concerning functions of import. We will follow Brian Conrad's expository paper *Impossibility Theorems for Elementary Integration*, [1].

2. What does "elementary" mean?

So, what do we really mean by an elementary function? It seems reasonable to define the standard calc 1 functions like arithmetic functions, polynomials, rational functions, logarithms and exponentials, roots, and trig functions and their inverses should be "elementary". Thus

$$\frac{x^{1/3} + \pi x \log(x^{e^x})}{2x - \sin(x)}$$

should be elementary, *on an open interval on which it is defined.* Liouville, who was a very well-known French mathematician, proved a result about such elementary functions in the form fe^g for rational functions f, g in terms of the solution of a simple differential equation in rational functions. In particular, this setting settles the issues of the elementarity of the two functions $\mathrm{Li}(x)$ and e^{-x^2} we're concerned with here. Conrad develops the result in slightly higher levels of generality.

Aside on notation. Here I will include a brief discussion of polynomial rings and rational function fields over $\mathbb{R}$ and $\mathbb{C}$, which are Googleable. I will include definitions of fields like $\mathbb{C}(x, f_1, ..., f_n)$.

It will often be convenient to discuss complex functions, i.e. functions of the form $u(x) + iv(x)$ for real-valued functions u, v, of a real variable x. We expand notions of continuity, analyticity, etc. of real functions of real variables to complex functions by examining real and imaginary parts.

Observe that via formulas of the form

$$\cos^{-1}(x) = \tan^{-1}\left(\sqrt{\frac{1}{x^2} - 1}\right), \qquad \sin^{-1}(x) = \tan^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right)$$

and the standard facts

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \qquad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

defined on suitable intervals, we may recover trigonometric functions as exponentials, logarithms, and rational functions of analytic functions. Further, we leave as an exercise to show that if a $\mathbb{C}$-valued function f is analytic and non-vanishing, then f'/f is also analytic. Then appropriately choosing x_0 we obtain the analytic function $(\log f)(t) = \int_{x_0}^{t}(f'(s)/f(s))ds$ called the *logarithm* of f. The definition of this logarithm depends on x_0 up to additive constant ("$+C$"), but we won't be concerned with this slight ambiguity. When $x_0 = 1$ and $f(t) = t$ we recover the standard logarithm (on what interval?). We can arrange things (by choosing appropriate constants) as well so that $e^{\log f} = f$, justifying the notation.

Another exercise: check that $2i\tan^{-1}(x) + i\pi$ is a logarithm of the non-vanishing function $(x - i)/(x + i)$ (remember we are only using real variables x!). Thus using the formulas above, we really have reduced things to non-trigonometric functions. (Note: I did not do this, so if you want to do it, I will include your computations in this document and cite your good work.)

3. Elementary fields and functions

Some important definitions and notions.

Definition 3.1. If $f_1, ..., f_n$ are meromorphic functions (that is, quotients of analytic functions), then $\mathbb{C}(f_1, ..., f_n)$ denotes the set of meromorphic functions h of the form

$$h = \frac{p(f_1, ..., f_n)}{q(f_1, ..., f_n)} = \frac{\sum \alpha_{e_1,...,e_n} f_1^{e_1} \cdots f_n^{e_n}}{\sum b_{j_1,...,j_n} f_1^{j_n} \cdots f_n^{j_n}}$$

for polynomials p, q in n variables, with $q(f_1, ..., f_n) \neq 0$.

Example. The field $K = \mathbb{C}(x, \sin(x), \cos(x))$ is the set of ratios

$$\frac{p(x, \sin(x), \cos(x))}{q(x, \sin(x), \cos(x))}$$

for polynomials $p, q \in \mathbb{C}[X, Y, Z]$ such that $q(x, \sin(x), \cos(x)) \neq 0$. For example, the polynomial $Y^2 + Z^2 - 1$ won't do.

