Math 592: Algebraic topology

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These are the course notes for Math 592, taught by Bhargav Bhatt at the University of Michigan in the Winter semester, 2018. Here is the link to the course webpage:

http://www-personal.umich.edu/~bhattb/teaching/mat592w18/

These notes are compiled by Ben Gould, and all mistakes are mine. By viewing this file you agree to send me all typos and mistakes you find. My email is brgould [at] umich [dot] edu, and these notes are posted on my personal webpage. It is

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1 Introduction

First, some notation. I will denote all categories by bolded words and terms, and all fields/spaces with blackboard letters. For example, the category of graded Abelian groups will be denoted **grAbGroup**, and the real numbers will be denoted $\mathbb{R}$. Following Professor Bhatt, I will use capital letters $X, Y, Z, ...$ for topological spaces, and lower-case letters $x, y, z...$ for points. $I = [0,1]$ will denote the unit interval throughout. The set $\text{Maps}(X, Y)$ will denote the set of (continuous) maps between objects X and Y, which in our class will normally denote topological spaces. The word "map" will nearly always implicitly assume continuity. For a point $y \in Y$, we will denote by $c_y : X \to Y$ the constant map with value y. All notation will be made clear, or will be clear from context.

We will start by motivating the ensuing discussions. Some of the very broad goals of algebraic topology include

- studying topological spaces. Some of the significant ones in this course will be S^1, $\mathbb{R}$, $S^1 \times S^1$, S^2.
- More precisely, studying topological spaces via "algebraic invariants". That is, functors **Top** $\to$ {algebraic objects}, where an algebraic object could be a group, an Abelian group, a commutative ring, etc.

  An example is a functor $H_* :$ **Top** $\to$ **grAbGrp** called singular homology which satisfies:
$$S^1 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus 0 \oplus \cdots$$
$$\mathbb{R} \longrightarrow \mathbb{Z} \oplus 0 \oplus \cdots$$
$$S^1 \times S^1 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}^{\oplus 2} \oplus \mathbb{Z} \oplus 0 \oplus \cdots$$
$$S^2 \longrightarrow \mathbb{Z} \oplus 0 \oplus \mathbb{Z} \oplus 0 \oplus \cdots .$$

We will study singular homology more in depth later in the course.

Some of the main functors we will study in 592 include

- fundamental groups: $\pi_1 :$ **pointedTop** $\to$ **Group**, which takes as an argument a topological space with a choice of point. These are studied in two main ways, via loops, and via covering spaces. We will introduce both viewpoints. There are other, higher homotopy groups (π_n for any n), and these are also important, but notoriously difficult to compute.
- singular homology: $H_* :$ **Top** $\to$ **grAbGrp**.
- singular cohomology $H^* :$ **Top** $\to$ **grRing**.

All of these functors will be defined on a homotopy category, i.e. the space of topological spaces up to homotopy equivalence. We will see this notion in the first proposition in the next section.

2 Fundamental Groups and Covering Spaces

2.1 Homotopy

Definition 2.1. Given maps $f, g : X \to Y$, a *homotopy* $h : f \simeq g$ is a map $h : X \times I \to Y$ such that for each $x \in X$, $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$. We say that two maps f, g are *homotopic* when there exists a homotopy $h : f \simeq g$.

Definition 2.2. A map $f : X \to Y$ is *nullhomotopic* if $f \simeq c_y$ for some $y \in Y$.

Definition 2.3. A space X is *contractible* if the identity map on X is nullhomotopic.

Example. We claim that $\mathbb{R}^n$ is a contractible topological space. Indeed, let $f(x) = x$ and $g(x) = 0$ for $x \in \mathbb{R}^n$; define the map $h : \mathbb{R}^n \times I \to \mathbb{R}^n$ that takes $x \mapsto (1-t)x$. h satisfies all of the necessary requirements of a homotopy, so the claim follows.

Example. Let X be a one-point space and Y be a two-point space. The two obvious maps $X \to Y$ are *not* homotopic, and proving this is left as an exercise.

Example. Let $f : S^1 \to S^1$ be the map taking $(x, y) \mapsto (-x, -y)$. We claim that this is homotopic to the identity map on S^1. Indeed, viewing things in polar coordinates, a desired homotopy is $h : S^1 \times I \to S^1$ taking $(e^{it}, \theta) \mapsto e^{i(t+\pi\theta)}$, where t ranges over $\mathbb{R}$.

Remark. f is *not* homotopic to the map $(x, y) \mapsto (-x, y)$. We will be able to prove this when we have more tools.

Remark. X is contractible implies that X is path-connected. This is also left as an exercise.

Proposition 2.4. *Homotopy equivalence defines an equivalence relation on the set Maps*(X, Y).

Proof. To show the relation is reflexive, consider the trivial homotopy $(x, t) \mapsto x$. To show it is symmetric, given two homotopic maps $f, g : X \to Y$ and a homotopy $h : f \simeq g$, replace h with the map $h' : (x, t) \mapsto (x, 1-t)$. One readily checks that this is a desired homotopy $g \simeq f$. To see that it is also transitive, let $f \simeq g$, $g \simeq q$ be homotopic and choose corresponding homotopies h_1 and h_2. Then we define the map

$$H : x \mapsto \begin{cases} h_1(x, 2t) & 0 \le t \le 1/2, \\ h_2(x, 2t-1) & \text{else.} \end{cases}$$

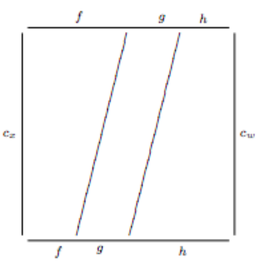
Again, one readily checks that this is a desired homotopy $f \simeq q$. □

2.1.1 Paths and loops

Let $X \in \mathbf{Top}$ and $x, y \in X$.

Definition 2.5. A *path* f from x to y, which we will always denote $f : x \rightsquigarrow y$, is a map $f : I \to X$ such that $f(0) = x$ and $f(1) = y$. A *loop based at* x is a path from x to itself. A *homotopy of paths* between paths $f, g : x \rightsquigarrow y$ is a homotopy $h : I \times I \to X$ such that for each $s \in I$, $h(s, 0) = f(s)$, $h(s, 1) = g(s)$, and $h(0, s) = x$, $h(1, s) = y$.

A brief remark on notation: there is a useful diagrammatic way of approaching statements about homotopy. Namely, a homotopy of maps $h : f \simeq g$ can be represented by a square in the following way[1]: In the figure, f, g, and h are maps $x \rightsquigarrow w$, and the sloped lines represent a



homotopy between the compositions $f(gh) \simeq (fg)h$; see below.

Remark. Homotopy of paths defines an equivalence relation on the set of paths between points x and y. This is left as an exercise. See Figure 1.

Example. Let $X = \mathbb{R}^2$ and $Y = S^1$, and consider the points (-1,0) and (1,0), with paths α and β between them, which trace out, respectively, upper and lower semicircles between the two points. In X, these maps are homotopic, but in Y they are not. It will take us some time to build up the machinery for the proof of the second statement, while the first is clear.

[1]All such images are lifted from Peter May's *A Concise Course in Algebraic Topology*, available on the web.

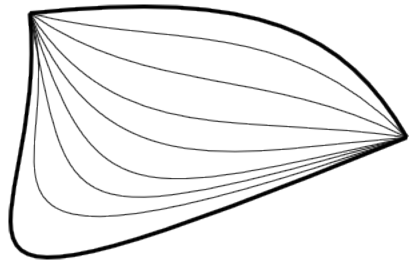


Figure 1: A homotopy between two paths (bolded lines) between two points (cusps).

2.1.2 Operations on Paths

1. Composition: given $f : x \rightsquigarrow y$, $g : y \rightsquigarrow z$ paths in X, get a new path $gf = g \circ f : x \rightsquigarrow z$. Equations for this are given by

$$(gf)(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2, \\ g(2t-1) & \textit{else}. \end{cases}$$

2. Inversion: given $f : x \rightsquigarrow y$, get a new path $f^{-1} : y \rightsquigarrow x$ by reversing the direction. Symbolically this is $f^{-1}(t) = f(1-t)$.

Remark. This composition law is *not* associative. This is because in the two compositions $f(gh)$ and $(fg)h$ for compatible paths f, g, h in X, paths are traversed in different times. In the first, f takes place over time 0 to 1/2, and in the second, in time 0 to 1/4.

For a path f in X, we will denote by $[f]$ its homotopy class.

Theorem 2.6. *1. Composition is well-defined, and associative up to homotopy. That is, $[h(gf)] = [(hg)f]$. This implies that there is a well-defined map*

$$\frac{\{paths\ x \rightsquigarrow y\}}{homotopy} \times \frac{\{paths\ y \rightsquigarrow z\}}{homotopy} \longrightarrow \frac{\{paths\ x \rightsquigarrow z\}}{homotopy}$$

2. Inversion factors through homotopy: $[f]^{-1} := [f^{-1}]$ is well-defined. Similarly, we have a map

$$\frac{\{paths\ x \rightsquigarrow y\}}{homotopy} \longrightarrow \frac{\{paths\ y \rightsquigarrow x\}}{homotopy}$$

3. Constant maps give left and right identities: for a path $f : x \rightsquigarrow y$ in X, $[f] \cdot [c_x] = [f \cdot c_x] = [f] = [c_y \cdot f] = [c_y] \cdot [f]$.

4. Inversion gives inverses: $[f] \cdot [f^{-1}] = [c_y]$ and $[f^{-1}] \cdot [f] = [c_x]$.

Corollary 2.7. *The set of loops based at a fixed point x up to homotopy on a topological space X forms a group under composition, with identity element c_x, with inverses given by inverting paths.*

Definition 2.8. $\pi_1(X, x)$ is the group defined in the above corollary. It is called the *fundamental group* of X based at x.

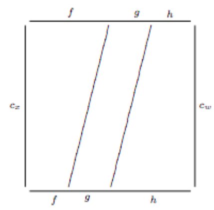
Proof of Theorem. In order:

1. We need to prove that given paths $f, g : x \rightsquigarrow y$, $i : y \rightsquigarrow z$, if $f \simeq g$ then $if \simeq ig$. To do so, choose a homotopy of paths $h : f \simeq g$ and let $k : i \simeq i$ be the constant homotopy. Then the desired homotopy $H : if \simeq ig$ is given by

$$H(s,t) = \begin{cases} h(2s,t) & s \leq 1/2 \\ k(2s-1,t) & s \geq 1/2. \end{cases}$$

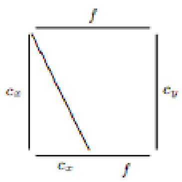
Diagrammatically, if one has a homotopy square for h and one for k, this is equivalent to placing them next to one another.

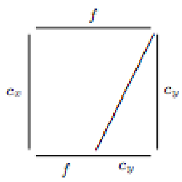
Now to prove associativity, we are given paths $f : x \rightsquigarrow y$, $g : y \rightsquigarrow z$, $i : z \rightsquigarrow w$, and we need to show that $[i(gf)] = [(ig)f]$. The diagram and corresponding formula for the desired homotopy are then given by the following.



$$H(s,t) = \begin{cases} f(2s,t) & s \leq t/4 + 1/4 \\ g(s,t) & t/4 + 1/4 \leq s \leq t/4 + 1/2 \\ h(s/2,t) & s \geq t/4 + 1/2 \end{cases}$$

2. We need to prove that given $f, g : x \rightsquigarrow y$, then $[f] = [g]$ implies $[f^{-1}] = [g^{-1}]$. To do so, choose a homotopy $h : f \simeq g$. Then the diagram is: and we leave the symbolic expression





as an exercise.

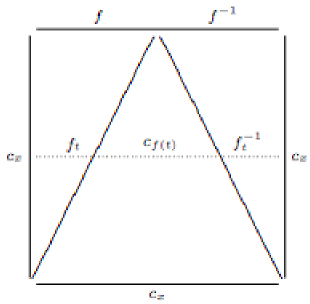
3. For a path $f : x \rightsquigarrow y$ we want: $c_y \cdot f \simeq f$ and $f \cdot c_x \simeq f$. To do this, use the same trick as in 2).

4. Given $f : x \rightsquigarrow y$, we need to show that $f^{-1} \cdot f \simeq c_x$ and $f \cdot f^{-1} = c_y$. For the first composition, we have a homotopy given by the formula

$$h : I \times I \longrightarrow X$$
$$(s,t) \longmapsto \begin{cases} f(2s) & s \leq t/2 \\ f(2t) & t/2 \leq s \leq 1 - t/2 \\ f(2-2s) & s \geq 1 - t/2 \end{cases}$$

The diagram is:

□



2.2 Fundamental groups

Theorem 2.9 (Properties of π_1.). *1. π_1 is (essentially) independent of choice of basepoint: for fixed $x, y \in X$ and a path $a : x \rightsquigarrow y$, there is an isomorphism $\Phi_a : \pi_1(X, x) \to \pi_1(X, y)$ given by conjugating elements of $\pi_1(X, x)$ by $[a]$. That is, $[\alpha] \mapsto [a] \cdot [\alpha] \cdot [a]^{-1}$ is an isomorphism.*

2. π_1 is functorial: for a map $f : X \to Y$ and a point $x \in X$, we obtain a map $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ by $f_*(\alpha) = f \circ \alpha$.

3. Homotopy invariance: fix $f, g : X \to Y$ such that $h : f \simeq g$ is a homotopy. We set $a = h(x, -) : I \to Y$, which induces $a : f(x) \rightsquigarrow y$ making the following diagram commute:

$$\begin{array}{ccc} & \pi_1(X,x) & \\ {\scriptstyle f_*}\swarrow & & \searrow{\scriptstyle g_*} \\ \pi_1(Y, f(x)) & \xrightarrow{\quad \Phi_a \quad} & \pi_1(Y, g(x)) \end{array}$$

where the bottom arrow is an isomorphism, as in (1).

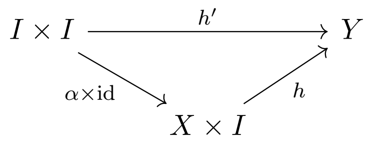
Proof. 1. It is obvious that Φ_a is well-defined. Φ_a is a homomorphism simply because conjugation is a homomorphism. (One might like to write out the statements necessary for this line to notice how the statements in **??** are being used.) Φ_a is an isomorphism since its inverse is clear, and it is given by $(\Phi_a)^{-1} = \Phi_{a^{-1}}$.

2. The given map is a homomorphism: given $\alpha, \beta \in \pi_1(X, x)$ parametrized by maps $h, g : I \to X$ respectively, we have

$$\begin{aligned} f_*(\alpha\beta) &= f \circ (\alpha\beta) \\ &= \begin{cases} f(h(2t)) & 0 \le t \le 1/2 \\ f(g(2t-1)) & 1/2 \le t \le 1 \end{cases} \\ &= \begin{cases} (f \circ h)(2t) & 0 \le t \le 1/2 \\ (f \circ g)(2t-1) & 1/2 \le t \le 1 \end{cases} \\ &= (f \circ h)(f \circ g) \\ &= f_*(\alpha) \cdot f_*(\beta) \end{aligned}$$

as required. Now we check that the composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ induces equal maps $(f \circ g)_* = f_* \circ g_*$. This follows from a similar string of equalities as given above, where the key step is simply that function composition is associative.

3. Fix $\alpha : x \rightsquigarrow x$. We need to show that $\Phi_a f_*(\alpha) = g_*(\alpha)$. This is equivalent to $af_*(\alpha)a^{-1} = g_*(\alpha)$, i.e. $af_*(\alpha) = g_*(\alpha)a$ as paths $f(x) \rightsquigarrow g(x)$, up to homotopy. That is, we require a homotopy realizing this equality. To do this, define a homotopy h' as follows



Symbolically, this is $h'(s,t) = h(\alpha(s), t)$. This gives a desired homotopy $af_*(\alpha) \simeq g_*(\alpha)a$. □

Corollary 2.10. *If X is path-connected, $\pi_1(X, x)$ is independent of x up to isomorphism. Thus we may write $\pi_1(X)$ for the fundamental groups of path-connected spaces.*

We are interested now in computing $\pi_1(X, x)$ for the topological spaces of interest.

Example (The fundamental group of $\mathbb{R}^n$). Let $X = \mathbb{R}^n$ and $x = \mathbf{0} \in \mathbb{R}^n$. We claim that $\pi_1(X, x) = 0$. To show this, choose a path $\alpha : \mathbf{0} \rightsquigarrow \mathbf{0}$. By transitivity of homotopy equivalence, it is enough to show that $\alpha \simeq c_{\mathbf{0}}$. Set $h : I \times I \to \mathbb{R}^n$ to be $h(s,t) = \alpha(s) \cdot (1-t)$. h gives the desired homotopy.

This is true more generally for all contractible spaces, with the same proof carrying over.

Example (The fundamental group of S^1). Let $X = S^1$. Then $\pi_1(S_1) = \mathbb{Z}$. In particular, id_{S^1} is a generator. This is our first example of a space with a nontrivial fundamental group.

The facts necessary to the proof are the following: $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. With this definition it is clear that S^1 is a topological group under multiplication (of complex numbers). We have a covering map given by $\exp : \mathbb{R} \to S^1$ where $t \mapsto e^{2\pi i t}$; it is a group homomorphism from the additive group of $\mathbb{R}$. Its kernel is (isomorphic to) $\mathbb{Z}$. See Figure 2. Further we have $\exp^{-1}(S^1 \setminus \{1\}) = \mathbb{R} \setminus \mathbb{Z} = \coprod_{\mathbb{Z}}(S^1 \setminus \{1\})$. We have $\exp : (i, i+1) \xrightarrow{\sim} S^1 \setminus \{1\}$. Note that the choice of 1 here is not special, and choosing any other point $x \in S^1$ gives the analogous statement (since exp is a homomorphism).

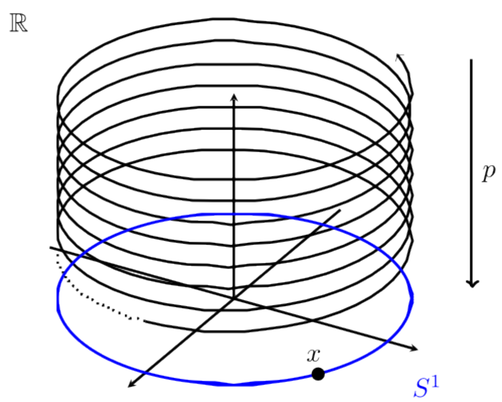
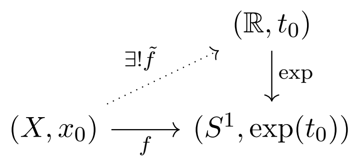


Figure 2: The covering $\exp = p : \mathbb{R} \to S^1$.

We isolate a key lemma.

Lemma 2.11. *Let $X \subset \mathbb{R}^n$ be compact and convex about $x_0 \in \mathbb{R}^n$. Fix $f : X \to S^1$, $t_0 \in \mathbb{R}$ such that $\exp(t_0) = f(x_0)$. Then there exists a unique map $\tilde{f} : X \to \mathbb{R}$ satisfying: $\exp \cdot \tilde{f} = f$ and $\tilde{f}(x_0) = t_0$.*

The diagram is



Remark. The same statement is true when "convex" is replaced with "star-convex around $x \in X$".

Let $f : I \to S^1$ be a loop through $1 \in S^1$. Now, we have that there is a unique $\tilde{f} : I \to \mathbb{R}$ lifting $1 \in S^1$ to $0 \in \mathbb{R}$ such that $\exp \tilde{f} = f$ and $\tilde{f}(0) = 0$. Moreover, $\tilde{f}(1) \in \mathbb{R}$ is another lift of $1 \in S^1$, since $(\exp \tilde{f})(1) = f(1) = 1$. As $\exp^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$, we obtain a map

$$\widetilde{\deg} : \{\text{loops at } 1 \in S^1\} \longrightarrow \mathbb{Z}$$
$$f \longmapsto \tilde{f}(1).$$

Step 1. $\widetilde{\deg}$ factors through homotopy. To show this, say $f, g : I \to S^1$ are loops through 1 which are homotopic. Choose $h : f \simeq g$ realizing this. We need to show that $\tilde{f}(1) = \tilde{g}(1)$.

Applying the lemma to $X = I \times I$ with $x_0 = (0,0)$ and $t_0 = 0$, we obtain a unique map $\tilde{h} : I \times I \to \mathbb{R}$ such that $\exp \tilde{h} = h$ with $\tilde{h}(0,0) = 0$.

Via the uniqueness statement in the lemma (applied twice), we obtain that $\tilde{f}(1) = \tilde{g}(1) = c_1$. We obtain after this a map $\deg : \pi_1(S^1, 1) \to \mathbb{Z}$ which is well-defined up to homotopy.

Step 2. deg is a group homomorphism. For two loops $f, g : I \to S^1$ based at 1, we need to show that $\tilde{f}(1) + \tilde{g}(1) = \widetilde{(gf)}(1)$. Set $g' = \tilde{g} + \tilde{f}(1)$. This is the unique path lifting g with base point $\tilde{f}(1)$ (instead of 0). Now we can compose $g' \cdot \tilde{f}$, which is a path lifting gf based at 0. The uniqueness statement in the lemma dictates that $g' \cdot \tilde{f} = \widetilde{(gf)}$.

We obtain now that $\widetilde{(gf)}(1) = (g' \cdot \tilde{f})(1) = \tilde{g}(1) + \tilde{f}(1)$ as required.

Step 3. deg is an isomorphism. To show that it is injective, choose $f : I \to S^1$ is a loop through 1 such that $\tilde{f}(1) = 0$ (i.e. $f \in \ker(\deg)$). We want to show $f \simeq c_1$. As $\tilde{f}(1) = 0 = \tilde{f}(0)$, the map $\tilde{f}$ is a loop through $0 \in \mathbb{R}$. Since $\mathbb{R}$ is contractible (see 2.2) we have that $\tilde{f} \simeq c_0$, and applying exp we obtain a homotopy $f \simeq c_1$.

To show surjectivity, fix $n \in \mathbb{Z}$. Consider the map $F : I \to \mathbb{R}$ defined by $t \mapsto nt$. $F(0) = 0, F(1) = n$. Set $f = \exp \cdot F$. $f(0) = f(1) = 1$, so f is a loop. By uniqueness in the key lemma we obtain $F = \tilde{f}$. Then $\deg(f) = \tilde{f}(1) = F(1) = n$.

We still have to prove the lemma 2.11. We do that now.

Proof of Lemma. By translation, we allow $x_0 = 0 \in X$ and $f(0) = 1$, and we aim to conclude the uniqueness and existence of $\tilde{f}$ with $\exp \cdot \tilde{f} = f$ and $\tilde{f}(0) = 0$.

We prove uniqueness first. Suppose that $\tilde{f}_1$ and $\tilde{f}_1$ are two lifts as in the lemma. Consider the difference $\tilde{f}_1 - \tilde{f}_2$, which has image contained in $\exp^{-1}(1) = \ker(\exp) = \mathbb{Z}$. Since X is connected (it is convex) and $\mathbb{Z}$ is discrete, continuity implies that $\tilde{f}_1 - \tilde{f}_2$ is constant. To show that this is constantly zero, recall that $\tilde{f}_1(0) = \tilde{f}_2(0)$ by hypothesis.

Now we show existence. We apply the intuition given by the description of the kernel of exp given above. Fix $\epsilon > 0$ such that for all points $x, y \in X$ we have

$$|x - y| < \epsilon \Rightarrow |f(x) - f(y)| < 2$$

which we can choose by appealing to compactness and then uniform continuity. The second inequality implies that $f(x)$ and $f(y)$ are not antipodal, i.e. that $f(x) \neq -f(y)$. Fix $N \geq 0$ such that $|x/N| \leq \epsilon$ for all $x \in X$. Observe that when $\alpha \in I$, $x \in X$, then $\alpha \cdot x \in X$ by convexity (we implicitly use a parametrization between 0 and x here). For all $0 \leq j \leq n-1$, define

$$g_j(x) = \frac{f\left(\frac{j+1}{n} x\right)}{f\left(\frac{jx}{n}\right)} \in S^1.$$

By hypothesis, $g_j(x) \neq -1 \in S^1$, since we prohibited antipodal images. We obtain a map $g_j : X \to S^1 \setminus \{-1\}$. Now consider $g_0(x) \cdot g_1(x) \cdots g_{n-1}(x) = \frac{f(nx/n)}{f(0)} = f(x)/f(0) = f(x)$.

Now we are done: define $\tilde{f}(x) := \sum_i \tilde{g}_i(x) = \sum_i \log(g_i(x))$ where $\log : S^1 \setminus \{-1\} \to (-1/2, 1/2)$ is inverse to $\exp|_{(-1/2,1/2)} : (-1/2, 1/2) \xrightarrow{\sim} S^1 \setminus \{-1\}$. □

Remark. Observe that $S^1 \xrightarrow{z \mapsto z^n} S^1$ is a degree n loop on S^1. We will use this in the proof of the fundamental theorem of algebra.

Now we consider applications of the example.

Theorem 2.12 (Brouwer fixed point theorem)**.** *Let $D \subset \mathbb{R}^2$ be the closed unit disk. Then any (continuous) $f : D \to D$ has a fixed point.*

Proof. Assume not, i.e. that there is f such that $f(x) \neq x$ for all $x \in D$. Define $F : D \to S^1$ by Figure 3, i.e. by continuing the ray connecting $f(x)$ to x to the boundary S^1.

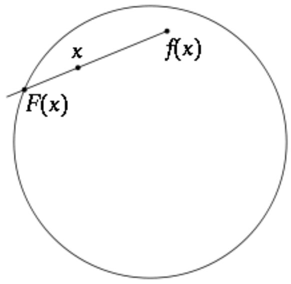


Figure 3: The Brouwer fixed point theorem.

We have further that $F(x) = x$ for each $x \in S^1$. Now we have a retraction $S^1 \hookrightarrow D \twoheadrightarrow S^1$ whose composite is the identity. Applying the functor π_1 we obtain $\mathbb{Z} \to 0 \to \mathbb{Z}$ whose composite is the identity, a contradiction. This finishes the proof. $\square$

Theorem 2.13 (Fundamental theorem of algebra)**.** *Every nonconstant polynomial $p \in \mathbb{C}[x]$ with complex coefficients has a root in $\mathbb{C}$.*

Proof. Fix $n > 0$ and write $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ for the coefficients $a_i \in \mathbb{C}$. It suffices by induction to find a single root.

For a contradiction, assume there is no root, i.e. that $p(z) \neq 0$ for any $z \in \mathbb{C}$. Then we may think of p as a map $\mathbb{C} \to \mathbb{C} \setminus \{0\}$, and we will use the fact that the punctured plane is homotopic to a circle. We set $X_R = \{z \in \mathbb{C} : |z| = R\}$. p restricts to a map $p_R : X_R \to \mathbb{C} \setminus \{0\} \simeq S^1$.

We know that $\pi_1(X_r) = \pi_1(\mathbb{C} \setminus \{0\}) = \mathbb{Z}$, and we set $(p_R)_*(1) =: \deg(p_R) \in \mathbb{Z}$. We will calculate this in two different ways.

First, p_R factors as $X_R \subset \mathbb{C} \xrightarrow{p} \mathbb{C} \setminus \{0\}$, so applying π_1 we obtain $\mathbb{Z} \to 0 \to \mathbb{Z}$, from which we conclude that $(p_R)_*$ is the zero map, from which it follows that $\deg(p_R) = 0$.

Second, we will show that $\deg(p_R) = \deg(p) = n$ to finish the contradiction. Consider $h' : X_R \times I \to \mathbb{C}$ defined by $h'(z,t) = z^n + t(p(z) - z^n)$. We see that $h'(z,0) = z^n$, and $h'(z,1) = p(z)$. We would like to know that h' takes values in $\mathbb{C} \setminus \{0\}$; elementary analysis implies that for sufficiently large R, $h'(z,t) \neq 0$. Thus we obtain a similar map $h : X_R \to \mathbb{C} \setminus \{0\}$, and h gives a homotopy $(z \mapsto z^n) \simeq p$, and therefore $\deg(p_R) = \deg(z \mapsto z^n) = n$. $\square$

2.2.1 Calculating π_1

In this section we introduce some tools for computing fundamental groups. Two versions of Van Kampen's theorem will tell us how to calculate fundamental groups of spaces that are obtained by gluing two spaces (whose fundamental groups we know) together along a shared subspace, when that subspace is path-connected in one case and in general in the other. It is an important theorem for calculating fundamental groups. We will state and prove Van Kampen's theorem below.

A non-example where Van Kampen's theorem does not apply, then, is the gluing of two intervals at their endpoints to obtain S^1. As the intersection along which we glue is not connected, Van Kampen's theorem does not hold, and we required other tools for computing π_1. For this reason, we introduce groupoids.

Definition 2.14. A category $\mathbf{C}$ is a *groupoid* if all maps in $\mathbf{C}$ are isomorphisms. A map (equivalence) of groupoids $\mathbf{C}_1 \to \mathbf{C}_2$ is a functor (equivalence) of the underlying categories. We denote by **Grpd** the category of groupoids with maps of groupoids.

We begin by giving some examples of groupoids.

Example. For G a group we obtain BG a groupoid such that: $\text{ob}(BG) = \{*\}$, with $\text{Aut}(*) = \text{Hom}_{BG}(*, *) = G$. Morphisms compose via enforcing the commutativity of

$$\begin{array}{ccc} \text{Hom}_{BG}(*,*) \times \text{Hom}_{BG}(*,*) & \xrightarrow{\text{composition}} & \text{Hom}_{BG}(*,*) \\ \downarrow = & & \downarrow = \\ G \times G & \xrightarrow{\text{multiplication}} & G \end{array}$$

It is left as an exercise to prove an equivalence of categories between the category of groupoids with a single object and the category of groups.

Example. For X a topological space, we define $\tau_{\leq 1}X$ which we call the *fundamental groupoid of* X, where $\text{ob}(\tau_{\leq 1}) = \{x \in X\}$, and for $x, y \in X$, $\text{Hom}(x, y) = \frac{\{\text{paths } x \rightsquigarrow y\}}{\text{homotopy}}$. The composition law is given by composing paths (up to homotopy). It follows from associativity of composition of paths that this is a category, and from inversion of paths that it is a groupoid.

We obtain a functor $\mathbf{Top} \to \mathbf{Grpd}$ taking $X \to \tau_{\leq 1}X$, with the obvious way of sending continuous maps $X \to Y$ to paths. We call this functor the fundamental groupoid functor.

Note also that for $x \in X$ we have an isomorphism $\pi_1(X, x) \cong \text{Aut}_{\tau_{\leq 1}X}(x)$.

Definition 2.15. For $\mathbf{C}$ a groupoid and $x \in \mathbf{C}$ an object, we define the sets $\pi_0(\mathbf{C}) = \text{ob}(\mathbf{C})/\text{isomorphism}$, and for $x \in X$, $\pi_1(\mathbf{C}, x) = \text{Aut}_{\mathbf{C}}(x)$.

We isolate a lemma that we will use later.

Lemma 2.16. *For $\mathbf{C}$ a connected groupoid and $x \in \mathbf{C}$, the natural map*

$$F : B\pi_1(\mathbf{C}, x) \to \mathbf{C}$$

is an equivalence of categories.

Proof. We use the following fact: a functor $F : \mathbf{C} \to D$ is an equivalence if it satisfies the following two conditions. For every $d \in D$ there is $c \in \mathbf{C}$ such that $F(c) \cong d$ (F is essentially surjective), and the natural map $\text{Hom}_{\mathbf{C}}(c_1, c_2) \to \text{Hom}_D(F(c_1), F(c_2))$ is a bijection (F is fully faithful).

Both requirements are trivial to check. F is essentially surjective since $\mathbf{C}$ is connected, and F is fully faithful in a similar way. It follows that F is indeed an equivalence of categories.

One can also construct the inverse functor explicitly, as we do now. On the level of objects, we are forced to send each object in $\mathbf{C}$ to the unique object in $B\pi_1(\mathbf{C}, x)$. For morphisms, in every object $y \in \mathbf{C}$, choose an isomorphism $\gamma_y \in \text{Hom}_{\mathbf{C}}(y, x)$ such that $\gamma_x = id_x$. Now for a sufficient map on the level of morphisms, use

$$\text{Hom}_{\mathbf{C}}(y, z) \longrightarrow \text{Hom}_{B\pi_1(\mathbf{C},x)}(x, x)$$

defined by $(\alpha : y \to z) \mapsto \gamma_z \circ \alpha \circ \gamma_y^{-1}$. We leave it as an exercise to prove that this is a functor, i.e. it is compatible with composition. ☐

Corollary 2.17. *For X a path-connected space and $x \in X$, $B\pi_1(X, x) \simeq \tau_{\leq 1}(X)$ (an equivalence of categories) by applying the preceding lemma.*

Now we arrive at the Seifert-van Kampen theorem. We will present two versions, one in a standard topological setting and a version using groupoids. The second version will be useful when the space along which we are gluing the two larger spaces is not path-connected, and it does not depend on a choice of base point. We will need the following construction.

2.2.2 Pushouts of groups

Theorem 2.18. **Group** *is cocomplete, i.e. has all (small) colimits.*

In what follows we will only need that **Group** has all coproducts and cofibered coproducts, so that is what we will prove.

Proof. For coproducts: choose groups $G, H \in \mathbf{Group}$. We define $G * H$, the coproduct of G and H, to be the free product of G and H. We require that for any $K \in \mathbf{Group}$, giving a map $G * H \to K$ is equivalent to giving maps $G \to K$ and $H \to K$. That is, $\mathrm{Hom}_{\mathbf{Group}}(G * H, K) \simeq \mathrm{Hom}_{\mathbf{Group}}(G, K) \times \mathrm{Hom}_{\mathbf{Group}}(H, K)$.

We leave it as an exercise to verify the universal property, so that $G * H$ gives the coproduct of G and H.

For cofibered coproducts (pushouts): given a diagram

$$\begin{array}{ccc} K & \longrightarrow & G \\ \downarrow & & \\ H & & \end{array}$$

we need to be able to extend it to a diagram

$$\begin{array}{ccc} K & \longrightarrow & G \\ \downarrow & & \downarrow \\ H & \longrightarrow & G *_K H \end{array}$$

such that $G *_K H$ is universal with respect to this property. Note that when K is trivial, this is given by the free product of G and H. In general, for maps as above $\alpha : K \to G, \beta : K \to H$, and inclusions $i_G : G \to G * H$ and $i_H : H \to G * H$, we set

$$G *_K H := \text{“amalgamated product of } G \text{ and } H \text{ over } K\text{”} = \frac{G * H}{\langle i_G\alpha(k) = i_H\beta(k)\rangle}.$$

By construction, and as one can check, we obtain the necessary extension of the first diagram, and this extension commutes.

In general one would need to check that **Group** has all filtered colimits as well, but we will not need this. The existence of all colimits is a formal consequence of the existence of these smaller cases. ☐

Example. $\mathbb{Z} * \mathbb{Z} = a^{\mathbb{Z}} * b^{\mathbb{Z}}$ is the free group on two generators. More generally, for any set S, we obtain a free group $F(S)$ on the set S: at least when S is finite, $F(S) = \mathbb{Z} * \cdots * \mathbb{Z}$, on $|S|$-many copies of $\mathbb{Z}$. Now for the projections $\mathbb{Z} \to \mathbb{Z}/2, \mathbb{Z}/3$, we obtain $\mathbb{Z}/2 *_{\mathbb{Z}} \mathbb{Z}_3 \cong 0$; using the universal property, we see that the amalgamated products needs to be generated by an element of order 2 that also has order 3, so the equality follows. We won't prove this, but it is also true that $\mathbb{Z}/2 * \mathbb{Z}/3 \cong PSL_2(\mathbb{Z})$. (This is a straighforward application of something called the ping-pong lemma, which we will not see in this course.)

To prove the groupoid version of Van Kampen, we will use the following lemma.

Lemma 2.19. *Given a diagram of groups*

$$\begin{array}{ccc} K & \xrightarrow{\alpha} & G \\ \downarrow{\scriptstyle\beta} & & \\ H & & \end{array}$$

the diagram

$$\begin{array}{ccc} BK & \xrightarrow{\alpha} & BG \\ \downarrow{\scriptstyle\beta} & & \downarrow \\ BH & \longrightarrow & B(G *_K H) \end{array}$$

is a pushout of groupoids. That is, the functor B takes pushouts to pushouts.

Theorem 2.20 (Seifert-van Kampen). *For a space $X = U \cup V$ for $U, V \subset X$ open subsets, we have the following two statements.*

(1) (Groupoid version) The square

$$\begin{array}{ccc} \tau_{\leq 1}(U \cap V) & \longrightarrow & \tau_{\leq 1}(U) \\ \downarrow & & \downarrow \\ \tau_{\leq 1}(V) & \longrightarrow & \tau_{\leq 1}(X) \end{array}$$

is a pushout in the category **Grpd**.

(2) (Group version) If $U, V, U \cap V$ are all path-connected, and $x \in U \cap V$, then

$$\begin{array}{ccc} \pi_1(U \cap V, x) & \longrightarrow & \pi_1(U, x) \\ \downarrow & & \downarrow \\ \pi_1(V, x) & \longrightarrow & \pi_1(X, x) \end{array}$$

is a pushout in **Group**. *That is, $\pi_1(X, x) \simeq \pi_1(U, x) *_{\pi_1(U \cap V, x)} \pi_1(V, x)$.*

Before proving the theorem, we give some applications.

Example (Fundamental group of S^n). Choose $U =$ northern hemisphere "$+\epsilon$" (which passes a bit below the equator), and $V =$ southern hemisphere $+\epsilon$. When $n \geq 2$ the intersection $U \cap V$ is a path-connected annulus (or a higher-dimensional analogue). Each of U and V is homotopic to $\mathbb{R}^n$ via stereographic projection, and the intersection is homotopic to S^{n-1}. Applying the group version of the theorem, we obtain $\pi_1(S^n, x) \simeq \pi_1(\mathbb{R}^n, x) *_{\pi_1(S^{n-1}, x)} \pi_1(\mathbb{R}^n, x) \simeq 0$, for $x \in U \cap V$, since $\mathbb{R}^n$ is contractible.

Example (Fundamental group of the figure-8). For U and V we take one circle "plus ϵ" (which is open) in the figure-8. $U \cap V$ is an "open X". We have $\pi_1(U, x) \simeq \pi_1(V, x) \simeq \mathbb{Z}$ for $x \in U \cap V$. The intersection has trivial fundamental group. The group version of the theorem gives us $\pi_1(S^1 \vee S^1, x) \simeq F_2$, the free group on two generators.

Now we prove the theorem for groupoids. We will specify how this version will imply the group version. We will need a lemma from point-set topology.

Lemma 2.21. *For a compact metric space (Y, d) and an open cover $\{U_i\}_{i \in I}$ of Y, there exists a $\delta > 0$ such that $A \subset Y$ with diam$(A) < \delta$, then $A \subset U_i$ for some $i \in I$.*

The proof is left as an exercise.

Proof of Seifert-van Kampen for groupoids). We directly verify the universal property. Fix a diagram

$$\begin{array}{ccc} \tau_{\leq 1}(U \cap V) & \longrightarrow & \tau_{\leq 1}(U) \\ \downarrow & & \downarrow{\scriptstyle g} \\ \tau_{\leq 1}(V) & \xrightarrow{f} & C \end{array}$$

for a groupoid C. We need to show that there is a unique map $h : \tau_{\leq 1}(X) \to C$ making the following diagram commute:

$$\begin{array}{ccccc}
\tau_{\leq 1}(U \cap V) & \longrightarrow & \tau_{\leq 1}(U) & & \\
\downarrow & & \downarrow & \searrow^{g} & \\
\tau_{\leq 1}(V) & \longrightarrow & \tau_{\leq 1}(X) & \overset{h}{\dashrightarrow} & C \\
& \searrow_{f} & & & \nearrow
\end{array}$$

We specify h on objects: let x be an object in $\tau_{\leq 1}(X)$. We define $h(x) = f(x)$ if $x \in \tau_{\leq 1}(V)$ and $g(x)$ if $x \in \tau_{\leq 1}(U)$. The commutativity of the above diagram shows that his is well-defined, i.e. both definitions agree on $\tau_{\leq 1}(U \cap V)$.

To specify h on morphisms, we apply the lemma. Fix a path $\alpha : x \to y$ in X. Via the lemma, we may write $\alpha = \alpha_0 \cdot \alpha_1 \cdots \alpha_m$ such that the image of each α_i lies entirely in either U or V. The metric space in the lemma is the unit interval. Therefore, define $h(\alpha_i) = f(\alpha_i)$ or $g(\alpha_i)$ depending on the containment $\alpha_i \subset U, V$. This is well-defined again by commutativity of the first given diagram.

We define h on morphisms via $h(\alpha) = h(\alpha_0) \cdot h(\alpha_1) \cdots h(\alpha_m)$. One needs to check that this is really a functor, i.e. compatible with composition. This is left as an exercise.

Now we show that h factors through homotopy. Fix $\phi : \alpha \simeq \beta$ a homotopy of paths in X. Then ϕ is a map $I \times I \to X$ satisfying the necessary properties. The lemma implies that one may subdivide $I \times I$ into subsquares $M_{i,j}$ such that $\phi(M_{i,j}) \subset U$ or V. $M_{i,j}$ is defined by a bottom path a and a path b which is the composition of the left, top, and right paths, and these paths are homotopic. As $\phi(M_{i,j})$ lies in either U or V, $h(a) = h(b)$ for each subsquare $M_{i,j}$. Repeating this many times we obtain $h(\alpha) = h(\beta)$, so h does indeed factor through homotopy. The result follows. □

Now for the group version:

Proof. We are in the same setting as in the previous proof, with a choice of $x \in U \cap V$. We verify the universal property directly again.

Step 1. We have a natural diagram

$$\begin{array}{ccccc}
B\pi_1(U, x) & \longleftarrow & B\pi_1(U \cap V, x) & \longrightarrow & B\pi_1(V, x) \\
\downarrow{\scriptstyle i_U} & & \downarrow{\scriptstyle i_{U \cap V}} & & {\scriptstyle i_V}\downarrow \\
\tau_{\leq 1}(U) & \longleftarrow & \tau_{\leq 1}(U \cap V) & \longrightarrow & \tau_{\leq 1}(V)
\end{array}$$

We showed above in 2.16 that the downward arrows are equivalences of categories, via path-connectedness. We choose inverses P_{-} for each i_{-} such that the induced diagram

$$\begin{array}{ccccc}
B\pi_1(U, x) & \longleftarrow & B\pi_1(U \cap V, x) & \longrightarrow & B\pi_1(V, x) \\
{\scriptstyle P_U}\uparrow\downarrow{\scriptstyle i_U} & & {\scriptstyle P_{U \cap V}}\uparrow\downarrow{\scriptstyle i_{U \cap V}} & & {\scriptstyle i_V}\downarrow\uparrow{\scriptstyle P_V} \\
\tau_{\leq 1}(U) & \longleftarrow & \tau_{\leq 1}(U \cap V) & \longrightarrow & \tau_{\leq 1}(V)
\end{array}$$

commutes. To do so: for all $y \in X$ choose a path $\gamma_y : y \rightsquigarrow x$ such that $\gamma_x = c_x$, and $\gamma_y \subset U$ if $y \in U$, $\gamma_y \subset V$ if $y \in V$, and similarly for the intersection. In this way we obtain

$$\begin{array}{rc}
& P_U : \tau_{\leq 1}(U) \longrightarrow B\pi_1(U, x) \\
\text{objects:} & y \longmapsto x \\
\text{morphisms:} & (y \xrightarrow{\alpha} z) \longmapsto (x \xrightarrow{\gamma_y^{-1}} y \xrightarrow{\alpha} z \xrightarrow{\gamma_y} x).
\end{array}$$

and similarly for $P_V, P_{U\cap V}$. The diagram commutes by construction, as one can check.

Now we aim to show that the diagram

$$\begin{array}{ccc} \pi_1(U\cap V, x) & \longrightarrow & \pi_1(U,x) \\ \big\downarrow & & \big\downarrow{\scriptstyle \beta} \\ \pi_1(V,x) & \xrightarrow{\ \alpha\ } & \pi_1(X,x) \end{array}$$

is a pushout. That is, given a commuting diagram

$$\begin{array}{ccc} \pi_1(U\cap V, x) & \longrightarrow & \pi_1(U,x) \\ \big\downarrow & & \big\downarrow{\scriptstyle g} \\ \pi_1(V,x) & \xrightarrow{\ f\ } & G \end{array}$$

there is a unique $h : \pi_1(X,x) \to G$ such that $h\alpha = f$ and $h\beta = g$. Applying the functor $B : \mathbf{Group} \to \mathbf{Grpd}$, we obtain

$$\begin{array}{ccc} B\pi_1(U\cap V, x) & \longrightarrow & B\pi_1(U,x) \\ \big\downarrow & & \big\downarrow{\scriptstyle Bg} \\ B\pi_1(V,x) & \xrightarrow{\ Bf\ } & BG \end{array}$$

and now precomposing with the P_-, obtain

$$\begin{array}{ccc} \tau_{\leq 1}(U\cap V) & \longrightarrow & \tau_{\leq 1}(U) \\ \big\downarrow & & \big\downarrow{\scriptstyle Bg\circ P_U} \\ \tau_{\leq 1}(V) & \xrightarrow{\ Bf\circ P_V\ } & BG. \end{array}$$

Applying the groupoid version of the theorem, we obtain via the universal property a map $h' : \tau_{\leq 1}(X) \to BG$ making the necessary diagram commute. Take the map on automorphism groups induced by h' at the point $x \in \tau_{\leq 1}(X)$ to obtain $h : \pi_1(X,x) \to G$ as required. Uniqueness of h is still not proved, but one can readily check it using uniqueness in the groupoid version, which we've already shown. □

We give some examples of Seifert-van Kampen in action. In the following $\langle a_1, ..., a_n\rangle$ will denote the free group on n letters. The free group with relations will be denoted $\langle a_1, ..., a_n \mid f_1 = g_1, ..., f_m = g_m\rangle$, i.e. the free group on n letters quotiented out by a normal subgroup enforcing the relations $f_i = g_i$. For example, $\langle a, b \mid ab = ba\rangle \cong \mathbb{Z}^2$ is the abelianization of the free group on two letters.