Definition 3.2. A field K of meromorphic functions is an *elementary field* if $K = \mathbb{C}(x, f_1, ..., f_n)$ with each f_j either an exponential or a logarithm of an element of $K_{j-1} = \mathbb{C}(x, ..., f_{j-1})$ or else algebraic over K_{j-1} in the sense that $P(f_j) = 0$ for some $P(T) \in K_{j-1}[T]$ with all coefficients lying in K_{j-1}. A meromorphic function is an *elementary function* if it lies in an elementary field of meromorphic functions.

Example. Recall the function given above,

$$f(x) = \frac{x^{1/3} + \pi x \log(x^{e^x})}{2x - \sin(x)}.$$

Then f is elementary, and an elementary field containing f is $\mathbb{C}(x, x^{1/3}, \sin(x))$.

Example. An example of an "algebraic" function in the definition 3.2 is given by $g(x) = \sqrt{x} + \sqrt[3]{x}$. We think of g as lying in the field $\mathbb{C}(x, \sqrt{x}, \sqrt[3]{x})$ with $\sqrt{x}$ algebraic over $\mathbb{C}(x)$ (as a root of $T^2 - x \in \mathbb{C}(x)[T]$), and $\sqrt[3]{x}$ algebraic over $\mathbb{C}(x, \sqrt{x})$ (similarly).

The following theorem will give us much of the tools we need to prove that functions are *not* elementary, which at present does not seem like an obvious task to go about.

Theorem 3.3. *If K is an elementary field, then it is closed under the operation of differentiation.*

Proof. We write $K = \mathbb{C}(x, f_1, ..., f_n)$ and induct on n. The case of $n = 0$ is the case of $K = \mathbb{C}(x)$, and the standard formulas for differentiating sums, products, and quotients of (polynomial) functions show that this field is indeed closed under differentiation. For the general case we have by induction that $K_0 = \mathbb{C}(x, f_1, ..., f_{n-1})$ is closed under differentiation, and $K = K_0(f_n)$ with f_n either algebraic over K_0 or a logarithm or exponential of an element of K_0. We check now that it suffices to show that $f_n' \in K_0(f_n)$.

Under this assumption, for any polynomial $P(T) = \sum_{j\geq 0} a_j T^j \in K_0[T]$ we have

$$P(fn)' = a_0' + \sum_{j\geq 1}(a_j' f_n^j + j a_{j-1} f_n^{j-1} f_n') \in KK_0(f_n)$$

since $a_j' \in K_0$ by assumption. Thus when $P, Q \in K_0[T]$ are polynomials over K_0 with $Q(f_n) \neq 0$, we have

$$\left(\frac{P(f_n)}{Q(f_n)}\right)' = \frac{Q(f_n)P(f_n)' - P(f_n)Q(f_n)'}{Q(f_n)^2} \in K_0(f_n) = K.$$

So, it just remains to check that f_n' lies in $K_0(f_n)$ when f_n is either algebraic over K_0 or the exponential or logarithm of an element of K_0. Standard formulas for differentiating logarithms and exponentials (of functions!) take care of the latter two cases.

To treat the case of f_n being algebraic over K_0, suppose we have $P(f_n) = 0$ for some $P = T^m + a_{m-1}(x)T^{m-1} + \cdots + a_0(x) \in K_0[T]$. Further choose P of minimal degree, so that $P'(T) = mT^{m-1} + (m-1)a_{m-1}(x)T^{m-1} + \cdots + 2a_2(x)T + a_1(x)$ with degree $m-1$ satisfies $P'(f_n) \neq 0$. However:

$$0 = P(f_n)' = \sum_{j>0} j a_j(x) f_n^{j-1} fn' + \sum_{j<m} a_j'(x) f_n^j = P'(f_n)f_n' + \sum_{j<m} a'j(x) f_n^j,$$

so $P'(f_n)f_n' = -\sum_{j<m} a_j'(x) f_n^j \in K_0(f_n)$. Since $P'(f_n) \neq 0$ and lies in $K_0(f_n)$, we are done after dividing through by $P'(f_n)$. □

Definition 3.4. A meromorphic function field which is closed under differentiation is called a *differential field*.