Example (Fundamental group of the (punctured) 2-torus). $\pi_1(S^1 \times S^1)$. We've shown that π_1 commutes with products, so we know that this is $\mathbb{Z} \times \mathbb{Z}$. Now we will use van Kampen to determine the fundamental group of the *punctured* 2-torus.

Consider $(S^1 \times S^1) \smallsetminus \{\text{small disc}\}$. Observe that when constructing the 2-torus as a quotient of a unit square, up to homotopy we have the identity $(S^1 \times S^1) \smallsetminus \{\text{small disc}\} \simeq S^1 \vee S^1$. See Figure 4.

We still haven't used van Kampen. Now we do:

Example (Fundamental group of a genus 2 surface). Express the genus 2 surface as the connect sum of two punctured tori along an annulus (homotopy equivalent to S^1). We are in the setting of the group version of van Kampen: we have

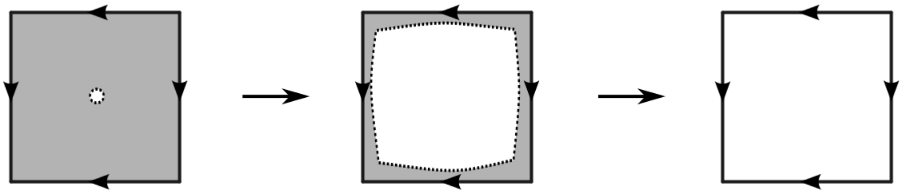


Figure 4: A retraction of the unit square (with identifications) minus a disk with the square (with identifications), i.e. the wedge of two circles. This image came from math.se.

$$\begin{array}{ccccc} \pi_1(V) & \longleftarrow & \pi_1(U\cap V) & \longrightarrow & \pi_1(U) \\ \Big\downarrow = & & \Big\downarrow = & & \Big\downarrow = \\ \langle a_2, b_2\rangle & \xleftarrow[{[a_2,b_2]\leftarrow c}]{} & c^{\mathbb{Z}} & \xrightarrow[{c\mapsto [a_1,b_1]^{-1}}]{} & \langle a_1, b_1\rangle \end{array}$$

Thus Seifert-van Kampen dictates that $\pi_1(X) = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] = 1\rangle$. We note for now that $\pi_1(X)^{\mathrm{ab}} = \mathbb{Z}^{\oplus 4}$.

2.3 Covering Spaces

We establish some notation: fix maps $f : X \to Y$ and $\alpha : Z \to Y$. A *lifting of α along f* is a map $\widetilde{\alpha} : Z \to X$ such that $f\widetilde{\alpha} = \alpha$. For example, we saw that any map $I \to S^1$ lifts along the exponential $\mathbb{R} \to S^1$.

Definition 2.22. A map $f : X \to Y$ is a *covering space* provided that f is surjective, and there exists a cover $\{U_i\}$ of Y such that there is a homeomorphism $f^{-1}(U_i) \cong \coprod_i U_i$ which is compatible with the projection maps to the U_i. That is, the following triangle commutes:

$$\begin{array}{ccc} f^{-1}(U_i) & \xrightarrow{\quad\sim\quad} & \coprod_n U_j \\ & {\scriptstyle f|_{U_i}}\searrow \quad \swarrow{\scriptstyle \mathrm{proj}_i} & \\ & U_i & \end{array}$$

for some $n \geq 2$.

Example. A cover of S^1 corresponding to the exponential is given by $S^1 \smallsetminus \{1\}$ and $S^1 \smallsetminus \{-1\}$.

Example. Consider $\alpha_n : S^1 \to S^1$ taking $z \mapsto z^n$. For $n = 2$, we see that $\alpha_n^{-1}(S^1 \smallsetminus \{0\}) = S^1 \smallsetminus \{1, -1\} \cong \coprod_{\mathbb{Z}/2}(S^1 \smallsetminus \{x\})$ (where $\cong$ denotes a homeomorphism). In general, $\alpha_n^{-1}(S^1) = S^1 \smallsetminus \{n\text{th roots of } 1\} = \coprod_{\mathbb{Z}/n}(\text{intervals})$.

Example. For any set S and any space X, the obvious map $\coprod_S X \to X$ is a covering space.

Example. Let $X = S^n$ and $Y = \mathbb{RP}^n = S^n/(\mathbb{Z}/2)$ where the $\mathbb{Z}/2$-action is given by $x \mapsto -x$. Then the quotient map $X \to Y$ is a covering space.

Theorem 2.23 (Path/homotopy lifting property.)**.** *Let $f : X \to Y$ be a covering space, with $x \in X$ and $y = f(x) \in Y$. Then*

(1) For $\alpha : I \to Y$ with $\alpha(0) = y$, there is a unique lift $\widetilde{\alpha} : I \to X$ such that $\widetilde{\alpha}(0) = x$.

(2) For $h : I \times I \to Y$ such that $h(0, 0) = y$, there is unique $\widetilde{h} : I \times I \to X$ such that $\widetilde{h}(0, 0) = x$.

Remark. For a bit of motivation, recall 2.2.

Proof. Fix a cover $\{U_i\}$ of Y that splits f (i.e. $f^{-1}(U_i) = \coprod U_i$).

First we lift paths. The topological lemma above, applied to the open cover $f^{-1}(U_i)$ of I, implies that there is a factorization $0 = s_1 \le s_1 \le \cdots \le s_m = 1 \in I$ such that $\alpha|_{[s_i,s_{i+1}]}$ has image in some U_j.

Step 1. Assume $\alpha(I) \subset U_j$ for some j. Then it is obvious that there exists a unique $\widetilde{\alpha} : I \to X$ as required.

Step 2. By step 1, there is a unique lift $\widetilde{\alpha}_0 : [s_0, s_1] \to X$ such that $\widetilde{\alpha}_0(s_0) = x$ and $\widetilde{\alpha}_0$ lifts $\alpha|_{[s_0,s_1]}$. Again by step 1, there exists unique $\widetilde{\alpha}_1$, analogous to $\widetilde{\alpha}_0$, such that $\widetilde{\alpha}_0(1) = \widetilde{\alpha}_1(0)$ and $\widetilde{\alpha}_1$ lifts $\alpha|_{[s_1,s_2]}$. Inductively we may construct such lifts for each $\alpha|_{[s_i,s_{i+1}]}$ which are mutually compatible. We set $\widetilde{\alpha} = \widetilde{\alpha}_{m-1} \cdots \widetilde{\alpha}_0$. Uniqueness essentially carries over from the uniqueness from step 1.

Now we lift homotopies. Avoiding the pain of being precise with indices, say $h : I \times I \to Y$ is such that $I \times I$ admits a decomposition into *four* squares $V_1, ..., V_4$, such that for all i, $h(V_i) \subset U_j$ for some j.

Construct $\tilde{h}_1 : V_1 \to X$ by choosing a sheet $f^{-1}(U_i)$ containing x. Proceeding as in the case of paths, we may glue such lifts together compatibly to obtain $\tilde{h}$ which lifts h as required. □

Corollary 2.24. *If $f : X \to Y$ is a covering space, and $x \in X$ with $y = f(x) \in Y$, then $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ is injective.*

Proof. Choose $\alpha : x \rightsquigarrow x$ such that $f_*([\alpha]) = 0$. Choose a homotopy $h : f_*(\alpha) \simeq c_y$; apply homotopy-lifting to obtain a homotopy $\tilde{h} : I \times I \to X$. Uniqueness of homotopy lifting gives that $\tilde{h}$ gives a homotopy $\alpha \simeq c_x$, as required. □

For what follows, choose a covering space $f : X \to Y$ with $x \in X$ and $y = f(x) \in Y$. If we suppose that the induced map $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ is injective, then we obtain a map $\widetilde{\phi} : \pi_1(Y, y) \to f^{-1}(y)$ given by $\alpha \mapsto \widetilde{\alpha}(1)$, where $\widetilde{\alpha}$ is the unique lift of α based at x. Indeed, we obtain a well-defined map (of sets) $\phi : \pi_1(Y, y)/f_*\pi_1(X, x) \to f^{-1}(y)$, as when $\alpha = f_*\beta$, we have $\widetilde{\alpha} = \beta$ so that $\widetilde{\alpha}(1) = \beta(1) = \beta(0) = x$.

Theorem 2.25. *ϕ is injective. When X is path-connected, ϕ is a bijection.*

Example. Consider again the cover $S^1 \to S^1$ given by $z \mapsto z^n$. We have $\#\pi_1(S^1)/(f_*\pi_1(S^1)) = \#f^{-1}(1) = n$.

Proof. Choose $\alpha_1, \alpha_2 \in \pi_1(Y, y)$ with $\phi(\alpha_1) = \phi(\alpha_2)$. Then $\phi(\alpha_i) = \widetilde{\alpha_i}(1)$; as α_1 and α_2 are paths $x \rightsquigarrow \widetilde{\alpha_i}(1)$, we have that $\widetilde{\alpha_2}^{-1}\widetilde{\alpha_1}$ is a loop in X based at x. In particular, we have $f_*([\widetilde{\alpha_2}^{-1}\widetilde{\alpha_1}]) = \alpha_2^{-1}\alpha_1 \in f_*\pi_1(X, x)$, as required.

For the second statement, assume that X is path-connected. Choose $x' \in f^{-1}(y)$ and a path $\beta : x \rightsquigarrow x'$ and set $\alpha = f_*(\beta) \in \pi_1(Y, y)$. Uniqueness of path-lifting dictates that $\widetilde{\alpha} = \beta$. Then we have $\widetilde{\alpha}(1) = \beta(1) = x'$, implying that $\phi(\alpha) = x'$. □

Definition 2.26. A space X is *simply connected* if X is path-connected and $\pi_1(X) = 0$.

Corollary 2.27. *If a covering space $f : X \to Y$ is given with X simply connected, then $\pi_1(Y, y)$ is in natural bijection with $f^{-1}(y)$.*

Example. For $X = S^n$, $n \geq 2$ and $Y = \mathbb{RP}^n$, we have a covering map $f : X \to Y$. The corollary implies that $\#\pi_1(\mathbb{RP}^n) = 2$, so we have $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2$.

For $Y \ni y$ a reasonable topological space, we aim to establish a dictionary

$$\left\{ \begin{array}{c} \text{covering spaces } f : X \to Y \\ x \in X, X \text{ path-connected} \end{array} \right\} \overset{\sim}{\longleftrightarrow} \{\text{subgroups of } \pi_1(Y, y)\}$$

$$(X, x) \longleftrightarrow f_*\pi_1(X, x)$$

As an application of such a dictionary, we would immediately see that all subgroups of free groups are free. We will have to develop more tools in order to construct this correspondence.

2.3.1 Classification of covering spaces

In what follows, all spaces will be path-connected and locally path-connected.

Definition 2.28. Two covering spaces $f_1 : X_1 \to Y$, $f_2 : X_2 \to Y$ are *equivalent* provided that there exists a homeomorphism g making the triangle

$$\begin{array}{ccc} X_1 & \xrightarrow{\;g\;} & X_2 \\ & {\scriptstyle f_1}\searrow \quad \swarrow{\scriptstyle f_2} & \\ & Y & \end{array}$$

commute. We have a similar definition for pointed topological spaces, where we require all maps to preserve the base points.

We will show that for $Y \ni y$ path-connected, we have

$$\begin{array}{ccc} \dfrac{\left\{\begin{array}{c}\text{pointed path-connected} \\ \text{covering spaces } (X,x)\to(Y,y)\end{array}\right\}}{\text{pointed equivalence}} & \xrightarrow{\;\sim\;} & \{\text{subgroups of } \pi_1(Y,y)\} \\ \downarrow & & \downarrow \\ \dfrac{\left\{\begin{array}{c}\text{path-connected} \\ \text{covering spaces } X\to Y\end{array}\right\}}{\text{equivalence}} & \xrightarrow{\;\sim\;} & \dfrac{\{\text{subgroups of } \pi_1(Y,y)\}}{\text{conjugation}} \end{array}$$

We have the following key lifting lemma.

Lemma 2.29. *Fix a (path-connected and locally path-connected) space Z pointed with $z \in Z$ and a map $\alpha : Z \to Y$. With X, Y as above, then we have a lift $\widetilde{\alpha} : Z \to X$ lifting α, with $\widetilde{\alpha}(z) = x$ if and only if we have inclusions $\alpha_*\pi_1(Z,z) \leq f_*\pi_1(X,x) \leq \pi_1(Y,y)$ of groups. The diagrams are*

$$\begin{array}{ccc} & & (X,x) \\ & {\scriptstyle\widetilde{\alpha}} \nearrow & \downarrow{\scriptstyle f} \\ (Z,z) & \xrightarrow{\;\alpha\;} & (Y,y) \end{array} \quad \Leftrightarrow \quad \begin{array}{ccc} & & \pi_1(X,x) \\ & \nearrow & \downarrow{\scriptstyle f_*} \\ \pi_1(Z,z) & \xrightarrow{\;\alpha_*\;} & \pi_1(Y,y) \end{array}$$

Further, when $\widetilde{\alpha}$ exists, it is unique.

Remark. When $Z = I$, this gives the path lifting lemma. When $Z = I \times I$, this gives the homotopy lifting lemma.

Proof. The forward implication is clear. For the backward implication, given $z' \in Z$, choose a path $\beta : z \rightsquigarrow z'$ in Z. We obtain a path $\alpha_*(\beta) : y \rightsquigarrow \alpha(z')$. Path lifting via f gives us a path $\widetilde{\alpha_*(\beta)}$ in X, lifting $\alpha_*(\beta)$, based at x. Set $\widetilde{\alpha}(z') = \widetilde{\alpha_*(\beta)}(1)$.

We want to show that this is independent of the choice of β. That is, given $\beta, \beta' : z \rightsquigarrow z'$, we want that $\widetilde{\alpha_*(\beta)}(1) = \widetilde{\alpha_*(\beta')}(1)$. So, consider the loop $(\beta')^{-1}\beta \in \pi_1(Z,z)$; we have $\alpha_*(\beta')^{-1}\beta \in \alpha_*\pi_1(Z,z) \leq f_*\pi_1(X,x)$. Now say that $\alpha_*(\beta')^{-1}\beta = f_*\gamma$ for $\gamma \in \pi_1(X,x)$. By uniqueness of path lifts, we see that $\widetilde{\alpha_*(\beta')^{-1}\beta} = \gamma$, as both lifts are based at x.

Thus, $\widetilde{\alpha_*(\beta')}^{-1} \cdot \widetilde{\alpha_*\beta}$ is a loop. It follows that $\widetilde{\alpha_*\beta}(1) = \widetilde{\alpha_*(\beta')}(1)$ as required. We obtain a well-defined lift $\widetilde{\alpha}$ of α. It follows immediately that $\widetilde{\alpha}(z) = x$ (use the constant path $\beta : z \rightsquigarrow z$) and that $\widetilde{\alpha}$ is continuous.

To prove the second statement, simply apply the uniqueness of path lifting. $\square$

Remark. The lemma implies that covering spaces are monomorphisms in the category of pointed topological spaces.

Theorem 2.30 (Classifying covering spaces via π_1)**.** *Let $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ be path-connected covering spaces, with compatible basepoints $x_i \in X_i, y \in Y$ such that $f_i(x_i) = y$. There exists an equivalence h of covering spaces preserving the basepoints if and only if $(f_1)_*\pi_1(X_1, x_1) = (f_2)_*\pi_1(X_2, x_2)$.*

In other words, we have an embedding of categories

$$\frac{\{\text{pointed covering spaces}\}}{\text{equivalence}} \longrightarrow \{\text{subgroups of } \pi_1(Y, y)\}.$$

Proof. The forward direction is clear, after noting that π_1 is functorial. For the reverse direction, we apply the key lemma 2.29 to obtain a lift $h : X_1 \to X_2$ of f_1; since $(f_1)_*\pi_1(X_1, x_1) = (f_2)_*\pi_1(X_2, x_2)$, we have a lift g going in the other direction as well. We need to show that this is an equivalence. However by uniqueness in the lifting lemma implies that $g \circ h$ and $h \circ g$ are both the necessary identities on the X_i, finishing the proof. □

We would like to excise basepoints from our arguments.

Lemma 2.31. *If $f : X \to Y$ is a covering space (where we assume all spaces are path-connected). Fix $y \in Y$ and $x_1, x_2 \in f^{-1}(y)$. If $\alpha : x_1 \rightsquigarrow x_2$ is a path, then $f_*(\alpha) \in \pi_1(Y, y)$ and $f_*(\alpha) \cdot f_*\pi_1(X, x_1) \cdot f_*(\alpha)^{-1} = f_*\pi_1(X, x_2)$. Moreover, if $f_*\pi_1(X, x_1) = gHg^{-1}$ for $g \in \pi_1(Y, y)$ and $H \leq \pi_1(Y, y)$, then there is $x_3 \in f^{-1}(y)$ such that $H = f_*\pi_1(X, x_3)$.*

The proof is left as an exercise. With the lemma in hand, we have the following classification of non-pointed covering spaces.

Theorem 2.32. *Say $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ be covering spaces and fix $x_i \in X_i$ basepoints lying over $y \in Y$. Then f_1 is equivalent to f_2 if and only if $(f_1)_*\pi_1(X_1, x_1) = g(f_2)_*\pi_1(X_2, x_2)g^{-1}$ for some $g \in \pi_1(Y, y)$.*

In other words, we have an embedding of categories

$$\frac{\{\text{covering spaces}\}}{\text{equivalence}} \longrightarrow \frac{\{\text{subgroups of } \pi_1(Y, y)\}}{\text{conjugation}}.$$

Proof. For the forward direction, let $h : X_1 \to X_2$ be an equivalence. Then $(f_1)_*\pi_1(X_1, x_1) = (f_2)_*\pi_1(X_2, h(x_1))$. The second group is conjugate to $(f_2)_*\pi_1(X_2, x_2)$ by the previous lemma.

For the reverse direction, assume that $(f_1)_*\pi_1(X_1, x_1)$ is conjugate to $(f_2)_*\pi_1(X_2, x_2)$. The previous lemma implies that $(f_1)_*\pi_1(X_1, x_1) = (f_2)_*\pi_1(X_2, x_3)$ for some $x_3 \in X$. The previous theorem says that there exists pointed equivalence $(X_1, x_1) \simeq (X_2, x_3)$, and forgetting basepoints gives a non-pointed equivalence. □

Example. If Y is simply connected, the above imply that Y has no nontrivial covering spaces.

Example. For $Y = S^1$ and a path-connected covering space $f : X \to Y$, the above imply that $(Y, f) = (S^1, z \mapsto z^n)$ or $(\mathbb{R}, t \mapsto \exp(t))$.

Theorem 2.33. *Let $f : X \to Y$ be a covering space with X simply connected and basepoints $x \in X$, $y = f(x) \in Y$. Then $\pi_1(Y, y)^{op} \simeq Aut_Y(X) := \{h : X \xrightarrow{\sim} X : fh = f\}$. The right-hand side we call the group of deck transformations of Y over X.*

Proof. Given $\psi \in \mathrm{Aut}_Y(X)$, define $\rho(\psi) \in \pi_1(Y, y)$ as follows: $\rho(\psi) = f_*(\beta : x \rightsquigarrow \psi(x))$ for some path β. Then $\rho(\psi) \in \pi_1(Y, y)$ as $f\psi(x) = f(x) = y$. $\rho(\psi)$ is well-defined, that is, independent of choice of β, as X is simply connected. We obtain a map of sets $\rho : \mathrm{Aut}_Y(X) \to \pi_1(Y, y)$.

We need to check that this is an isomorphism into the opposite group of $\pi_1(Y, y)$. First, note that $\rho(1_{Y/X}) = f_*(c_x) = c_y$. Now, to check the homomorphism property, choose paths $\beta_1 : x \rightsquigarrow \psi_1(x)$ and $\beta_2 : x \rightsquigarrow \psi_2(x)$. Then

$$\begin{aligned}\rho(\psi_2)\rho(\psi_1) &= f_*(\beta_2)f_*(\beta_1)\\ &= f_*(x \overset{\beta_1}{\rightsquigarrow} \psi_1(x) \overset{\psi_1(\beta_2)}{\rightsquigarrow} \psi_1\psi_2(x)) \qquad (f_*\psi_1(\beta_2) = f_*(\psi_1)_*(\beta_2) = f_*\beta_2)\\ &= \rho(\psi_1\psi_2).\end{aligned}$$

To show that ρ is injective, choose $\psi \in \mathrm{Aut}_Y(X)$ such that $\rho(\psi) = c_y$. It suffices to show that $\psi(x) = x$ by uniqueness of lifting. For any path $\beta : x \rightsquigarrow \psi(x)$, $\rho(\psi) = c_y$, so $\beta = c_x$ by uniqueness. It follows that $x = \psi(x)$. Surjectivity implies existence of lifts rather than uniqueness, and this argument is left as an exercise.

We may also construct an inverse, namely an isomorphism $\pi_1(Y, y)^{op} \to \mathrm{Aut}_Y(X)$, where $\alpha \mapsto$ the unique automorphism of X taking α to $\widetilde{\alpha}(1)$, where $\widetilde{\alpha}$ is the unique lift of α to X guaranteed by the "key lifting lemma". □

Remark. We have the following.

1. Bijections $\mathrm{Aut}_Y(X) \overset{\sim}{\to} \pi_1(Y, y)$ and $\pi_1(Y, y) \overset{\sim}{\to} f^{-1}(y)$ (via $\alpha \mapsto \widetilde{\alpha}(1)$). We obtain a bijection $\mathrm{Aut}_Y(X) \overset{\sim}{\to} f^{-1}(y)$ via $\psi \mapsto \psi(x)$.