Remark. The preceding theorem says that elementary fields are differential fields, but the proof says more. Namely, we've shown that expanding a differential field by adjoining a function which is the logarithm or exponential of an element of that field, or is algebraic over that field, gives yet another differential field.

Definition 3.5. A meromorphic function f can be *integrated in elementary terms* if $f = g'$ for an elementary function g (and so f is necessarily elementary, by 3.3).

3.5 captures the reasonable intuition for what gives an "elementary formula" for finding an anti-derivative of a function you might find in a calculus class, with the added generality of considering $\mathbb{C}$-valued functions instead of the more typical $\mathbb{R}$-valued ones.

It is important to note, however, that if we did not include the addition of complex coefficients, that we would have the *wrong* definition of elementary integrability. A key example witnessing this necessity is the meromorphic function $1/(1+x^2)$. Under any reasonable definition we would like to say that this function admits an elementary integral (such as $\tan^{-1}(x)$). This is the case in the $\mathbb{C}$-valued setting, since as noted above we have $\tan^{-1}(x) = \log((x+i)/(x-i))$. However if we work only in the $\mathbb{R}$-valued setting, then it can be proved that $1/(1+x)^2$ is *not* integrable over $\mathbb{R}$ in elementary terms (see [2, p. 968] for a rigorous proof).

One might want instead now to just throw in *all* of the usual trigonometric functions and their inverses in the definition of an elementary field, but this turns out also to be the wrong thing to do. This is because in (the proofs of) the results we will state in the following section, one needs to work out elementary functions as solutions to simple first-order differential equations, and trigonometric functions aren't.

4. Two integrability criteria, and applications

For the results in this section that we do not prove, see the homepage for documents containing proofs.

The main theorem we will state (and punt the proof of) is due to Liouville, and it asserts that if an elementary function is integrable in elementary terms then there are severe constraints on the possible form of an antiderivative.

Theorem 4.1 (Liouville)**.** *Let f be an elementary function and let K be an elementary field containing f. The function f can be integrated in elementary terms if and only if there exist nonzero $c_1, ..., c_n \in \mathbb{C}$, nonzero $g_1, ..., g_n \in K$, and an element $h \in K$ such that*

$$f = \sum c_j \frac{g_j'}{g_j} + h'.$$

The key feature is that the g_j's and h can be found in *any* elementary field containing f. In this setting $\sum c_j \log(g_j) + h$ gives an antiderivative of f in terms of elementary functions.

A consequence of 4.1 is the following, extra criterion.

Theorem 4.2. *Choose $f, g \in \mathbb{C}(x)$ with $f \neq 0$ and g nonconstant. The function $f(x)e^{g(x)}$ can be integrated in elementary terms if and only if there exists a rational function $R \in \mathbb{C}(x)$ such that $R'(x) + g'(x)R(x) = f(x)$ in $\mathbb{C}(x)$.*

The content of this theorem isn't that one can find a solution to the given differential equation as a $\mathbb{C}$-valued differentiable function of x, indeed via integrating factors we can *always* write down such a solution. Rather, we obtain a solution with the very special property that it is a *rational function* in x.

Modulo the proof of these results, we now prove the non-integrability of the two examples given in the opening section.

Example. Taking $f = 1$ and $g = -x^2$ in the statement of 4.2, we need only to show that the differential equation $R'(x) - 2xR(x) = 1$ in $\mathbb{C}(x)$ has no solution (in $\mathbb{C}(x)$). The method of integrating factors gives a formula for the general function solution, namely $R_c(x) = e^{-x^2} \int e^{-x^2} dx + c$ with $c \in \mathbb{C}$, but we cannot show by inspection that this is never a rational function, since we don't know how to describe $\int e^{-x^2} dx$ in the first place! However this does yield a significant simplification of our original problem of finding an elementary antiderivative for f, since now we have concluded that any such antiderivative must have the form $e^{-x^2} r(x)$ for some *rational* function $r \in \mathbb{C}(x)$.