2. Any X as in the above theorem is unique up to equivalence. Such a cover is called the *universal cover* of Y.

Example. $\exp : \mathbb{R} \to S^1$ is a universal cover. So is $\pi : S^n \to \mathbb{RP}^n$ for $n \geq 2$.

We use universal covers to construct all covering spaces:

Theorem 2.34. *Let $f : X \to Y$ be a universal cover, with basepoints $x \in X$, $y = f(x) \in Y$.*

1. *$Aut_Y(X)$ acts properly discontinuously on X, i.e. for each $x \in X$ there exists $U \ni x$ open, such that $\psi(U) \cap U = \emptyset$ for all $\psi \in Aut_Y(X)$.*

2. *If $H \leq Aut_Y(X)$ is any subgroup, then we obtain the map $\pi : X/H \to Y$, which is also a covering space, and $\pi_*\pi_1(X/H, \bar{x}) = H$, under the correspondence $Aut_Y(X) \cong \pi_1(Y, y)^{op}$.*

3. *Every connected covering space of X arises by the construction in (2).*

Corollary 2.35. *We obtain a correspondence*

$$\frac{\{\textit{pointed covering spaces}\}}{\textit{pointed equivalence}} \longleftrightarrow \{\textit{subgroups of } \pi_1(Y, y)\}.$$

Proof. In order:

1. We have a bijective correspondence $\mathrm{Aut}_Y(X) \simeq f^{-1}(y)$. Choose an open neighborhood V of y such that $f^{-1}(V) \cong \coprod_{x' \in f^{-1}(y)} U_{x'}$, where $U_{x'} \overset{\sim}{\to} V$ is a homeomorphism. We claim that U_x provides the desired neighborhood of x. Note now that $\psi(U_x) = U_{\psi(x)}$, and as the automorphism group acts simply transitively on the fiber of y, whenever $\psi \neq \mathrm{id}$, we have that $\psi(U_x) \cap U_x = \emptyset$.

2. We have $H \leq \mathrm{Aut}_Y(X)$. As the action of H on X is properly discontinuous, the quotient map $q : X \to X/H$ is, by a result from homework, a covering space. We also have the induced map $\pi : X/H \to Y$. Fix $y \in V \subset Y$ as in (1). We saw that $f^{-1}(V) = \coprod_{x' \in f^{-1}(y)} U_{x'}$. Thus $\pi^{-1}(V) = f^{-1}(V)/H \cong \coprod_{\bar{x}' \in f^{-1}(Y)/H} U_{x'}$. It follows that π is a covering space.

   Additionally we have

$$\begin{array}{ccc} \pi_*\pi_1(X,x) & \xrightarrow{\subseteq} & \pi_1(Y,y) \\ \downarrow{=} & & \\ \{\alpha \in \pi_1(Y,y) \mid \text{unique lift of } \alpha \text{ to } X/H \text{ based at } \bar{x} \text{ is a loop}\} & & \\ \downarrow{=} & & \\ \{\alpha \in \pi_1(Y,y) \mid \text{unique lift of } \alpha \text{ to } X \text{ based at } x \text{ ends at } H \cdot x\} & & \\ \downarrow{=} & & \\ H & & \end{array}$$

where the vertical equalities follow from identifying $\mathrm{Aut}_Y(X) \cong \pi_1(Y,y) \cong f^{-1}(y)$.

3. Say $g : Z \to Y$ is any connected covering space, $z \in g^{-1}(y)$. We want to show that $Z \cong X/H$ for some $H \leq \mathrm{Aut}_Y(X)$, where $\cong$ denotes covering space equivalence. Choose $H = g_*\pi_1(Z,z) \leq \pi_1(Y,y)$. Apply the lifting lemma to the triangle

$$\begin{array}{ccc} Z & & X/H \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & Y & \end{array}$$

in both directions to obtain $a : Z \to X/H$ and $b : X/H \to Z$, and we use the lifting lemma again to obtain that these are mutually inverse homeomorphisms.

□

Example. We may classify all covering spaces of $S^1 \times S^1$ in this way: they are all of the form $(\mathbb{R} \times \mathbb{R})/H$ for $H \leq \mathbb{Z}^{\oplus 2} \subset \mathbb{R} \times \mathbb{R}$. Compactness of the cover is dependent on the index of H in $\mathbb{Z}^{\oplus 2}$, as one can check.

2.4 Applying covering space theory

2.4.1 Existence of universal covers

The fundamental question will be: for a path-connected space Y, when does there exist a universal cover $Z \to Y$? We have the following necessary condition: for all $y \in Y$, there must exist $U \ni Y$ an open neighborhood such that $\pi_1(U,y) \to \pi_1(Y,y)$ is the zero map. This follows from the fact that when there exists a universal cover $f : Z \to Y$, then there exists an open neighborhood U of y in Y such that $f^{-1}(U) = \coprod_I U$ so that we have a factorization

$$\begin{array}{ccc} & & Z \\ & {\scriptstyle \exists}\nearrow & \downarrow{\scriptstyle f} \\ U & \hookrightarrow & Y. \end{array}$$

In this case we say that Y is *semi-locally simply connected.* We have, without proof:

Theorem 2.36. *If Y is semi-locally simply connected then it admits a universal cover.*

2.4.2 Galois theory and covering space theory

Here is a brief dictionary. Let k be a field, and L/k an extension. Let X be a topological space, and $Y \to X$ a covering space.

	Algebra	*Topology*
1.	the separable closure $\overline{k}/k$	the universal cover $\widetilde{X} \to X$
2.	finite extensions of k	connected covering spaces over X
3.	$\mathrm{Gal}(L/k)$	$\mathrm{Aut}_Y(X)$
4.	open subgroups of $\mathrm{Gal}(L/k)$	subgroups of $\pi_1(Y, y)$
5.	correspondence between 2. and 4.	(same)
6.	Galois extensions L/k	regular/Galois covering spaces
7.	each finite, separable L/k embeds into the algebraic closure $\overline{k}$	all covering spaces are quotients of universal cover$\widetilde{X}$

2.4.3 Subgroups of free groups are free

We prove that any subgroup of a free group is itself free; however we note as a warning that the rank is not bounded: F_2 contains free groups of arbitrarily high rank as subgroups.

We realize F_n as the fundamental group of the wedge $\bigvee_{i=1}^n S^1$. We "break down" this wedge as a graph. We will largely play it fast and loose with the terminology in this section, assuming some background or intuition for simple graphs.

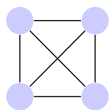
Definition 2.37. A *graph* G is a triple (V, E, ϕ) where V and E are sets, and $\phi : E \to \{\text{subsets of } V \text{ with 1 or 2 elements}\}$. V is the *vertex set*, E the *edge set*, and $\phi(e)$ is the *set of endpoints* of e.

Construction. For a graph $G = (V, E, \phi)$, we obtain a *geometric realization* of the graph, $|G|$, the top space obtained from V by attaching edges for all $e \in E$. That is,

$$|G| = \frac{V \sqcup (\coprod_{e \in E} I_e)}{0 \in I_e \sim x \in V \quad 1 \in I_e \sim y \in V}$$

where $I_e \cong I = [0, 1]$, and $\phi(e) = \{x, y\}$.

Example. Consider the graph $|G|$ given by



Definition 2.38. A graph is a *tree* if it is connected and (any two vertices are linked by a sequence of edges) has no cycles. (We haven't defined a cycle, but we mean the intuitive thing.)

Examples abound. We establish a few lemmas.

Lemma 2.39. *Every connected graph G contains a maximal tree T, in the sense that no larger tree properly contains T as a subgraph. Moreover, every vertex of G lies in T.*

Proof. We will assume that G has finite vertex and edge sets. In particular, this means that $|G|$ is compact as a topological space (exercise). In this case G has only finitely many subgraphs, so the existence of a maximal tree T is clear. However we need to show that T contains each vertex.

To do so, suppose not. First we note that T is nonempty, as then any tree defined by a single vertex would be supermaximal. If T does not contain every vertex. If T does not contain a vertex u, then there is an edge between a vertex in T and u, as G is connected. Extending T via this edge gives a supermaximal tree, a contradiction.

In the infinite case, one can apply Zorn's lemma for a similar result. □

Lemma 2.40. *If G is a graph and T is a maximal subtree, then $|G| \to |G|/|T|$, the retraction of $|T|$ in $|G|$ to a single vertex, is a homotopy equivalence.*

We will not prove this. One can find a proof in Hatcher's book *Algebraic Topology*.

Lemma 2.41. *For G a connected graph and $T \subset G$ a maximal subtree, then $|G|/|T| \cong \bigvee_{e\in J} S^1$, where $\cong$ is a homeomorphism, and J is the set of edges not contained in T.*

We haven't proved a lot in this section and we won't start now.

Lemma 2.42. *For G a connected graph with a covering space $f : X \to |G|$, there exists a graph G' such that $X \cong |G'|$.*

Proof. We set

$$V' := f^{-1}(V(G)) \qquad E' := \{\text{lifts of paths given by edges in } |G|\} \qquad \phi' = \text{obvious}.$$

We let $G' = (V', E', \phi')$, and leave it as an exercise to check that $X \cong |G'|$. □

And now:

Corollary 2.43. *Subgroups of free groups are free.*

Proof. We restrict ourselves to the finite case. Say F_n is the free group on n letters, and $H \leq F_n$ is a subgroup. We know that $F_n \cong \pi_1(\bigvee^n S^1) \cong |G|$, where G has a single vertex and n edges. Covering space theory implies that H is realized as $\pi_1(X)$ for some covering space $X \to |G|$. By the lemmas above, X is itself realized as a geometric realization of a graph G'. Contracting a maximal subtree in G' and applying an above lemma, we see that $\pi_1(G')$ is free, and the result follows. □

3 Homology

We still have basic questions in topology that we are not equipped to answer with covering spaces and fundamental groups. Such as:

Question: How to show $S^n \not\cong S^{n+1}$ for n large? We may use π_1 up to $n = 2$, but then we get stuck.

We aim to construct generalizations of π_1. We could construct π_n, i.e. homotopy classes of (pointed) maps $S^n \to X$ for $n \geq 0$, however these are famously difficult to compute. It turns out that some funny things happen. For example, it is true that $\pi_3(S^2) = \mathbb{Z}/2$. Instead, we will introduce the homology groups H_n. For an initial observation, we note that π_1^{ab} is easier to compute, in general, than π_1: this is how we will construct homology groups. (Warning: $H_n \neq \pi_n^{ab}$ in general.)

3.1 Introduction to homological algebra

The setting is the following: **Ab** will denote the category of abelian groups, $\mathbf{Vect}_k$ will denote the category of k-vector spaces, and $\mathbf{Mod}_R$ will denote the category of R-modules.

Definition 3.1. Say we have maps $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ in **Ab** (so this will work in $\mathbf{Mod}_R$ as well). We say that this is a *sequence* when $\beta\alpha = 0$, i.e. image$(\alpha) \subset \ker(\beta)$.

Example. Consider $\mathbb{Z}/4 \to \mathbb{Z}/4 \to \mathbb{Z}/4$ where both maps are multiplication by 2.

Definition 3.2. We say a sequence is *exact* when image$(\alpha) = \ker(\beta)$.

Example. The above example is exact.

Example. If $A = 0$, then the sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is exact if and only if β is surjective, and if $C = 0$, if and only if α is surjective.

Definition 3.3. A set of maps

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n+1}} A_n \xrightarrow{f_{n+2}} A_{n+1}$$

is a *sequence* provided that the composition of any two adjacent maps is zero, and is *exact at the mth entry* provided that $\text{image}(f_{m-1}) = \ker(f_m)$.

Definition 3.4. A *short exact sequence* is a sequence

$$0 \to A \to B \to C \to 0$$

which is exact at each entry.

Remark. In this case, we have that the first (nonzero) map is injective, and the second is surjective.

Example. The doubly-composed multiplication by 2 map in $\mathbb{Z}/4$ is *not* a short exact sequence.

Example. $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/p \to \mathbb{Z}/p \to 0$ with the obvious maps is exact. This is an example of a *split exact sequence*, where the analogue of B decomposes as the direct sum $A \oplus C$.

Definition 3.5. A *chain complex* $K_\bullet$ is a sequence of the following sort

$$\cdots \xrightarrow{d_i} K_{i+1} \xrightarrow{d_{i+1}} K_i \to K_{i-1} \to \cdots$$

in $\mathbf{Ab}$, where K_i is called the *degree i term*, and the d_i are called the *differentials* of $K_\bullet$.

Remark. By abuse of notation, we will often denote each d_i by d. Now a chain complex is defined by the equation $d^2 = 0$. Unwritten entries will always be 0, for example:

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2$$

is a chain complex with zeros populating the left- and right-hand sides.

Definition 3.6. A *morphism of chain complexes* $K_\bullet \to L_\bullet$ consists of morphisms $p_i : K_i \to L_i$ for each i such that the diagram obtained from $K_\bullet, L_\bullet$, and the p_i commutes.

We obtain then a category $\mathbf{Ch}(\mathbf{Ab})$ of chain complexes.

Example. Given $A \in \mathbf{Ab}$, obtain a chain complex $A[0] \in \mathbf{Ch}(\mathbf{Ab})$ with A in the degree 0 term.

Example. In the same way, for a map $A \to B \in \mathbf{Ab}$, we obtain a chain complex $A \to B \in \mathbf{Ch}(\mathbf{Ab})$.

Example. Consider the sequence

$$\cdots \to \mathbb{Z}/4 \to \mathbb{Z}/4 \to \mathbb{Z}/4 \to \cdots$$

with each map multiplication by 2. This is a chain complex.

Example. Say $M_1 \in \text{Mat}_{m,n}(\mathbb{Z}), M_2 \in \text{Mat}_{m,\ell}(\mathbb{Z})$ such that $M_1 \cdot M_2 = 0$. We obtain a chain complex

$$\mathbb{Z}^{\oplus \ell} \xrightarrow{M_2} \mathbb{Z}^{\oplus m} \xrightarrow{M_1} \mathbb{Z}^{\oplus n}.$$

3.1.1 Homology of chain complexes

We aim to measure the "failure of complexes to be exact".

Definition 3.7. For $K_\bullet \in \mathbf{Ch}(\mathbf{Ab})$, $i \in \mathbb{Z}$. We define $Z_i(K_\bullet) = \ker(d : K_i \to K_{i-1})$, the *$i$-cycles of $K_\bullet$*, and $B_i = \text{image}(d : K_{i+1} \to K_i)$, the *$i$-boundaries of $K_\bullet$*. Then in general we have $B_i \subset Z_i$. We define $H_i(K_\bullet) = Z_i(K_\bullet)/B_i(K_\bullet)$, the *ith homology of K*.

Remark. $H_i(K_\bullet) = 0$ if and only if $K_\bullet$ is exact in degree i.

Example. Consider $K = A \xrightarrow{\alpha} \underbrace{B}_{\text{deg } 0} \in \mathbf{Ch}(\mathbf{Ab})$. Then $H_0(K) = \text{coker}(\alpha) = B/\text{image}(\alpha)$, and $H_1(K) = \ker(\alpha)$.

Example. Consider

$$K = \cdots \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\text{pr}} \mathbb{Z}/2 \to 0 \to \cdots .$$

Then $H_0(K) = H_1(K) = H_2(K) = 0$. All other homologies are also trivially zero.

Definition 3.8. We say that K is *acyclic* or *exact* when all of its homology groups are 0.

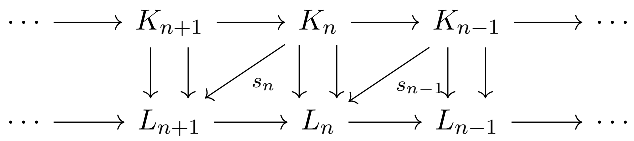
Example. Consider the multiplication-by-2 complex $0 \to \mathbb{Z}/4 \to \mathbb{Z}/4 \to \mathbb{Z}/4 \to 0$. $H_0 = \mathbb{Z}/2$, $H_1 = 0$, and $H_2 = \mathbb{Z}/2$.

Remark. Say $f : \mathbf{Ab} \to \mathbf{Ab}$ is an *additive functor*, i.e. $\text{Hom}(X, Y) \to \text{Hom}(f(X), f(Y))$ is a homomorphism. This induces a functor $F := \mathbf{Ch}(f) : \mathbf{Ch}(\mathbf{Ab}) \to \mathbf{Ch}(\mathbf{Ab})$.

Warning: F does not commute with taking homology.

Example. Say $f(A) = A/2A$. Then $K = 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\text{pr}} \mathbb{Z}/2 \to 0$ becomes $0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0$ under f; $H_2(K) = 0$ however $H_2(f(K)) = \mathbb{Z}/2$.

Definition 3.9. Given morphisms $f, g : K \to L$ of chain complexes, a *homotopy* $h : f \simeq g$ is given by maps $s_n : K_n \to L_{n+1}$ such that $f_n - g_n = ds_n + s_{n-1}d$.



Definition 3.10. A morphism $f : K \to L$ is called *nullhomotopic* provided that it is homotopic to 0.

Definition 3.11. A morphism $f : K \to L$ is a *homotopy equivalence* provided that there exists $g : L \to K$ such that $g \circ f \simeq \text{id}_K$ and $f \circ g \simeq \text{id}_L$.

Lemma 3.12. *If $f, g : K \to L$ are homotopic, then for all $i, \in \mathbb{Z}$, $H_i(f) = H_i(g)$ (as maps of abelian groups $H_i(K) \to H_i(L)$).*

Proof. Pick $(s_n : K_n \to L_{n+1})_n$ such that $f_n - g_n = ds_n - s_{n-1}d$, and a cycle $\alpha \in Z_i(K)$. Then $d\alpha = 0$. Then $f_i(\alpha) = g_i(\alpha) + ds_i(\alpha) + \underbrace{s_{i-1}d(\alpha)}_{=0}$. It follows that $f_i(\alpha) - g_i(\alpha) = ds_i(\alpha)$, so that $[f_i(\alpha)] = [g_i(\alpha)]$ in homology. □

Corollary 3.13. *$f : K \to L$ is nullhomotopic implies that $H_i(f) = 0$ for all i. Also, when $f : K \to L$ is a homotopy equivalence, the $H_i(f)$ give isomorphisms.*

The proof of the second statement follows after applying the lemma, so that $H_i(fg) = H_i(f)H_i(g) = H_i(\text{id})$, and symmetrically.

Example. Consider $K = (\mathbb{Z} \xrightarrow{1} \mathbb{Z})$; then $\mathrm{id}_K \simeq 0$. (Note: if we extend K via 0 on the left and right, it is *not* nullhomotopic.)

Example. Consider $K = (\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\mathrm{pr}_2} \mathbb{Z})$, $L = (\mathbb{Z} \to 0)$, with the map $(f : K \to L) : \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\mathrm{pr}_1} \mathbb{Z}$, $\mathbb{Z} \to 0$. We claim that f is a homotopy equivalence.

We specify maps $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ by inclusion into the first factor, and $0 \to \mathbb{Z}$ by the only possible map. We use $s : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ defined by $m \mapsto (0, -m)$. Then $(gf - \mathrm{id})(a, b) = (0, -b)$, and $ds(a, b) = (0, -b)$.

Remark. The converse to the corollary is false: when $H_i(f) = 0$, f is not necessarily nullhomotopic. However when we consider $\mathbf{Ch}(\mathbf{Vect}_k)$, it becomes true.

Theorem 3.14. *Let*

$$0 \to M \xrightarrow{\alpha} K \xrightarrow{\beta} L \to 0$$

be a SES of chain complexes. Then there exist canonical maps $\delta : H_i(L) \to H_{i-1}(M)$ such that

$$\begin{array}{ccccc} H_i(M) & \longrightarrow & H_i(K) & \longrightarrow & H_i(L) \\ & & & & \big\downarrow \delta \\ H_{i-1}(M) & \longrightarrow & H_{i-1}(K) & \longrightarrow & H_{i-1}(L) \end{array}$$

$$H_i(L) \xrightarrow{\delta} H_{i-1}(M)$$

is exact.

***Warning**: the following proof needs to be corrected. Do not read it.*

Proof. First we check exactness at $H_i(K)$. The inclusion image $\subset$ ker follows from functoriality of H_i. Take $x \in Z_i(K)$ such that the image of x in $H_i(K)$ maps to zero in $H_i(L)$. Then $\beta(x) \in B_i(L)$, so we may write $\beta_i(x) = dy$ for some $y \in L_{i+1}$. Choose $\hat{x} \in K_{i+1}$ such that $\beta(\hat{x}) = y$, and consider $\beta_i(x - d\hat{x}) = \beta_i(x) - d\beta_i(\hat{x}) = dy - dy = 0$. Thus by exactness in the given sequence, $x - d\hat{x} \in \mathrm{image}(\alpha_i)$, so write $x - d\hat{x} = \alpha_i(z)$ for some $z \in M_i$. Then we have $[x - d\hat{x}] = [x] = H_i(\alpha)([z])$ in homology, as required.

Now we define δ. For $[y] \in H_i(L)$, we may choose $y \in L_i$ to represent $[y]$, with $d_L(y) = 0$. We refer to the following diagram throughout:

$$\begin{array}{ccccccccc} & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_i & \xrightarrow{\alpha_i} & K_i & \xrightarrow{\beta_i} & L_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_M} & & \downarrow{\scriptstyle d_K} & & \downarrow{\scriptstyle d_L} & & \\ 0 & \longrightarrow & M_{i-1} & \xrightarrow{\alpha_{i-1}} & K_{i-1} & \xrightarrow{\beta_{i-1}} & L_{i-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \end{array}$$

Since β_i is surjective, there is $x \in K_i$ such that $\beta_i x = y$; we would like to show that $d_K x \in \mathrm{image}(\alpha_{i-1})$ so that we could pull it back along α_{i-1}. By exactness of the rows it suffices to show that $\beta_{i-1}(d_K x) = 0$. This follows from commutativity of the right-inner square in the diagram, after noting that $d_L \beta_i x = d_L y = 0$. Now choose $z \in M_{i-1}$ such that $\alpha_{i-1} z = d_K x$. We would like to prove that the assignment $[y] \mapsto [z]$ suffices to define δ.

First, we check that $d_M z = 0$, i.e. that z determines an element of $H_{i-1}(M)$. For this, it suffices to show that $d_K \alpha_{i-1} z = 0$, as α is injective. This is $d_K \alpha_{i-1} z = d_K^2 x = 0$, as required (recall the choices of x, y, z in the above paragraph).