We argue by contradiction. If there does exist a solution $R \in \mathbb{C}(x)$ then R is certainly nonconstant, and we claim that R is not a polynomial in x. If it were, then $R'(x) - 2xR(x)$ is a polynomial of degree 1 greater than the degree of R, so it is not 1. Thus after reducing we must have $R(x) = p(x)/q(x)$ for nonzero relatively prime polynomials $p, q \in \mathbb{C}[x]$ with q nonconstant. The defining differential equation dictates that $(p(x)/q(x))' - 2x(p(x)/q(x)) = 1$.

By the fundamental theorem of algebra, q has a root $z_0 \in \mathbb{C}$. Relative primality of p and q implies that $p(z_0) \neq 0$. Hence if z_0 is a root of q with multiplicity $m \geq 1$, then $p(x)/q(x) = h(x)/(x - z_0)^m$ for $h(x) \in \mathbb{C}(x)$ having numerator and denominator nonvanishing at z_0. Now we differentiate:

$$\left(\frac{p(x)}{q(x)}\right)' = \frac{-mh(x)}{(x - z_0)^{m+1}} + \frac{h'(x)}{(x - z_0)^m}$$

so passing to a limit $z \to z_0$ in $\mathbb{C}$ we see that $(p(x)/q(x))'|_{x=z_0}$ has absolute value that blows up like $A/|z - z_0|^{m+1}$ with $A = |mh(z_0)| \neq 0$. But, $|-2z \cdot (p(z)/q(z))|$ has growth bounded by a constant multiple of $1/|z - z_0|^m$ as $z \to z_0$ in $\mathbb{C}$, so

$$\left| \left(\left(\frac{p(x)}{q(x)}\right)' - 2x \cdot \left(\frac{p(x)}{q(x)}\right) \right) |_{x=z} \right| \sim \frac{A}{|z - z_0|^{m+1}}$$

in the limit $z \to z_0$ in $\mathbb{C}$. But this contradicts the identity $(p(x)/q(x))' - 2x(p(x)/q(x)) = 1$.

Example. We return now to the logarithmic integral $\mathrm{Li}(x) = \int dt/\log(t) \approx \int (e^u/u) du$. Taking $f = 1/x$ and $g = x$ in 4.2, to prove that the logarithmic integral cannot be expressed in elementary terms it suffices to show that the differential equation $R'(x) + R(X) = 1/x$ has no solution in $\mathbb{C}(x)$. Clearly such a solution R is not a polynomial in x, so writing $R = p/q$ in reduced form with $q \in \mathbb{C}[x]$ nonconstant, there is a root $z_0 \in \mathbb{C}$ of q with multiplicity $m \geq 1$. Thus $R'(x)$ has a zero of order $m + 1$ at z_0 and so as in the previous example we can see that $1/x = R'(x) + R(x)$ blows up like a nonzero constant multiple of $1/|z - z_0|^{m+1}$ in the limit $z \to z_0$. However the only $a \in \mathbb{C}$ for which $|1/z|$ has explosive growth in absolute value as z approaches a is $a = 0$. Hence $z_0 = 0$. But as $z \to 0$ we would then have that $|1/z|$ grows like a nonzero constant multiple of $z/|z - z_0|^{m+1} = 1/|z|^{m+1}$, a contradiction.

References

[1] Brian Conrad. *Impossibility Theorems for Elementary Integration.* `http://math.stanford.edu/~conrad/papers/elemint.pdf`

[2] M. Rosenlicht. *Integration in finite terms.* American Math. Monthly 79 (1972)