Second, we check that $[z]$ is independent of the choices of x and y. Say we have $x, x' \in K_i$ such that $\beta_i x = \beta_i x' = y$. Then $\beta_i(x - x') = 0$, so $x - x' \in \ker(\beta_i) = \mathrm{image}(\alpha_i)$, so there is $u \in M_i$ such that $x - x' = \alpha_i u$. We have $d_K x - d_K x' = d_K(\alpha_i(u)) = d_M u \in B_{i-1}(M)$. We obtain z via the equation $\alpha_{i-1} z = d_K x$, and similarly a z' via $\alpha_{i-1} z' = d_K x'$, so since α

is injective we have $\alpha_{i-1}(z-z') = \alpha_{i-1}d_M u$, so that $[z] = [z']$ in $H_{i-1}(M)$. Now if there are $[y] = [y'] \in H_i(L)$, then $y - y' = d_L v$ for some $v \in L_{i+1}$. We write $y = \beta_i x$ and $y' = \beta_i x'$, as β is surjective, so $y - y' = \beta(x - x')$; we write $v = \beta_{i+1} w$ for some $w \in K_{i+1}$. Then $\beta_i d_K w = d_L \beta_{i+1} w = d_L v = y - y'$. Thus $x - x' - d_K w \in \ker(\beta) = \text{image}(\alpha)$. Choose a lift $s \in M_i$ for $x - x' - d_K w$

We leave it as an exercise to check that δ is a homomorphism of abelian groups.

Now we check exactness at L_i. To see that $\text{image}(H_i\beta) \subset \ker\delta$, i.e. $\delta H_i(\beta[x]) = 0 \in H_{i-1}(M)$. We note that $\delta[\beta x] = z$ for some z such that $\alpha z = d_K x$. ...

For the reverse inclusion, say $\delta([y]) = [0] \in H_{i-1}(M)$. We want to show that $[y] \in \text{image}H_i(\beta)$. We have $x \in K_i$ such that $\beta x = y$, and $u \in M_i$ such that $d_K x = \alpha d_M u$. We naively claim that $H_i(\beta)([x]) = [y]$; this is always true when $[x]$ is actually well-defined, so we proceed in checking this. However we have $d_K x = \alpha d_M u \neq 0$, so we modify x. We have $d_K x = \alpha d_M u = d_K \alpha u$, so $d_K(x - \alpha u) = 0$. The correct claim, thus, is that $H_i(\beta)([x - \alpha u]) = [y]$. This is well-defined by construction, and we have $[\beta(x - \alpha u)] = [\beta(x)] - [\beta\alpha u] = [y]$, as $\beta\alpha = 0$.

We leave the rest of the exactness checks as exercises as well, to close the proof. ☐

Example. If we have the SES $0 \to A \to B \to C \to 0$ of abelian groups, we consider the induced SES of chain complexes

$$0 \to (A \xrightarrow{\cdot 2} A) \to (B \xrightarrow{\cdot 2} B) \to (C \xrightarrow{\cdot 2} C) \to 0.$$

We let $X[2] = \{x \in X \mid 2x = 0\}$ be the 2-torsion part of X. Then we obtain the LES

$$\begin{array}{l} 0 \longrightarrow A[2] \longrightarrow B[2] \longrightarrow C[2] \\ \hookrightarrow A/2A \longrightarrow B/2B \longrightarrow C/2C \longrightarrow 0. \end{array}$$

When we let $A = B = C = \mathbb{Z}$, we obtain

$$0 \to 0 \to 0 \to \mathbb{Z}/2 \xrightarrow{\delta = \text{id}} \mathbb{Z}/2 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0$$

and you may check that indeed $\delta = \text{id}$ here.

Corollary 3.15 (Snake Lemma)**.** *When we have two SESs and maps between them in the following arrangement:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

then we have the associated long exact sequence

$$\begin{array}{l} 0 \longrightarrow \ker(f) \longrightarrow \ker(g) \longrightarrow \ker(h) \\ \hookrightarrow \mathit{coker}(f) \longrightarrow \mathit{coker}(g) \longrightarrow \mathit{coker}(h) \longrightarrow 0. \end{array}$$

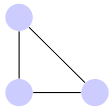
3.2 Simplicial Homology

Definition 3.16. An *n-simplex in* $\mathbb{R}^m$ is the convex hull of $n+1$ points $\{v_0, ..., v_n\}$ not all lying in a hyperplane. Equivalently, and sometimes preferably, $\{v_1 - v_0, ..., v_n - v_0\}$ is a linearly independent set.

Definition 3.17. An n-simplex is *oriented* when an ordering on the v_i has been specified.

Example. Here is a 1-simplex in $\mathbb{R}^2$:

And here is a 2-simplex in $\mathbb{R}^2$ (which we think of as being filled in):



A 0-simplex is a point.

Example. The *standard n-simplex in $\mathbb{R}^{n+1}$* is $\Delta^n = \{t_0, ..., t_n \in \mathbb{R}^{n+1} \mid \sum t_i = 1,\ 0 \le t \le 1\}$.

Remark. We write $[v_0, ..., v_n]$ for the (oriented) n-simplex given by the convex hull of $v_0, ..., v_n$ not lying in a hyperplane. In this case there is a canonical affine homeomorphism $\Delta^n \cong [v_0, ..., v_n]$ given by $(t_i) \mapsto \sum t_i v_i$ (alternatively, $e_i \mapsto v_i$). In what follows, we will canonically identify all oriented n-simplices with Δ^n.

Definition 3.18. A *face* of an oriented n-simplex $[v_0, ..., v_n]$ is a subsimplex $[w_0, ..., w_k]$ for some subset $\{w_0, ..., w_k\} \subset \{v_0, ..., v_n\}$.

Example. $[v_i]$ is a face of $[v_0, ..., v_n]$. So is $[v_i, v_j]$, for $i \neq j$. These are the *vertices* and the *edges*, respectively, of $[v_0, ..., v_n]$. $[v_0, ..., v_n]$ is a face of itself. We will not require that the empty set is a face.

Remark. Some notation: if $[v_0, ..., v_n]$ is an n-simplex, and $0 \le i \le n$, then $[v_0, ..., \hat{v}_i, ..., v_n]$ is the $(n-1)$-simplex obtained by removing v_i.

For Δ^n, we write $\partial_i \Delta^n = [e_1, ..., \hat{e}_i, ..., e_{n+1}]$, where e_j is the jth basis vector in $\mathbb{R}^{n+1}$.

3.2.1 Δ-complexes

We want to creates spaces by gluing simplices together along their faces; these are called Δ-complexes.

Remark. Our convention for defining the *interior* of a simplex will be $\mathrm{Int}(\Delta^n) = \Delta^n \smallsetminus \bigcup_i \partial_i \Delta^n$.

Definition 3.19. A *Δ-complex structure* on a topological space X is a collection A of maps $\{\sigma_\alpha^n : \Delta^n \to X\}_{\alpha \in A}$ satisfying the following conditions:

1. $\sigma_\alpha^n|_{\mathrm{Int}(\Delta^n)}$ is a homeomorphism onto its image, and each $x \in X$ lies in the image of exactly one such map.

2. $U \subset X$ is open if and only if $(\sigma_\alpha^n)^{-1}(U) \subset \Delta^n$ is open, for all α.

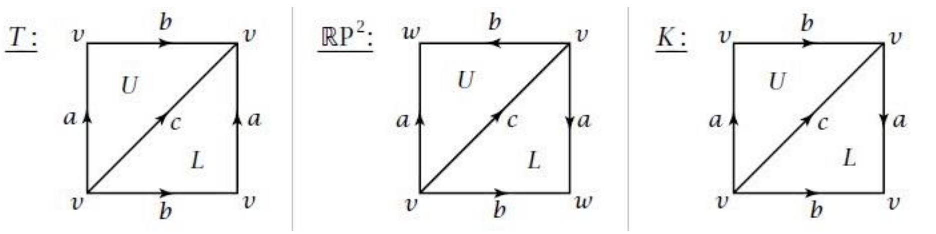
3. If $\Delta^m \subset \Delta^n$ is a face, then $\sigma_\alpha^n|_{\Delta^m} \in A$.

That is, X has the quotient topology induced by the map $\coprod_{\alpha \in A} \Delta^n \to X$ induced by the σ_α^n.

Example. Here are Δ-complex structures on the 2-torus $S^1 \times S^1$, $\mathbb{RP}^2$, and the Klein bottle K, all given by standard edge identifications of a unit square.

The specification of the 0-, 1-, and 2-simplices is left as an exercise. (Image stolen from math.se.)

Let A be a Δ-complex structure on X.



Definition 3.20. $C_\bullet^\Delta(X) \in \mathbf{Ch}(\mathbf{Ab})$ is the simplicial chain complex associated to (X, A), defined to be the free abelian group on n-simplices in A, with differential maps given by

$$d : C_\bullet^{\Delta^n}(X) \longrightarrow C_\bullet^{\Delta^{n-1}}(X)$$
$$(\sigma_\alpha^n : \Delta^n \to X) \longmapsto \sum_{i=0}^{n} (-1)^i \sigma_\alpha^n|_{\partial_i \Delta^n}.$$

Example. If $\Delta^2 = [v_0, v_1, v_2] \xrightarrow{\sigma_\alpha^2} X$ then $d(\sigma_\alpha^n) = \sigma_\alpha^2|_{[v_1,v_2]} - \sigma_\alpha^2|_{[v_0,v_2]} + \sigma_\alpha^2|_{[v_0,v_2]}$.

Definition 3.21. We define $H_i^\Delta(X) := H_i(C_\bullet^\Delta(X))$.

Lemma 3.22. *We need to check that $C_\bullet^\Delta(X)$ is a complex, i.e. that $d^2 = 0$.*

Proof. Let $A = \{\sigma_\alpha^n : \Delta^n \to X\}$ be a Δ-complex structure on X.

$$\begin{aligned} d(d(\sigma_\alpha^n)) &= d\left(\sum_{i=0}^{n} (-1)^i \sigma_\alpha^n|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}\right) \\ &= \sum_{i=0}^{n} (-1)^i d(\sigma_\alpha^n|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}) \\ &= \sum_{i=0}^{n} (-1)^i \left(\sum_{j=0}^{i-1} \sigma_\alpha^n|_{[v_0,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n]} + \sum_{j=i+1}^{n} (-1)^j(-1)\sigma_\alpha^n|_{[v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]}\right) \\ &= 0 \end{aligned}$$

where the final equality comes after cancelling like terms. □

Example. Consider $X = S^1$. We give X a Δ-complex structure by specifying a point $v = * \in X$ (a 0-simplex) and $a = X \setminus \{*\}$ (a 1-simplex). Then

$$C_\bullet^\Delta(X) = \mathbb{Z} \cdot a \xrightarrow{d} \mathbb{Z} \cdot v$$
$$a \mapsto v - v = 0.$$

Thus we have

$$H_i^\Delta(X) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} & i = 1 \\ 0 & \text{else.} \end{cases}$$

Example. Consider $Y = S^1 \times S^1$. We recall the Δ-complex structure on Y given above. We label the 2-simplices by T_1 and T_2; we have

$$\begin{aligned} C_\bullet^\Delta &= \mathbb{Z} \cdot T_1 \oplus \mathbb{Z} \cdot T_2 \to \mathbb{Z} \cdot a \oplus \mathbb{Z} \cdot b \oplus \mathbb{Z} \cdot c \to \mathbb{Z} \cdot v \\ d(T_1) &= b - c + a \qquad d(T_2) = a - c + b \\ d(a) &= d(b) = d(c) = v - v = 0 \\ d(v) &= 0. \end{aligned}$$

Forgetting generators, we see that we have

$$C_\bullet^\Delta = \mathbb{Z}^{\oplus 2} \to \mathbb{Z}^{\oplus 3} \to \mathbb{Z}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} \quad 0$$

Thus

$$H_i^\Delta(Y) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^{\oplus 2} & i = 1 \\ \mathbb{Z} \cdot (T_1 - T_2) & \end{cases}$$

where the second entry follows from the following fact: $(1, 1, -1)^T \in \mathbb{Z}^{\oplus 3}$ is *unimodular*, i.e. the greatest common denominator of its entries is 1. It follows that $\mathbb{Z}^{\oplus 3}/(\mathbb{Z} \cdot (1, 1, -1)^T)$ is a torsion-free abelian group, and comparing ranks we see that it must be $\mathbb{Z}^{\oplus 2}$.

3.3 Singular Homology

Simplicial homology requires choices, and it was not clear that it is a functor. As this is unsatisfactory, we want a better program. Thus we introduce simplicial homology, with the goal of making it independent of all choices. To do so, we will "quantify over all possible choices."

Definition 3.23. For X a topological space, we define

1. $C_n(X)$ = *n-chains* on X = the free abelian group on all maps $\sigma^n : \Delta^n \to X$, with $d : C_n(X) \to C_{n-1}(X)$ defined by the same equation as above:

$$d(\sigma^n) = \sum_{i=0}^{n} (-1)^i \sigma^n|_{\partial_i \Delta^n}$$

   Note that we still have $d^2 = 0$.

2. $B_n(X) = \text{image}(d : C_{n+1}(X) \to C_n(X))$ = *n-boundaries* on X
3. $Z_n(X) = \ker(d : C_n(X) \to C_{n-1}(X))$ = *n-cycles* on X
4. $H_n(X) = Z_n(X)/B_n(X)$ = nth *singular homology* of X.

Example. Consider a one-point space $X = \{*\}$. Then $C_n(X)$ is the free abelian group on a 1-element set, so $C_n(X) = \mathbb{Z} \cdot \sigma_n^i$ with

$$C_\bullet(X) = (\cdots \to \mathbb{Z}\sigma_2 \to \mathbb{Z}\sigma_1 \to \mathbb{Z}\sigma_0 \to \cdots)$$

where

$$d(\sigma_n) = \begin{cases} 1 & n \text{ even} \\ 0 & \text{else.} \end{cases}$$

Thus we see that we have

$$C_\bullet(X) = (\cdots \to \mathbb{Z}\sigma_2 \xrightarrow{0} \mathbb{Z}\sigma_1 \xrightarrow{1} \mathbb{Z}\sigma_0 \xrightarrow{0} \cdots)$$

whereafter computing the singular homology is easy.

Remark. We may choose the coefficients in any abelian group A (where above it was done with $\mathbb{Z}$); we write $C_*(X; A)$ for the *singular chains with coefficients in A*. In this setting we have $C_n(X; A) = \bigoplus_{\sigma:\Delta^n \to X} A \cdot \sigma$.

Remark. $C_*(X)$ and $H_*(X)$ only depend on the homeomorphism type of A.

Remark. H_* and C_* are functorial, in that if $f : X \to Y$ is a continuous map of spaces, composing with f gives maps $f_* : C_*(X) \to C_*(Y)$ a map of chain complexes and $H_i(f_*) : H_i(X) \to H_i(Y)$. Checking the functoriality conditions is left as an exercise.

We obtain functors

$$\begin{array}{ccccc} \mathbf{Top} & \xrightarrow{C_*(-)} & \mathbf{Ch} & \xrightarrow{H_i(-)} & \mathbf{Ab} \\ \uparrow & & & & \uparrow {\scriptstyle H_i(-)} \\ \Delta - \mathbf{Top} & & \longrightarrow & & \mathbf{Ch} \end{array}$$

where the dotted arrows represent functors which are as-of-yet undefined ($\Delta - \mathbf{Top}$ represents the category of topological spaces with Δ-complex structures).

Remark. If $X = \coprod_i X_i$, then $C_*(X) = \bigoplus_i C_*(X_i)$. As Δ^n is connected, we obtain

$$\{n\text{-simplices } \Delta^n \to X\} = \coprod_i \{n\text{-simplices } \Delta^n \to X_i\}$$

Proposition 3.24. *For any locally path-connected space X, we have $H_0(X) = \bigoplus_{\pi_0(X)} \mathbb{Z}$ (where π_0 denotes the set of path-connected components).*

Note that applying the final remark above, it is enough to show that path-connectedness of X implies that $H_0(X) = \mathbb{Z}$.

Proof. We define the map

$$C_0(X) = \bigoplus_{x \in X} \mathbb{Z} \cdot x \xrightarrow{\phi} \mathbb{Z} \qquad \sum_{x \in X} a_x \cdot x \mapsto \sum_{x \in X} a_x \in \mathbb{Z}$$

as we work with direct sums, the summations written are all finite. We claim that ϕ induces an isomorphism $H_0(X) := C_0(X)/dC_1(X) \xrightarrow{\sim} \mathbb{Z}$.

First we check that the image of the boundaries lies in the kernel of ϕ. Given $\sigma : \Delta^1 \to X \in C_1(X)$ (corresponding to basis elements $0, 1 \in C_0(X)$), $\phi(d\sigma) = \phi(\sigma(0) - \sigma(1)) = 1 - 1 = 0$. Thus we denote the induced map on the quotient by $\bar{\phi}$.

Second we show that $\bar{\phi}$ is surjective. Pick $x \in X$; then $\bar{\phi}(x \in C_0(X)) = 1$. Third we check injectivity. For $\sum_i a_i x_i \in C_0(X)$ such that $\sum_i a_i = 0$, we need to prove that $\sum_i a_i x_i = d(\tau)$ for some $\tau \in C_1(X)$. Choose $x_0 \in X$ and paths $\sigma_i : x_i \rightsquigarrow x_0$. We obtain a 1-simplex $\tau := \sum_i a_i \sigma_i \in C_1(X)$. We see that $d\tau = \sum_i a_i d\sigma_i = \sum_i a_i(x_i - x_0) = \sum_i a_i x_i - (\sum_i a_i)x_0 = \sum_i a_i x_i$, as required. ∎

Remark. The proof of the proposition gives a chain complex

$$\widetilde{C_*(X)} = (\cdots \to C_2(X) \to C_1(X) \to C_0(X) \xrightarrow{\phi} \mathbb{Z})$$

where $\mathbb{Z}$ sits in the degree -1 position. We define $\widetilde{H}_i(X) := H_i(\widetilde{C_*(X)})$, the *reduced homology of* X. You may check that when X is nonempty,

$$\widetilde{H}_i(X) = \begin{cases} H_i(X) & i \geq 1 \\ \ker(\bar{\phi} : H_0(X) \to \mathbb{Z}) & i = 0 \\ 0 & \text{else.} \end{cases}$$

therefore, when X is path-connected,

$$\widetilde{H}_i(X) = \begin{cases} H_i(X) & i \geq 1 \\ 0 & i = 0. \end{cases}$$

It follows that $\widetilde{H}_i(\{\text{pt}\}) = 0$ for all i.

Remark. If $\alpha : I \to X$ is a loop based at $x \in X$, then $\alpha : \Delta^1 \to X$ is a cycle (as $d\alpha = \alpha(0) - \alpha(1) = 0$). We obtain a map

$$\{\text{loops based at } x\} \longrightarrow \{H_1(X)\}$$
$$\alpha \longmapsto \overline{\alpha}$$

Further, we will prove a theorem (due to Hurewicz) that this map induces an isomorphism $\pi_1(X)^{ab} \xrightarrow{\sim} H_1(X)$ for X path-connected.

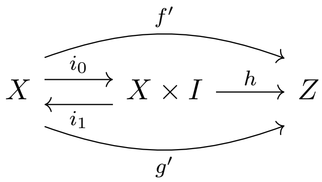
Theorem 3.25 (Homotopy invariance). *If $f, g : X \to Y$ are homotopic maps, then $f_*, g_* : C_*(X) \to C_*(Y)$ are homotopic (as maps of chain complexes). It follows that the induced maps $H_i(f_*)$ and $H_i(g_*)$ on homology are the same, via 3.12.*

Corollary 3.26. *It follows that if X is contractible then*

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \textit{else.} \end{cases}$$

We will use the fact that homotopic chain maps $a, b : K \to L$ induce homotopic maps $ca \simeq cb$ when $c : L \to M$ is a chain map.

Proof. First, we reduce to the case where $Y = X \times I$ and $f = i_0 : X \to X \times I$ is the map $x \mapsto (x, 0)$ and $g = i_1 : X \to X \times I$ is the map $x \mapsto (x, 1)$. To deduce the general case from this, given homotopic $f', g' : X \to Z$, let $h : X \times I \to Z$ be a homotopy realizing this equivalence. We obtain that $h(x, 0) = f'(x)$ and $h(x, 1) = g'(x)$. The maps in question are



So given a homotopy $(i_0)_* \simeq (i_1)_*$ we obtain a homotopy $f'_* \simeq h_*(i_0)_* \simeq h_*(i_1)_* = g'_*$.

Second, we will construct a homotopy $h : (i_0)_* \simeq (i_1)_*$ functorially in X. That is we construct maps $h_n : C_n(X) \to C_{n+1}(X \times I)$ such that $dh + hd = (i_0)_* - (i_1)_*$. It suffices to check that $dh(\sigma) + hd(\sigma) = (i_0)_*(\sigma) - (i_1)_*(\sigma)$, for each $\sigma : \Delta^n \to X \in A_X$.

We establish some notation: write $\Delta^n = [v_0, ..., v_n]$ generally, and $\Delta^n \times \{0\} = [v_0, , ..., v_n] \subset \Delta^n \times I$ and $\Delta^n \times \{1\} = [w_0, ..., w_n] \subset \Delta^n \times I$. We leave it as an exercise to check that each $[v_0, ..., v_i, w_i, ..., v_n]$ is an $(n+1)$-simplex in $\Delta^n \times I$.[2]

We define

$$h_n : C_n(X) \longrightarrow C_{n+1}(X \times I)$$
$$\sigma \longmapsto \sum_{i=0}^{n} (-1)^i (\sigma \times \mathrm{id})|_{[v_0, ..., v_i, w_i, ..., w_n]}.$$

To show that this works as required, we show (as above) that $(i_0)_* - (i_1)_* = dh + hd$[3].

We close the proof now in lieu of actually working though the messy details. See Hatcher. ☐

Remark. Speaking categorically, this is a special case of the Yoneda Lemma.

Remark. These homotopies are *universal* in the sense that if $f : X \to Y$ is a map of spaces, then

[2] Or one can find this in Hatcher.

[3] One can find detailed examples of these computations in low dimensions in Hatcher.

$$\begin{CD}
C_n(X) @>{h_n^X}>> C_{n+1}(X) \\
@V{f_*}VV @VV{(f\times \mathrm{id})_*}V \\
C_n(Y) @>{h_n^Y}>> C_{n+1}(Y \times I)
\end{CD}$$

commutes.

3.4 Relative Homology and Excision

Definition 3.27. A *pair* (X, A) is a space X with a subspace $A \subset X$. A *map of pairs* $(X, A) \to (X, B)$ is a map $X \to Y$ such that $f(A) \subset B$. A *homotopy* $h : f \simeq g$ between two maps of pairs $(X, A) \to (Y, B)$ is a homotopy of maps $f, g : X \to Y$ that restricts to a homotopy of maps $f|_A, g|_A : A \to B$.

Example. We've seen many examples of these while looking at pointed topological spaces, where the pair in question is $(X, \{x \in X\})$. π_1 classifies maps of pairs $(S^1, 1) \to (X, x)$.

Definition 3.28. For a pair (X, A), define $C_n(X, A) = C_n(X)/C_n(A)$. Observe that $d : C_n(X) \to C_{n-1}(X)$ takes $C_n(A) \to C_{n+1}(A)$, so we obtain a chain complex $C_*(X, A)$. We define $H_i(X, A) = H_i(C_*(X, A))$.

Theorem 3.29. *Given a pair (X, A) we have a LES*

$$\cdots \to H_i(A) \to H_i(X) \to H_i(X, A) \xrightarrow{\partial} H_{i-1}(A) \to \cdots$$

Proof. Use the LES induced by

$$0 \to C_*(A) \to C_*(X) \to C_*(X)/C_*(A) \to 0.$$

□

Example. We consider the example of $A = \{x\}$. In this case $H_i(X, x) = \widetilde{H}_i(X)$ for each i. One sees this from the LES in homology, which is

$$\cdots \longrightarrow H_1(x) = 0 \hookrightarrow H_1(X) \longrightarrow H_1(X, x) \xrightarrow{\partial} H_0(x) \hookrightarrow H_0(X) \longrightarrow H_0(X, x) \longrightarrow \cdots$$

As $H_i(x) = 0$ for all $i \geq 2$, we obtain that $\ker(H_i(X) \to H_i(X, x)) = \operatorname{coker}(H_i(X) \to H_i(X, x)) = 0$, so $H_i(X) \simeq H_i(X, x)$ for each $i \geq 2$.

As $H_0(x) \to H_0(X)$ is injective (as $H_0(X) = \mathbb{Z}^{\oplus \pi_0(X)}$) and $H_1(x) = 0$, we obtain that $H_1(X) \simeq H_1(X, x)$. It follows as well that $H_0(X, x) = H_0(X)/\mathbb{Z} \simeq \widetilde{H}_0(X)$. To show this last statement we claim that the composition in the following diagram is an isomorphism.

$$\begin{array}{ccc}
\widetilde{H}_0(X) & \hookrightarrow & \mathbb{Z}^{\oplus \pi_0(X)} \\
& \searrow & \downarrow \\
& & \mathbb{Z}^{\oplus \pi_0(X)}/\mathbb{Z} \cdot x
\end{array}$$

Remark. We have the following.

1. If $f(X, A) \to (Y, B)$ is a map of pairs, then we obtain an induced map $f_* : C_*(X, A) \to C_*(Y, B)$, and thus further induces $f_* : H_i(X, A) \to H_i(Y, B)$.

2. If $f, g : (X, A) \to (Y, B)$ are homotopic, then $f_*, g_* : C_*(X, A) \to C_*(Y, B)$ are also homotopic.

   Proof. The homotopies $h_n : C_n(X) \to C_{n+1}(Y)$ giving a homotopy between f_* and g_* : $C_*(X) \to C_*(Y)$ were functorial in $X \to Y$. It follows that the diagram

$$\begin{array}{ccc} C_n(A) & \xrightarrow{h_n^A} & C_{n+1}(B) \\ \downarrow & & \downarrow \\ C_n(X) & \xrightarrow{h_n^X} & C_{n+1}(Y) \end{array}$$

   commutes. Passing to the quotients by the complexes $C_*(A), C_*(B)$, we obtain the commutative square

$$\begin{array}{ccc} C_n(X, A) & \xrightarrow{h_n^{(X,A)}} & C_{n+1}(Y, B) \\ \downarrow & & \downarrow \\ C_n(X)/C_n(A) & \xrightarrow{h_n^{(X,A)}} & C_{n+1}(Y)/C_{n+1}(B) \end{array}$$

   Now one checks that $f_* - g_* = dh^{(X,A)} + h^{(X,A)}d$ as maps $C_*(X, A) \to C_*(Y, B)$, using the fact that the corresponding equality held before passing to quotients. □

3. If $A \subseteq B \subseteq X$ is a tower of spaces, then we obtain the LES

$$\cdots \xrightarrow{\partial} H_i(B, A) \to H_i(X, A) \to H_i(X, B) \xrightarrow{\partial} \cdots$$

   Proof. We have a SES

$$0 \to C_*(B)/C_*(A) \to C_*(X)/C_*(A) \to C_*(X)/C_*(B) \to B$$

   And proceed as is evident. We call the induced LES the *LES of a triple.* □

Theorem 3.30 (Excision)**.** *Given $Z \subseteq A \subseteq X$ a tower of spaces such that $\overline{Z} \subset Int(A)$, the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces an isomorphism on homology.*

Before proving this, we will see some applications. In particular, we will identify $H_*(X, A)$ in terms of the quotient space X/A. We recall the following definition.

Definition 3.31. For an inclusion of spaces $A \subseteq U$, A is a *deformation retract* of U provided that there is a homotopy $h : U \times I \to U$ such that:

- $h(U, 0) = \mathrm{id}_U$
- $h(a, t) = a$ for all $a \in A$, $t \in I$
- the map $h(-, 1) : U \to Y$ takes image in A.

Remark. If $i : A \hookrightarrow U$ is a deformation retract, then i is a homotopy equivalence. The map $h(-, 1)$ provides the arrow in the reverse direction to i. It follows that $H_i(U, A) = 0$ for all i, from the LES on homology.

Example. The following inclusions $A \subseteq U$ are deformation retracts: $A = \{*\}$ with $U = \mathbb{R}^n$; $A =$ small disc with $B =$ big disc.

For the time being, we say a pair (X, A) is *good* provided that there exists an open neighborhood U of A in X such that the inclusion of A into U is a deformation retract.

Example. The following pair is good: $X = D^n$ with $A = S^n = \partial X$. For any (smooth) manifold X with (embedded) submanifold $A \subseteq X$, the pair (X, A) is good; this follows from the tubular neighborhood theorem.

Theorem 3.32 (LES of a good pair)**.** *For (X, A) a good pair, we have isomorphisms $H_i(X, A) \simeq \widetilde{H}_i(X/A)$ for each i. In particular, this is compatible with the LES in homology, so that there is a LES*

$$\cdots \to H_i(A) \to H_i(X) \to \widetilde{H}_i(X/A) \xrightarrow{\delta} H_{i-1}(A) \to \cdots$$

We proceed now with the proof of 3.32, assuming excision.

Proof. Using the LES of a pair, it suffices to show that $(X, A) \to (X/A, A/A)$ induces an isomorphism $H_*(X, A) \simeq H_*(X/A, A/A) \overset{\text{def}}{\simeq} \widetilde{H}_*(X/A)$. Choose a tower $A \subset U \subset X$ of subspaces with U open such that $A \hookrightarrow U$ is a deformation retract.

We have the following commutative diagram:

$$\begin{array}{ccccc}
H_i(X, A) & \xrightarrow{\quad d \quad} & H_i(X, U) & \xleftarrow{\quad e \quad} & H_i(X \setminus A, U \setminus A) \\
\downarrow a & & \downarrow b & & \downarrow c \\
H_i(X/A, A/A) & \xrightarrow{f} & H_i(X/A, U/A) & \xleftarrow{g} & H_i(X/A \setminus U/A, U/A \setminus A/A)
\end{array}$$

We need to show that a is an isomorphism.

We first note that via excision, e and g are isomorphisms. So is d, since $A \hookrightarrow U$ is a homotopy equivalence; to see this, one can use the LES of a triple $A \subset U \subset X$ to conclude that each $H_i(U, A) = 0$. It also follows that f is an isomorphism: we use the fact that $A/A \hookrightarrow U/A$ is also a deformation retract, and use the same reasoning. c is an isomorphism because the underlying spaces are homeomorphic (realized by the map of spaces inducing c). Commutativity now implies that a is an isomorphism. $\square$

Example. We have the following examples employing the theorem. All pairs, as you can check, are good.

1. $A = S^0 = \partial I \subseteq X = I$. The theorem implies there is a LES

$$H_i(A) \to H_i(X) \to \widetilde{H}_i(X/A) \to \cdots$$

with $H_i(A) = 0$ for $i > 0$ and $H_0(A) = \mathbb{Z}^{\oplus 2}$, $H_i(X) = 0$ for $i > 0$ and $H_0(X) = \mathbb{Z}$. The quotient X/A is homeomorphic to a circle, so we obtain that $\widetilde{H}_i(S^1) = 0$ for $i \geq 2$, $H_0(S^1) = 0$, and there is a SES

$$0 \to \widetilde{H}_1(S^1) \to \mathbb{Z}^{\oplus 2} \to \mathbb{Z} \to 0$$

with the third map being given by summing coordinates. This implies immediately that $\widetilde{H}_1(S^1) \simeq \mathbb{Z}$.

2. $A = S^{n-1} \subseteq X = D^n$. This gives $X/A \cong S^n$. We have that

$$H_i(D^n) = \begin{cases} 0 & i \neq 0 \\ \mathbb{Z} & i = 0 \end{cases}$$

so we may repeat our analysis in (1) to obtain that

$$H_i(D^n) = \begin{cases} 0 & i \neq 0, n \\ \mathbb{Z} & i = 0, n \end{cases}$$

(and this is done inductively).

Corollary 3.33. *If $\mathbb{R}^n \cong \mathbb{R}^m$ is a homeomorphism, then $n = m$.*

Proof. Choose such a homeomorphism ϕ. We obtain a homeomorphism $\psi : \mathbb{R}^n \smallsetminus \{*\} \to \mathbb{R}^m \smallsetminus \{*'\}$ which induces a homotopy equivalence $S^{n-1} \simeq S^{m-1}$. Comparing homologies, we see that $m = n$. □

Example. If $\emptyset \neq U \subset \mathbb{R}^n$ and $\emptyset \neq V \subset \mathbb{R}^m$, then $U \cong V$ implies that $m = n$. To show this, we establish a lemma.

Lemma 3.34. *In this setting, with $x \in U$, we have*

$$H_i(U, U \smallsetminus \{x\}) = \begin{cases} \mathbb{Z} & i = m \\ 0 & \textit{else.} \end{cases}$$

Proof. The map of pairs $(U, U \smallsetminus \{x\}) \to (\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{x\})$ induces an isomorphism on H_* via excision for the triple $Z = \mathbb{R}^n \smallsetminus U$, $A = \mathbb{R}^n \smallsetminus \{x\}$, and $X = \mathbb{R}^n$. Excision implies that $(X \smallsetminus Z, A \smallsetminus Z) \to (X, A)$ is an isomorphism on H_*. This is $(U, U \smallsetminus \{x\}) \to (\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{x\})$, where the second term's homology is given by the above example, and the lemma follows. □

The example now follows.

Definition 3.35. We define the *local homology of X at x* to be $H_*(X, X \smallsetminus \{x\})$.

Example. For an n-manifold M, we have

$$H_*(M, M \smallsetminus \{x\}) \simeq H_*(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{0\}) = \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{else} \end{cases}$$

following the above example. Thus, we see that homology can detect the dimension of a manifold.

Definition 3.36. The generators $\omega_x \in H_n(X, X \smallsetminus \{x\})$ are called *local orientations of X at x.*

Remark. A topological manifold M is orientable if and only if there are local orientations ω_x for each $x \in M$ which can be chosen "compatibly", somehow, in x.

Now we will prove excision (3.30). The first step is to introduce the notion of "small chains", which we will use in the proof. Let X be a space and $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover such that the interiors of the U_i also cover X. We define $C_n^{\mathcal{U}}$ to be the free abelian group on all $\sigma : \Delta^n \to X$ such that $\sigma(\Delta^n) \subset U_i$ for some i. One checks that d takes $C_n^{\mathcal{U}}(X)$ into $C_{n-1}^{\mathcal{U}}(X)$, so that we obtain a subcomplex $C_\bullet^{\mathcal{U}}(X)$ of $C_n(X)$. Denote the inclusion of complexes $C_\bullet^{\mathcal{U}}(X) \to C_\bullet(X)$ by ϕ.

We have the following theorem (which we use as a lemma).

Theorem 3.37 (Theorem of small chains). *ϕ induces an isomorphism on homology. In fact, ϕ is a homotopy equivalence.*

Example. For $X = S^1$ let $\mathcal{U}$ be the open cover whose elements are the upper and lower (closed) hemispheres. The theorem of small chains implies that the generator of $H_1(S^1)$ comes from the generators of the homologies of the (oriented) hemispheres.

We prove excision assuming small chains:

Proof (Excision). Let $Z \subset A \subset X$ be as in the statement of excision. Define $\mathcal{U} = \{A, X \smallsetminus Z\}$. Note that the interiors of A and $X \smallsetminus Z$ also cover Z, as each is open. The theorem of small chains implies that the morphism $C_\bullet^{\mathcal{U}}(X) \to C_\bullet(X)$ is an isomorphism in homology. We have commutative diagrams

$$
\begin{array}{ccc}
A \setminus Z & \longrightarrow & X \setminus Z \\
\big\downarrow & & \big\downarrow \\
A & \longrightarrow & X
\end{array}
\quad \overset{C_\bullet}{\rightsquigarrow} \quad
\begin{array}{ccc}
C_\bullet(A \setminus Z) & \longrightarrow & C_\bullet(X \setminus Z) \\
\big\downarrow & & \big\downarrow \\
C_\bullet(A) & \longrightarrow & C_\bullet(X)
\end{array}
$$

from which we obtain SESs

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C_\bullet(A \setminus Z) & \longrightarrow & C_\bullet(X \setminus Z) & \longrightarrow & C_\bullet(X \setminus Z, A \setminus Z) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow c & & \\
0 & \longrightarrow & C_\bullet(A) & \longrightarrow & C_\bullet^{\mathcal{U}}(X) & \longrightarrow & C_\bullet^{\mathcal{U}}(X)/C_\bullet(A) & \longrightarrow & 0 \\
 & & \big\downarrow = & & \big\downarrow \beta & & \big\downarrow \gamma & & \\
0 & \longrightarrow & C_\bullet(A) & \longrightarrow & C_\bullet(X) & \longrightarrow & C_\bullet(X, A) & \longrightarrow & 0
\end{array}
$$

We claim that c induces an isomorphism on homology. Assuming this, the theorem of small chains implies that β induces an isomorphism on homology. The 5-lemma now implies that γ is also an isomorphism on homology. Thus $\gamma \circ c$ is an isomorphism on homology, as required.

We show now that c does indeed induce an isomorphism of homology groups. We have

$$
\frac{C_\bullet^{\mathcal{U}}(X)}{C_\bullet(A)} \simeq \frac{C_\bullet(X \setminus Z) + C_\bullet(A)}{C_\bullet(A)} \simeq \frac{C_\bullet(X \setminus Z)}{C_\bullet(A) \cap C_\bullet(X \setminus Z)} \simeq \frac{C_\bullet(X \setminus Z)}{C_\bullet(A \setminus Z)}
$$

where the second map is induced by the second (group) isomorphism theorem.

□

Remark. A similar method shows that if $X = U \cup V$ is a union of open subsets, with $\mathcal{U} = \{U, V\}$, then we obtain a SES

$$
0 \to C_\bullet(U \cap V) \to C_\bullet(U) \oplus C_\bullet(V) \to C_\bullet^{\mathcal{U}}(X) \to 0
$$

which induces the LES

$$
\cdots \to H_i(U \cap V) \to H_i(U) \oplus H_i(V) \to H_i(X) \xrightarrow{\delta} H_{i-1}(U \cap V) \to \cdots
$$

known as the *Mayer-Vietoris sequence.*

We now prove the theorem of small chains, which will complete the proof of excision.

Proof (Small chains). Each simplex $[v_0, ..., v_n] \subset \mathbb{R}^n$ has a *barycentric subdivision* into $(n+1)!$ subsimplices. For an example in dimension 2, see the figure.

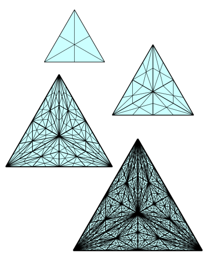


Figure 5: Four iterations of barycentric subdivision of a 2-simplex. Image from the Wikimedia Commons.

It is a fact taht the diameter of a "new" simplex appearing in a barycentric subdivision is $n/(n+1)$ times the diameter of the old one.

We define the operator $S : C_n(X) \to C_n(X)$ by

$$s(\sigma : \Delta^n \to X) = \sum_{\Delta'_n \text{ in a barycentric subdiv}} (-1)^? \sigma|_{\Delta'_n}$$

where the Δ'^n appear in a barycentric subdivision of the image of σ. We record some facts without proof.

1. $dS = Sd$, so we obtain a map on chain complexes.
2. S is homotopic to the identity, so that $S = \mathrm{id} + dh + hd$. (This comes from universal homotopies)
3. S preserves small chains, so we get a map $S^{\mathcal{U}} : C_*^{\mathcal{U}}(X) \to C_*^{\mathcal{U}}(X)$.
4. $S^{\mathcal{U}}$ is homotopic to id, with homotopy given by restricting those from (2).

We proceed now with the proof of the theorem. Consider $\phi : C_*^{\mathcal{U}}(X) \to C_*(X)$; we show it is surjective and injective on homology. Pick a cycle $z \in Z_i(X)$; for $m \gg 0$, $S^m(z)$ is small, as S makes things smaller and Δ^n is compact. Thus $S^m(z) \in Z_i^{\mathcal{U}}(X)$ for m sufficiently large.

We claim that $z = \phi(S^m(z))$ in $H_i(X)$, which follows since S is homotopic to id, so the same is true for S^m, so $S(z) - z = dh(z) + hd(z) = dh(z) \in B_i(X)$ as z is a cycle, and likewise for S^m. Thus ϕ is surjective on homology. ϕ is also injective on homology: say $w \in Z_i^{\mathcal{U}}(X)$ is such that $w = dz$ for some $z \in C_{i+1}(X)$; we need to show that $w = 0$ in homology. We have $S^m w = S^m dz$ for each m, so that from (1) $S^m w = dS^m z$. As S makes simplices smaller, we know that $S^m z \in C_{i+1}^{\mathcal{U}}(X)$, which implies that $w = S^m w = 0$ in homology, again as S is homotopic to the identity. □

3.5 Singular vs. Simplicial Homology

Let (X, A) be a pair. Assume that there exist compatible Δ-complex structures on X and A. We get a map $C_*^{\Delta}(A) \to C_*^{\Delta}(X)$. We define $C_*^{\Delta}(X, A) = X_*^{\Delta}(X)/C_*^{\Delta}(A)$.

Note that there is a natural map $C_*^{\Delta}(X, A) \to C_*(X, A)$ which sends a simplex to itself.

Theorem 3.38. *This induces an isomorphism on homology $H_i^{\Delta}(X, A) \to H_i(X, A)$.*

Proof. Define X^k to be the sub-Δ-complex of X spanned by simplices of dimension at most k. We have inclusions $X^{k-1} \subset X^k \subset X^{k+1} \subset \cdots \subset X$; we assume that this is a finite tower. Call the dimension of X the minimal such N with $X = X^N$. We get pairs (X^k, X^{k-1}) for each k.

We isolate a key lemma which we will prove after using it: $H_i^{\Delta}(X^k, X^{k-1}) \simeq H_i(X^k, X^{k-1})$. Now we prove the theorem assuming it.

We induct on the dimension of X (in the above sense). When $\dim(X) = 0$, X is a (finite) set of points, $A \subset X$ a subset. The theorem statement is clear. Assume the theorem holds for all pairs (Y, B) when $\dim(Y) < k$. Consider (X^k, X^{k-1}); we get long exact sequences

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & H_{i+1}^{\Delta}(X^k, X^{k-1}) & \longrightarrow & H_i^{\Delta}(X^{k-1}) & \longrightarrow & H_i^{\Delta}(X^k) & \longrightarrow & H_i^{\Delta}(X) & \longrightarrow & H_{i-1}^{\Delta}(X^{k-1}) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & H_{i+1}(X^k, X^{k-1}) & \longrightarrow & H_i(X^{k-1}) & \longrightarrow & H_i(X^k) & \longrightarrow & H_i(X) & \longrightarrow & H_{i-1}(X^{k-1}) & \longrightarrow \cdots \end{array}$$

The lemma implies that a and d are isomorphisms, and induction that b and e are. The 5-lemma now implies that c is an isomorphism.

If Y is a Δ-complex of dimension $\leq k$, then $H_i^{\Delta}(Y) \simeq H_i(Y)$. By a 5-lemma argument, we obtain the theorem for (X, A) with $\dim(X) \leq k$.

It remains to prove the lemma. Assume $k > 0$. Then

$$C_i^{\Delta}(X^k, X^{k-1}) = \begin{cases} 0 & i \neq k \\ \mathbb{Z}^{\oplus S_k} & i = k \end{cases}$$

where $S_k = \{\sigma : \Delta^k \to X\}$ all k-simplices in a given Δ-complex. Thus

$$H_i^{\Delta}(X^k, X^{k-1}) = \begin{cases} 0 & i \neq 0 \\ \mathbb{Z}^{\oplus S_k} & i = k \end{cases}.$$

On the other hand, $H_i(X^k, X^{k-1}) \simeq \widetilde{H}_i(X^k/X^{k-1})$ by the LES of a good pair. Observe that $X^k/X^{k-1} = \bigvee_{\sigma \in S_k} S^k$ as $\Delta^k/\Delta^{k-1} \cong S^k$. Comparing homologies, we have the desired isomorphisms. □

3.6 CW complexes and cellular homology

We examine[4] a homotopically and topologically significant set of spaces and the homology theory they require; these are *CW complexes* and *cellular homology.*

A CW complex is obtained by a structured gluing of spheres to one another:

Definition 3.39. A *CW complex* is a topological space X with a sequence of subspaces $X^0, X^1, ... \subset X$, where X^i is the *i-skeleton of* X, and a decomposition $X = \bigcup_{m \in \mathbb{Z}_{\geq 0}} X^m$, where:

- X^0 is a discrete set of points, and
- we obtain X^{m+1} from X^m as the pushout

$$\begin{array}{ccc} \coprod_{\alpha \in A} S^n & \xrightarrow{j^n_\alpha} & X^m \\ {\scriptstyle \amalg \partial}\big\downarrow & & \big\downarrow \\ \coprod_{\alpha \in A} D^{n+1} & \longrightarrow & X^{m+1} \end{array}$$

  where ∂ denotes the identification of S^n as ∂D^{n+1}; each map j^n is an *nth attaching map* for X.

This means that we obtain X^{n+1} as the topological quotient of $X^{n-1} \sqcup_\alpha D^n_\alpha$ for $\alpha \in A$, under the identificiations $x \sim j^n_\alpha(x)$ for each $x \in \partial D^n_\alpha$ for each α. Thus as a set, we have $X^{n+1} = X^n \sqcup_\alpha e^n_\alpha$, where each e^n_α is an open n-disk. Each e^n_α is called an *n-cell* in X.

We give X^n the weak[5] (or colimit) topology, where a subspace of X is closed if and only if its intersection with each X^n is; when $X = X^n$ for some n, then this is the topology on X. We will mostly be concerned with such spaces, where we write $n = \dim(X)$ and say that X is *n-dimensional.*

Example. A 1-dimensional CW complex is what is called a *graph* in topology. The attaching maps dictate which vertices are connected by a "0-disk", i.e. an edge.

Example. The sphere S^n has the structure of a CW complex with two cells, e^0 and e^n, where the attaching map $\partial D^n = S^{n-1} \to e^0$ is given by the constant map.

Example. Real and complex projective space can be given CW complex structures; this was on the homework. Can you work them out?

[4]We follow Hatcher obsequiously for the duration of the section, except where noted.

[5]This explains the 'W' in "CW complex". The 'C' stands for "closure finiteness", which references a topological fact about CW complexes which describes the behavior of their compact subspaces.

We depart from Hatcher for a moment to define a suitable category of CW complexes.

Definition 3.40. A *subcomplex* A of a CW complex X is a subspace $A \subset X$ of X and a CW complex such that the composite of each cell $D^n \to A \hookrightarrow X$ is a cell of X. Equivalently, A is the union of cells of X.

Definition 3.41. For CW complexes X and Y, a map $f : X \to Y$ is *cellular* provided that $f(X^n) \subset Y^n$ for each n.

Note that an inclusion of a subcomplex is a cellular map, and that composition of cellular maps is cellular. In this way we obtain the category **CW** of (finite) CW-complexes and (finite) cellular maps of CW complexes.

We aim to define a suitable homology theory on CW complexes; this is cellular homology. We establish a lemma.

Lemma 3.42. *Let X be a CW complex.*

1. *$H_k(X^n, X^{n-1})$ is zero if $k \neq n$, and free abelian when $k = n$, with basis in bijection with the n-cells of X.*

2. *$H_k(X^n) = 0$ for $k > n$. In particular, if X is finite-dimensional, then $H_k(X) = 0$ for $k > \dim(X)$.*

3. *The map $H_k(X^n) \to H_k(X^n)$ induced by the inclusion $X^n \to X$ is an isomorphism for $k < n$ and a surjection for $k = n$.*

Proof. The first claim follows after observing that (X^n, X^{n-1}) is a good pair, and that the quotient X^n/X^{n-1} is a wedge sum of n-spheres, one for each n-cell of X.

Now consider the following segment of the LES of the pair (X^n, X^{n-1}):

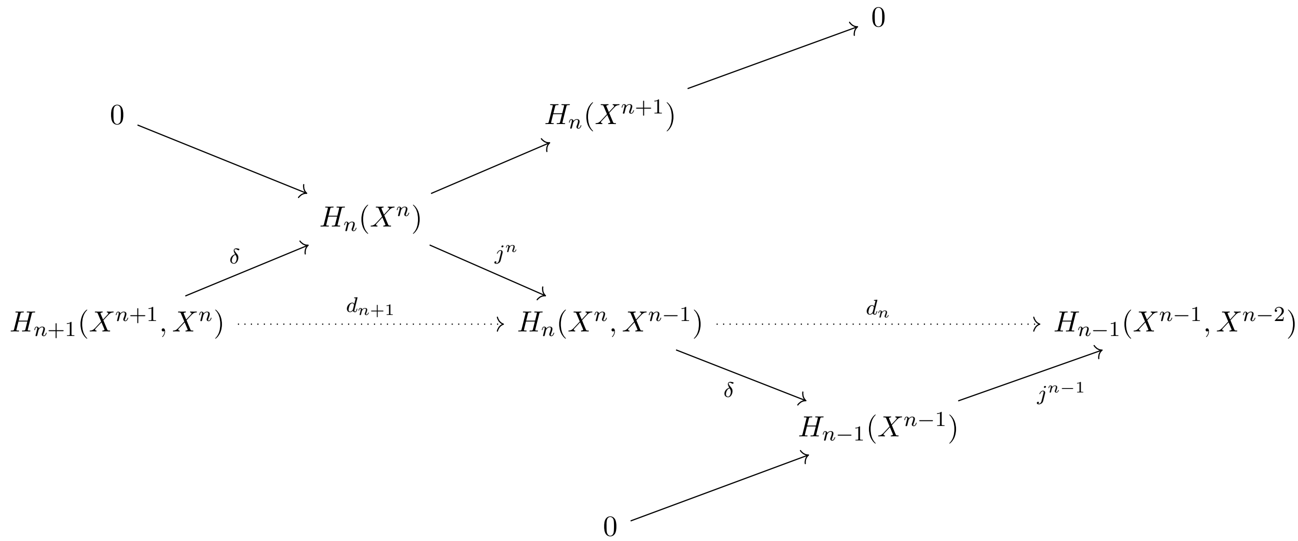
$$H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \to H_k(X^n) \to H_k(X^n, X^{n-1})$$

If $k \neq n$ then the final term is zero by the first claim, so the middle map is surjective. If $k \neq n - 1$, then the first term is zero, so the middle map is injective. Now we examine the inclusion-induced maps

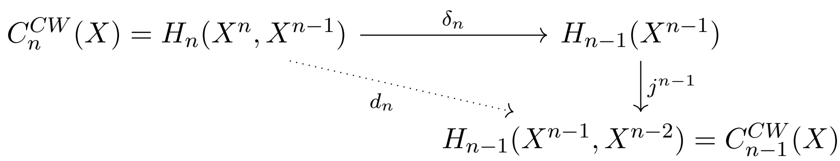
$$H_k(X^0) \to H_k(X^1) \to \cdots \to H_k(X^{k-1}) \to H_k(X^k) \to H_k(X^{k+1})$$

By the above paragraph each map is an isomorphism except for the map to $H_k(X^k)$, which might not be surjective, and the map from $H_k(X^k)$, which may not be injective. The second statement now follows from the fact that $H_k(X^0) = 0$ when $k > 0$. The last part of the sequence gives the third statement, when X is finite-dimensional. The infinite-dimensional case is more subtle, and can be found in Hatcher. $\square$

Using the LESs for the three pairs (X^{n+1}, X^n), (X^n, X^{n-1}), and (X^{n-1}, X^{n-2}), we construct the diagram



Using this diagram, we define a chain complex $C_\bullet^{CW}$ by $C_n^{CW} := H_n(X^n, X^{n-1})$, and define the differential as above as the following composition:



That is, $d_n = j^{n-1}\delta_n$. We define the *cellular homology* of X to be $H_n^{CW}(X) := H_n(C_\bullet^{CW})$. We still need to check that $d^2 = 0$, however this follows from the very definition of the cellular boundary map, and a diagram chase.

Theorem 3.43. *For X a CW-complex, the cellular and singular homologies of X are isomorphic.*

Proof. This is a diagram chase. As in the above diagram, $H_n(X)$ can be identified with $H_n(X^n)/\mathrm{im}(\delta_{n+1})$. as j^n is injective, it maps $\mathrm{im}(\delta_{n+1})$ isomorphically onto $\mathrm{im}(j_n\delta_{n+1}) = \mathrm{im}(d_{n+1})$, and $H_n(X^n)$ isomorphically onto $\mathrm{im}(j_n) = \ker(\delta_n)$. As j^{n-1} is injective, $\ker(\delta_n) = \ker(d_n)$. Thus j_n induces an isomorphism of the quotient $H_n(X^n)/\mathrm{im}(\delta_{n+1})$ onto $\ker(d_n)/\mathrm{im}(d_{n+1})$. □

There are a few immediate corollaries.

1. $H_n(X) = 0$ if X is a CW complex with no n-cells.
2. If X is a CW complex with k n-cells, then $H_n(X)$ is generated by at most k elements.
3. If X is a CW complex with no two of its cells in adjacent dimensions, then $H_n(X)$ is free abelian with basis in bijection with the n-cells of X (the boundary maps are zero). This applies, e.g., to $\mathbb{C}P^n$.

We give a formula for the cellular boundary maps.

Proposition 3.44 (Cellular boundary formula)**.** *$d_n(e_\alpha^n) = \sum_\beta d_{\alpha,\beta} e_\beta^{n-1}$, where β ranges over the $(n-1)$-cells in X, and $d_{\alpha,\beta}$ is the degree*[6] *of the map $S^{n-1} \to X^{n-1} \to S_\beta^{n-1}$, where the first arrow is the attaching map an the second is the quotient map collapsing $X^{n-1} \smallsetminus e_\beta^{n-1}$ to a point.*

The proof of the proposition is a rather large diagram chase, which can be found in Hatcher.

[6] We proved on homework that degree of a map $S^r \to S^r$ is identified with the integer k such that the induced map on top-dimensional homology is multiplication by k.

[Missing some lecture material here.]

Recall that for a finite CW complex X, we have the following:

1. We have a filtration $\emptyset = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^n = X$ such that $X^k/X^{k-1} \cong \bigvee S^k$, where this wedge is taken over all k-cells in X.

2. Hence $H_i(X^k, X^{k-1}) = \mathbb{Z}^{\oplus |k-\text{cells}|}$ for $i = k$, and 0 otherwise.

3. We obtain a cellular chain complex as above using the LESs of associated triples, using the cellular boundary formula.

4. We have $H_i(C_*^{CW}(X)) \cong H_i(X)$.

Example. Consider $X = S^1 \times S^1$. Of course we realize X as a unit square with standard edge identifications, from which we obtain a cell complex structure with one 0-cell, two 1-cells, and one 2-cell. Thus we obtain the cellular chain complex

$$0 \to \mathbb{Z} \to \mathbb{Z}^{\oplus 2} \to \mathbb{Z} \to 0$$

for which we need to compute the differentials. We apply the cellular boundary formula to calculate the hardest one.

Let $\partial D^2 = S^1 \to X^1 = S^1_a \vee S^1_b$ be the attaching map for our 2-cell, which identifies S^1 with $aba^{-1}b^{-1}$. Then $S^1 \to X^1 \to X^1/S^1_a \cong S^1_b$ is null-homotopic (as it is given by identifying S^1 to aa^{-1}), and likewise for b. It follows that $d(T) = (0, 0)$, so each of our homologies is given by the terms of the cell complex.

3.7 Eilenberg-Steenrod Axioms

We define the axiomatic formalism which underlies all homology theories we have described and many we have not. It is given by the *Eilenberg-Steenrod axioms*, which we lay out now.

A *homology theory* h_* is a functor

$$\text{pairs } (X, A) \to \mathbf{grAbGrp}$$

and maps $\partial : h_i(X, A) \to h_{i-1}(A, \emptyset) = h_i(A)$, satisfying the following axioms

1. (Homotopy) If $f, g : (X, A) \to (Y, B)$ are homotopic, then $h_*(f) = h_*(g)$.

2. (Exactness) For a pair (X, A), the map ∂ gives a LES

$$\cdots \to h_i(A) \to h_i(X) \to h_i(X, A) \to h_{i-1}(A) \to \cdots$$

3. (Excision) Given a tower of spaces $Z \subset A \subset X$ with $\overline{Z} \subset \text{Int}(A)$, the map $(X \setminus Z, A \setminus Z) \to (X, A)$ induces an isomorphism on h_*.

4. (Dimension) $h_i(\text{pt}) = \mathbb{Z}$ if $i = 0$ and is otherwise 0.

5. (Additivity) If $X = \coprod_\alpha X_\alpha$, then the map $\oplus_\alpha h_*(X_\alpha) \to h_*(X)$ induced by the inclusions is an isomorphism.

Remark. Excision implies the finite additivity axiom, so it is only necessary for infinite direct sums.

Remark. The dimension axiom is crucial! Dropping it opens up a wide world of strange homology theories. Further, we may replace $\mathbb{Z}$ in the dimension axiom with any abelian group, and obtain homology with coefficients in that group.

Proposition 3.45. *If h_* is a homology theory in the sense of the above axioms, then it satisfies the following:*

1. *LES of a triple: if $A \subset B \subset X$, there exists a LES*

$$\cdots \to h_i(B, A) \to h_i(X, A) \to h_i(X, B) \to h_{i-1}(B, A) \to \cdots$$

where the final map is built from excision. We leave the proof as an exercise.

2. *LES of a good pair: if (X, A) is a good pair, then $h_*(X, A) \cong \widetilde{h}_*(X/A) := h_*(X/A, A/A)$.*

3. *Mayer-Vietoris sequence: if $X = U \cup V$ where $U, V \subset X$ are open, then we have the LES*

$$\cdots \to h_i(U \cap V) \overset{std}{\to} h_i(U) \oplus h_i(V) \overset{std-std}{\longrightarrow} h_i(X) \overset{\delta}{\to} h_{i-1}(U \cap V) \to \cdots$$

4. *If X is a finite CW complex, then*

$$h_*(X^k, X^{k-1}) \simeq \begin{cases} \mathbb{Z}^{\oplus |k-cells|} & * = k \\ 0 & else. \end{cases}$$

Proof. We prove 3. We construct δ as follows. The map $(U, U \cap V) \to (X, V)$ induces a diagram

$$\begin{array}{ccccccccc} h_{i+1}(U, U \cap V) & \longrightarrow & h_i(U \cap V) & \overset{a}{\longrightarrow} & h_i(U) & \longrightarrow & h_i(U, U \cap V) & \overset{g}{\longrightarrow} & h_{i-1}(U \cap V) \\ \downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle f} & & \downarrow \\ h_{i+1}(X, V) & \longrightarrow & h_i(V) & \overset{c}{\longrightarrow} & h_i(X) & \overset{e}{\longrightarrow} & h_i(X, V) & \longrightarrow & h_{i-1}(V) \end{array}$$

Now $U/U \cap V \simeq X/V$ is a homeomorphism. Then a diagram chase gives

$$h_i(U \cap V) \overset{(a,b)}{\to} h_i(U) \oplus h_i(V) \overset{(d,-c)}{\to} h_i(X) \overset{gf^{-1}e}{\longrightarrow} h_{i-1}(U \cap V)$$

so we identify $\partial = gf^{-1}e$, and we leave it as an exercise to check that this is a LES.

The statement in 4. follows from the LES

$$C^h_*(X) = \cdots \to h_n(X^n, X^{n-1}) \overset{d}{\to} h^{(}_{n-1}X^{n-1}, X^{n-2}) \to \cdots$$

and the LES of a good pair implies the result. As the notion of degree is the same for h_* and H_*, we obtain an isomorphism $C^h_*(X) \cong C^{CW}_*(X)$, implying $h_i(X) \cong H_i(C^h_*(X))$ with the same computation as last time, and as we've shown $H_i(C^{CW}_*(X)) \cong H_i(X)$, we have the ... $\square$

Example. Consider the sphere S^n. Then for $n = 0$, the dimension and additivity axioms imply that $h_0(S^n) = \mathbb{Z}^2$, and when $n > 0$, we write $S^n = D^n \cup D^n$ where the intersection is (homotopic to) S^{n-1}. Apply Mayer-Vietoris and the homotopy axiom to determine the homology groups.

Proposition 3.46. 5. *If $f : S^n \to S^n$ is a map, then the degree of f with respect to h_* is equal to the degree of f with respect to H_* (singular homology).*

See May for a proof of the theorem. We work out some examples.

Example. If f is nullhomotopic, homotopy invariance combined with the dimension axiom force the degree of f (with respect to h_*) to be zero. If f is the identity map, as h_* is a functor, the degree of f is 1. (May's proof combines these two cases into a general argument.)

3.8 Three interesting theorems

3.8.1 Hurewicz Theorem for π_1

Theorem 3.47 (Hurewicz)**.** *Say X is path-connected, with $x \in X$. There exists a natural isomorphism $\pi_1(X,x)^{ab} \simeq H_1(X)$.*

Remark. There exists a variant of the theorem for π_n: for X path-connected, with $\pi_i(X) = 0$ for $i < n$, then $\pi_n(X) \cong H_n(X)$ if $n \geq 2$.

Proof of Hurewicz. Each map $f : (S^1, 1) \to (X, x)$ induces a map $f_* : H_1(S^1) \simeq \mathbb{Z} \to H_1(X)$, so we obtain $f_*(1) \in H_1(X)$. Homotopy invariance implies that $f_*(1)$ only depends on the homotopy class of f as a pointed map. We obtain a map

$$h : \pi_1(X, x) = \{\text{ptd homotopy classes of maps from } S^1\} \to H_1(X).$$

We prove that h is a homomorphism. Say $f, g : (S^1, 1) \to (X, x)$, and form $g * f : (S^1, 1) \to (X, x)$; we want $(g * f)_*(1) = f_*(1) + g_*(1)$. We consider the following picture: [picture goes here]

Observe the following:

1. The composite map is $f * g$.
2. Apply H_1:

$$\begin{array}{ccccc} & & H_1(S^1) & & \\ & & \downarrow{\scriptstyle (i_1)_*} & \searrow{\scriptstyle f_*} & \\ H_1(S^1) & \xrightarrow{a_*} & H_1(S^1 \vee S^1) & \longrightarrow & H_1(X) \\ & & \uparrow{\scriptstyle (i_2)_*} & \nearrow{\scriptstyle g_*} & \\ & & H_1(S^1) & & \end{array}$$

3. Under $H_1(S^1) = \mathbb{Z}$, $H_1(S^1 \vee S^1) = \mathbb{Z} \oplus \mathbb{Z}$ we have

$$a_*(1) = (1,1) \quad (i_1)_*(1) = (1,0) \quad (i_2)_*(1) = (0,1).$$

   Thus

$$\begin{aligned} (g * f)_*(1) &= (f \vee g)_* a_*(1) \\ &= (f \vee g)_*(1,1) \\ &= (f \vee g)_*((1,0) + (0,1)) \\ &= (f \vee g)_*(1,0) + (f \vee g)_*(0,1) \\ &= (f \vee g)_*(i_1)_*(1) + (f \vee g)_*(i_2)_*(1) \\ &= f_*(1) + g_*(1) \end{aligned}$$

   as required.

Now we show that this homomorphism induces an isomorphism $\pi_1(X)^{ab} \to H_1(X)$, in the case where X is a finite CW complex with $X^0 = \{\text{pt}\}$. We induct on $\dim(X)$: when $\dim(X) = 0$ or 1, h is an isomorphism.

1. $\pi_1(X^1, x) \to H_1(X)$ is the abelianization map. This follows from the fact that $X^1 \cong \bigvee S^1$.
2. Assume X^2 is obtained by attaching a single 1-cell, i.e.

$$X^2 = \operatorname{colim} \begin{array}{ccc} S^1 & \longrightarrow & D^2 \\ \downarrow{\scriptstyle \alpha} & & \\ X^1 & & \end{array}$$

where $\alpha \in \pi_1(X^1)$. Then the van Kampen theorem implies that $\pi_1(X^2, x) = \pi_1(X^1, x)/\overline{\langle \alpha_*(1) \rangle}$. Also, the LES of (X^2, X^1), get

$$H_2(X^2, X^1) \xrightarrow{\partial} H_1(X^1) \to H_1(X^2) \to H_1(X^2, X^1)$$

and observing that $X^2/X^1 \cong S^2$, we see that the first term is $\mathbb{Z}$ and the last is zero. We leave it as an exercise to show that the first arrow is multiplication by $\alpha_*(1)$, so that the second map is an isomorphism. Since $H_1(X^1) \simeq \pi_1(X, x)$, we have the result in dimension 2.

The higher dimensional cases follow from the fact that the inclusion $X^2 \hookrightarrow X$ induces an isomorphism on fundamental groups (proved on homework).

□

3.8.2 Lefschetz fixed point thoerem

Theorem 3.48. *If X is a finite simplicial complex (or even a retract of such), and $f : X \to X$ is an endomorphism, then one of the following is true:*

1. *f has a fixed point $x \in X$ such that $f(x) = x$.*
2. *The Lefschetz number $\tau(f) := \sum_i (-1)^i tr(f_*(H_i(X, \mathbb{Q})))$, is zero.*

Example. Consider X a finite set. We claim that $\tau(f) =$ the number of fixed points of f. We have $H_0(X, \mathbb{Q}) = \bigoplus_{x \in X} \mathbb{Q} \cdot x$, and the other homology groups vanish. Thus

$$\tau(f) = \operatorname{tr}(f_*(H_0(X, \mathbb{Q}))) = \operatorname{tr}(f_*(\bigoplus \mathbb{Q} \cdot x))$$

and we see that $f_*(x) = y$ if $f(x) = y$, so that f_* is a “permutation matrix" in some sense. In this case, the trace of (the matrix corresponding to) f is the number of fixed points of f.

Example (Brouwer fixed point theorem). Any endomorphism f of D^n has a fixed point. The homology of D^n is concentrated at zero, so that $f_* = 1$ on $H_0(D^n, \mathbb{Q})$, so $\tau(f) = 1$, implying that f has a fixed point. This holds for any contractible space which is also a finite simplicial complex.

Example. Any endomorphism f of $\mathbb{RP}^n$ for even n has a fixed point. We have

$$H_i(\mathbb{RP}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0 \\ 0 & \text{else} \end{cases}$$

as n is even. Now we can apply the previous argument.

This implies that any linear map $\mathbb{R}^{2k+1} \to \mathbb{R}^{2k+1}$ has a real eigenvalue. This is false for odd n, as you can see in the case $n = 1$, where $\mathbb{RP}^n \cong S^1$ has endomorphisms with no fixed points.

Example. Suppose G is a nontrivial path-connected topological group representable as a finite simplicial complex, e.g. $SO(n)$. We claim that $\chi(G) = \sum_i (-1)^i \dim_{\mathbb{Q}}(H_i(G, \mathbb{Q})) = 0$. This follows since the trace of the identity map on a vector space is the dimension of that space, so $\chi(X) = \tau(\mathrm{id}_X)$ for any space X.

Choose a nonidentity $g \in G$ to obtain $T_g : G \to G$ given by $h \mapsto hg$. T_g has no fixed points, so this implies that $\tau(T_g) = 0$. As G is path-connected we may drag T_g along a path between g and 1_G to give a homotopy between T_g and the identity map. As homotopic maps induce the same map on homology, the claim follows.

Remark. It is tempting to guess that the number of fixed points of an endomorphism f of some nice X equals $\tau(f)$, but this is not true. The identity map in the preceding example gives a contradiction.

This is almost true, however. There is a way to cleverly count fixed points (with "multiplicity") to obtain an equality.

Remark. Compactness of X is essential: for $X = \mathbb{R}$, the translation $x \mapsto x + 1$ has no fixed points (even in the sense of the above remark), however as X is contractible, $\tau(f) = 1$.

Remark. Hatcher proves that every CW complex is homotopic to a finite simplicial complex, so the Lefschetz fixed point theorem holds for CW complexes as well.

Remark. There is a very useful variant of Lefschetz used in algebraic geometry and number theory.

We prove the Lefschetz fixed point theorem. Observe that in the proof $\mathbb{Q}$ may be replaced with an arbitrary field.

We use the following lemma, which we do not prove.

Lemma 3.49 (Simplicial Approximation theorem.)**.** *Let L and K be finite simplicial complexes; we notate by $|K|$ and $|L|$ the associated topological spaces. Given a continuous map $F : |K| \to |L|$, there exists a map $f : K \to L$ such that*

1. *f is homotopic to F,*
2. *there is $n \gg 0$ such that $f : Bd^n(K) \to |L|$ is simplicial, where Bd^m denotes the barycentric subdivision of K applied m-times,*
3. *f is arbitrarily close to F.*

Example. Consider $F : S^1 \to S^1$ defined by $z \mapsto z^2$. We give S^1 a simplicial structure by gluing two intervals at their endpoints, so that in the context of the above lemma, $K = L$. Then letting $n = 2$ in the setting of statement 2 of the lemma, we have $f : \mathrm{Bd}^n(K) \to L$ obtained again by squaring is simplicial.

Proof of 3.48. Suppose $f : X \to X$ has no fixed points. Applying simplicial approximation, we obtain a subdivision L of X and $g : |L| \to X$ such that $g \simeq f$ and g is simplicial.

We may arrange that $g(\sigma) \cap \sigma = \emptyset$ for each simplex σ in L, as f has no fixed points. This is a compactness property: the distance between a point and its image, being distinct, is greater than zero, and as X is compact (finite simplicial complexes are) we may choose a sufficiently fine subdivision of L to separate all points from their f-images (this is known as the Lebesgue covering lemma).

Now as g is simplicial, we have $g(L^n) \subset X^n$ for each n. As L is a subdivision of X, we have $X^n \subset L^n$, so that $g(L^n) \subset L^n$ for each n. We obtain maps of pairs $(L^n, L^{n-1}) \to (L^n, L^{n-1})$ induced by g. We claim that

$$\tau(g) = \sum_i (-1)^i \mathrm{tr}(g_* H_i(X, \mathbb{Q})) = \sum_i (-1)^i \mathrm{tr}(g_* H_i(L^n, L^{n-1}, \mathbb{Q})).$$

Once we've established the claim, we will see that since $H_i(L^n, L^{n-1}, \mathbb{Q}) = \bigoplus_{\sigma \in L^n} \mathbb{Q} \cdot \sigma$, that as $g(\sigma) \neq \sigma$ for each $\sigma \in L^n$, the map g_* on the homology of pairs of L has no 1's appearing on the diagonal, hence has trace zero.

To prove the claim, we use another lemma. For any map of SESs of vector spaces over a field

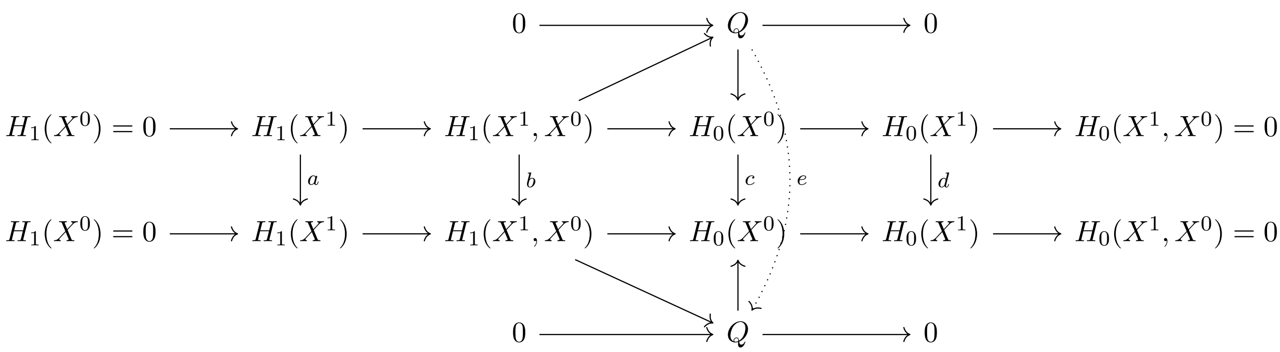
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

we have $\mathrm{tr}(\beta) = \mathrm{tr}(\alpha) + \mathrm{tr}(\gamma)$. The proof is left as an exercise; it uses the fact that such sequences are always split.

We show that the lemma implies the claim in dimension 1. In higher dimensions, an inductive procedure proves the implication in general. So we suppose that $X = X^1$ with $g : X \to X$. We want

$$\sum_i (-1)^i \mathrm{tr}(g_* H_i(X, \mathbb{Q})) = \mathrm{tr}(g_* H_0(X^0, X^{-1})) - \mathrm{tr}(g_* H_1(X^1, X^0)). \quad (1)$$

Consider the diagram



where we set $Q := \mathrm{coker}(H_1(X^1) \to H_1(X^1, X^0)) = \ker(H_0(X^0) \to H_0(X^1))$. We apply the lemma to the SESs ending in Q, with "vertical" maps given by a, b, e. By the lemma, we have $\mathrm{tr}(b) = \mathrm{tr}(a) + \mathrm{tr}(e)$. We apply the lemma also the the SESs starting with Q, with vertical maps given by e, c, d. By the lemma we have $\mathrm{tr}(c) = \mathrm{tr}(e) + \mathrm{tr}(d)$. The two equations together imply that $\mathrm{tr}(c) - \mathrm{tr}(b) = \mathrm{tr}(d) - \mathrm{tr}(a)$. This implies (1), by comparing the first term on the left with the first term on the right, and likewise for the second terms.

When $\dim(X) > 1$, one inducts on this procedure. The details are left as an exercise. This completes the proof. □

3.8.3 Vector fields on spheres

We pose the question: when does S^n admit a nonvanishing vector field? Equivalently, for which n does there exist a map $v : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ such that $v(x) \cdot x = 0$ for each $x \in S^n$? Normalizing v, we see that this is equivalent to finding $v : S^n \to S^n$ such that $v(x) \cdot x = 0$ for each $x \in S^n$. (We'll see that this is really an application of the second interesting theorem.)

Example. When $n = 1$, we obtain such a v by anchoring iz at the point $z \in S^1$. Checking that this works is left as an exercise.

Theorem 3.50. *For n even, this can not be done. That is, for n even, there does not exists a nowhere-vanishing vector field on S^n.*

Proof. Assume such a v exists, that $v(x) \cdot x = 0$ for all $v \in S^n$. Consider the homotopy

$$H : S^n \times I \to S^n \qquad H(x,t) = x\cos(\pi t/2) + v(x)\sin(\pi t/2).$$

We check that this is well-defined, i.e. maps into S^n. Consider

$$||H(x,t)|| = H(x,t) \cdot H(x,t) = ||x|| \cos^2(\pi t/2) + ||v(x)|| \sin^2(\pi t/2) = 1$$

as required, after applying that $v(x) \cdot x = 0$ and that $x \in S^n$ has norm 1. We have $H(x, 0) = x$ and $H(x, 1) = v(x)$, so v is homotopic to id_{S^n}.

It follows that $\tau(v) = \tau(\mathrm{id}) = \chi(S^n) = 2$, as n is even. The Lefschetz fixed point theorem tells us there is a fixed point, however as $v(x) \cdot x = 0$ for all $x \in S^n$, this is a contradiction. This concludes the proof. □

3.9 Kunneth formulas

The goal of this section is to compute the homology of a product of spaces in terms of the homology of its factors.

Example. We computed on homework $H_{i+n}(X \times S^n) = (H_i(X) \otimes H_n(S^n)) \oplus (H_{i+n}(X) \otimes H_0(S^n)) \simeq H_i(X) \oplus H_{i+n}(X)$, given a certain formula. We will generalize this construction with Kunneth formulas.

There are three steps.

Step 1. First, introduce tensor products of chain complexes: $\otimes : \mathbf{Ch}(\mathbf{Ab}) \times \mathbf{Ch}(\mathbf{Ab}) \to \mathbf{Ch}(\mathbf{Ab})$ taking $(K, L) \mapsto K \otimes L$.

Step 2. Obtain the algebraic Kunneth formula to calculate $H_*(K \otimes L)$ in terms of $H_*(K)$ and $H_(L)$.

Example. We may do this for any category of modules, in particular k-vector spaces. In this case, the Kunneth formula will be

$$H_n(K \otimes_k L) = \bigoplus_{i+j=n} H_i(K) \otimes H_j(L).$$

Step 3. Prove the Eilenberg-Zilber theorem: for X, Y spaces, then

$$C_*(X \times Y) \simeq C_*(X) \otimes C_*(Y)$$

is a homotopy equivalence.

Combining these steps, we obtain the topological Kunneth formula.

Example. For k a field, $H_n(X \times Y, k) \cong \bigoplus_{i+j=n} H_i(X, k) \otimes_k H_j(Y, k)$.

Recall that there exists a functor $\otimes : \mathbf{Ab} \times \mathbf{Ab} \to \mathbf{Ab}$ taking $(M, N) \to M \otimes N$ or $M \otimes_{\mathbb{Z}} N$ which satisfies several properties, such as (canonical) symmetry, $M \otimes -$ commutes with all colimits, etc.

Example. We have the following identities.

1. $M \otimes_{\mathbb{Z}} \mathbb{Z} = M$.
2. $M \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = M/nM$.
3. $M \otimes_{\mathbb{Z}} \mathbb{Q}$ is the "rationalization of M", which is a $\mathbb{Q}$-vector space. When $M = \mathbb{Z}^{\oplus r} \oplus T$ where T is torsion, then $M \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}^{\oplus r}$.
4. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$.

Definition 3.51. Set $\otimes : \mathbf{Ch}(\mathbf{Ab}) \times \mathbf{Ch}(\mathbf{Ab}) \to \mathbf{Ch}(\mathbf{Ab})$ taking $(K, L) \mapsto K \otimes_{\mathbb{Z}} L$ by

$$(K \otimes_{\mathbb{Z}} L)_n = \bigoplus_{i+j=n} K_i \otimes_{\mathbb{Z}} L_j$$

with, for $a \in K_i$, $b \in L_j$,

$$d_{K \otimes L}(a \otimes b) = d_K(a) \otimes b + (-1)^i a \otimes d_L(b).$$

We need to check that tensor products of complexes are complexes. Indeed:

$$\begin{aligned} d^2(a \otimes b) &= d(d(a) \otimes b + (-1)^i a \otimes d(b)) \\ &= d(d(a) \otimes b) + d(b + (-1)^i d(b)) \\ &= d^2(a) \otimes b + (-1)^{i+1} da \otimes db + (-1)^i (da \otimes db + (-1)^i a \otimes d^2(b)) \\ &= 0. \end{aligned}$$

Remark. We think of the tensor as a 2-complex:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & K_2 \otimes L_0 & \longrightarrow & K_1 \otimes L_0 & \longrightarrow & K_0 \otimes L_0 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & K_2 \otimes L_1 & \longrightarrow & K_1 \otimes L_1 & \longrightarrow & K_0 \otimes L_1 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & K_2 \otimes L_0 & \longrightarrow & K_1 \otimes L_0 & \longrightarrow & K_0 \otimes L_0 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

where the (direct) sums along the diagonals give the terms of the tensor product complex. The differentials, pictured this way, go up and to the right.

Example. We consider the following examples.

1. From $M, N \in \mathbf{Ab}$ we obtain $M[0], N[0] \in \mathbf{Ch}(\mathbf{Ab})$, with
$$M[0] \otimes N[0] \cong (M \otimes N)[0]$$
and more generally
$$M[i] \otimes N[j] \cong (M \otimes N)[i + j].$$

2. $K = (\mathbb{Z} \xrightarrow{p} \mathbb{Z})$, $L = N[0]$ for $N \in \mathbf{Ab}$. Then
$$K \otimes L = (N \xrightarrow{p} N).$$
We have also
$$H_i(K \otimes L) = \begin{cases} N/pN & i = 0 \\ \{x \in N : px = 0\} & i = 1 \\ 0 & \text{else.} \end{cases}$$
We remark here that if $N = \mathbb{Z}/p$, then both K and L have only zeroth nonvanishing homology, but $K \otimes L$ has H_i.

3. If k is a field, and $K, L \in \mathbf{Ch}(\mathbf{Vect}_k)$, then
$$H_n(K \otimes L) \cong \bigoplus_{i+j=n} H_i(K) \otimes H_j(L)$$
and the proof of this is left as an exercise.

3.9.1 Algebraic Kunneth formulas

Definition 3.52. For $M, N \in \mathbf{Ab}$, choose a SES
$$0 \to K \to P \to M \to 0$$
with P, K free abelian groups[7]. We define
$$\mathrm{Tor}(M, N) = \ker(K \otimes N \xrightarrow{d} P \otimes N).$$
This is a posteriori dependent on K and P.

[7]When we are not working over $\mathbb{Z}$, a PID, we cannot choose such a resolution. In this more general case, one needs projective resolutions.

Example. We have:

1. $\mathrm{Tor}(\mathbb{Z}/p, N)$:
$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to 0$$
so
$$\mathrm{Tor}(\mathbb{Z}/p, N) = \ker(N \xrightarrow{p} N) = N[p] = \{x \in N : px = 0\}.$$

2. N torsion free $\Rightarrow \mathrm{Tor}(M, N) = 0$. To see this, first suppose that N is finitely generated, so that $N \cong \mathbb{Z}^{\oplus r}$. Thus $K \otimes N \to P \otimes N$ is injective. In general write $N = \bigcup_i N_i$ where $N_i \subset N$ is finitely generated and torsion free, and reduce to the statement about N_i using the fact that $\otimes$ and ker commute with all direct limits.

Remark. $\mathrm{Tor}(M, N)$ is independent of the choice of presentation. This requires proof, which is left as an exercise.

We will need the following lemma for the construction of algebraic Kunneth formulas.

Lemma 3.53. *Tor is symmetric: Tor$(M, N) =$ Tor(N, M).*

Proof. Choose resolutions
$$0 \to K \to P \to M \to 0,$$
$$0 \to R \to Q \to N \to 0$$
for M and N. We obtain the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K \otimes R & \longrightarrow & P \otimes R & \longrightarrow & M \otimes R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle a} & & \\
0 & \longrightarrow & K \otimes Q & \longrightarrow & P \otimes Q & \longrightarrow & M \otimes Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & K \otimes N & \xrightarrow{b} & P \otimes N & \longrightarrow & M \otimes N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

after tensoring. The snake lemma[8] gives us a map $\ker(a) \to \ker(b)$ which fits into a long exact sequence
$$0 \to \ker(a) \to K \otimes N \xrightarrow{b} P \otimes N \to M \otimes N \to 0$$
so that $\ker(a) \simeq \ker(b)$, as required. □

Theorem 3.54 (Algebraic Künneth Formula)**.** *For $K, L \in$ **Ch**(**Ab**) chain complexes of free abelian groups, then for each n there is a SES functorial in K and L*
$$0 \to \bigoplus_{i+j=n} H_i(K) \otimes H_j(L) \to H_n(K \otimes L) \to \bigoplus_{i+j=n-1} \mathit{Tor}(H_i(K), H_j(L)) \to 0.$$

Remark. Note the following.

1. This sequence is non-functorially split, so that we may compute $H_*(K \otimes L)$ via $H_*(K)$ and $H_*(L)$.

[8]See 3.15 for the statement.

2. If K (or L) is exact, the theorem implies that so is $K \otimes L$.

3. If $K \to K'$ induces an isomorphism on homology, then so does $K \otimes L \to K' \otimes L$.

4. The proof of the theorem will show that when k is a field and $K, L \in \mathbf{Ch}(\mathbf{Vect}_k)$, we have $H_n(K \otimes L) \simeq \oplus_{i+j=n} H_i(K) \otimes H_j(L)$.

Example. Set $K = L = \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z}$. The homology of either chain complex is isolated at degree 0 as $\mathbb{Z}/p\mathbb{Z}$. The formula gives that $H_0(K \otimes L) = H_1(K \otimes L) = \mathbb{Z}/p\mathbb{Z}$ with all other homology vanishing.

Proof of 3.54. Let $K, L \in \mathbf{Ch}(\mathbf{Ab})$ be chain complexes of free abelian groups.

1. Assume that $d_L = 0$ so that each $H_i(L) = L_i$ is free. It follows that $\mathrm{Tor}(H_i(K), H_j(L)) = 0$ for each i, j. Thus

$$H_n(K \otimes L) = \frac{\ker(d_K \otimes 1 : (K \otimes L)_n \to (K \otimes L)_{n-1})}{\mathrm{im}(d_K \otimes 1 : (K \otimes L)_{n+1} \to (K \otimes L)_n)}.$$

As L_i is free, the functor $L_i \otimes -$ is exact. Thus

$$\begin{aligned} \ker(d_K \otimes 1 : (K \otimes L)_n \to (K \otimes L)_{n-1}) &\simeq \bigoplus_{i+j=n} \ker(d_K \otimes 1 : K_i \otimes L_j \to K_{i-1} \otimes L_j) \\ &\overset{\text{exactness}}{\simeq} \bigoplus_{i+j=n} \ker(d_K : K_i \to K_{i-1}) \otimes L_j, \end{aligned}$$

and similarly for $\mathrm{im}(d_K \otimes 1)$. We obtain

$$H_n(K \otimes L) = \bigoplus_{i+j=n} \frac{\ker(d_K : K_i \to K_{i-1})}{\mathrm{im}(d_K : K_{i+1} \to K_i)} \otimes L_j \simeq \bigoplus_{i+j=n} H_i(K) \otimes H_j(L).$$

2. We reduce to the first case. [Remainder of proof to be filled in later.]

□

Algebraic Kunneth formulas dictate that if K and L are chain complexes of free abelian groups then for all n there is a functorial SES

$$0 \to \bigoplus_{i+j=n} H_i(K) \otimes H_j(L) \to H_n(K \otimes L) \to \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(K), H_j(L)) \to 0.$$

Moreover that this sequence is non-functorially split. We now prove a topological version of this theorem.

Theorem 3.55 (Eilenberg-Zilber)**.** *For spaces X, Y there exist functorial maps*

$$\begin{aligned} \times : C_*(X) \otimes C_*(Y) &\longrightarrow C_*(X \times Y) \\ \theta : C_*(X \times Y) &\longrightarrow C_*(X) \otimes C_*(Y) \end{aligned}$$

such that $\times \circ \theta$ and $\theta \circ \times$ are homotopic to the identities on X and Y.

Corollary 3.56 (Künneth formula)**.** *We see from the above algebraic theorem that there is a functorial SES*

$$0 \to \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \to H_n(X \times Y) \to \bigoplus_{i+j=n-1} \mathit{Tor}(H_i(X), H_j(Y)) \to 0$$

which is non-functorially split.

Corollary 3.57. *If we suppose that $\bigoplus_i H_i(X)$ and $\bigoplus_j H_j(Y)$ are finitely generated, then $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.*

Proof of 3.57. First, it is an exercise to show that for $K, L \in \mathbf{Ch}(\mathbf{Vect}_k)$, where k is a field, then when $\bigoplus_i H_i(X)$ and $\bigoplus_j H_j(Y)$ finitely generated, then $\chi(K \otimes_k L) = \chi(K) \cdot \chi(L)$ (one can use the Künneth formula and notice that in this case, Tor vanishes).

Now apply that exercise to $K = C_*(X; \mathbb{F}_2)$ and $L = C_*(Y; \mathbb{F}_2)$. □

Example. For $X = \mathbb{RP}^2$, we compute the homology of $X \times X$. We have

$$H_i := H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2 & i = 1 \\ 0 & \text{else.} \end{cases}$$

Observe that $H_0(Y) = \mathbb{Z}$. Then

$$\begin{aligned} H_1(Y) &= H_0 \otimes H_1 \oplus H_1 \otimes H_0 \oplus \mathrm{Tor}(H_0, H_0) \\ &= \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus 0 \\ &= (\mathbb{Z}/2)^{\oplus 2}. \end{aligned}$$

We may do a similar calculation for $i = 3$, and obtain $H_3(X \times X) = \mathbb{Z}/2$.

Remark. If X and Y are CW complexes, one may prove the Künneth formula "by hand", which you can find in Hatcher.

Proof of 3.55. First we construct the map $\times$. Given $\sigma_p : \Delta^p \to X$, $\sigma_q : \Delta^q \to Y$ we aim to construct $\sigma_p \times \sigma_q \in C_{p+q}(X \times Y)$ by induction on $p+q$. Observe that if $p = 0$ then σ_0 represents some $x \in X$. Then set $\sigma_0 \times \sigma_q : \Delta^q \simeq \{*\} \times \Delta^q \xrightarrow{(x, \sigma_q)} X \times Y$, and likewise for $q = 0$.

Now we state the following proposition.

Proposition 3.58. *For all X, Y, there is a map*

$$\begin{aligned} \times : C_*(X) \otimes C_*(Y) &\longrightarrow C_*(X \times Y) \\ a \otimes b &\longmapsto a \times b \end{aligned}$$

such that

1. *$\times$ is the obvious map if one factor is a 0-chain.*
2. *$\times$ is compatible with the differential, so that $d(a \times b) = da \times b + (-1)^p a \times db$ for each $a \in C_p(X), b \in C_q(Y)$.*
3. *$\times$ is functorial in X and Y.*

Remark. We are not claiming that this map is unique; its construction will involve a choice (which does not mess up functoriality).

Proof of 3.58. We induct on $p + q = n$. When $n = 0, 1$, we use the obvious map. Now, assume we have suitably defined $\times$ on all spaces X, Y and p, q such that $p + q \leq n - 1$.

Consider the special case where X and Y are simplices, and $i_p : \Delta^p \to \Delta^p$ and $i_q : \Delta^q \to \Delta^q$ are the identities. We suppose $n \geq 2$. If $i_p \times i_q$ existed, then

$$d(i_p \times i_q) = di_p \times i_q + (-1)^p i_p \times di_q \in C_{n-1}(\Delta^p \times \Delta^q) \tag{2}$$

is well-defined by induction. Moreover, $d(\text{RHS of (2)}) = 0$. This is not given at present, but one can apply the Leibniz rule to derive it. Thus RHS of (2) $\in Z_{n-1}(\Delta^p \times \Delta^q)$. However $\Delta^p \times \Delta^q$

is contractible, and $n \geq 2$, so $B_{n-1}(\Delta^p \times \Delta^q) = Z_{n-1}(\Delta^p \times \Delta^q)$. Thus the RHS of (2) is given by $d(\alpha)$ for some $\alpha \in C_n(\Delta^p \times \Delta^q)$. We define $i_p \times i_q = \alpha \in C_n(\Delta^p \times \Delta^q)$.

In general, for spaces X and Y, given $\sigma : \Delta^p \to X$ and $\tau : \Delta^q \to Y$, we have $\sigma = \sigma_*(i_p) \in C_p(X)$. Similarly $\tau = \tau_*(i_q)$. We apply functoriality and the definitions of $i_p \times i_q$ to define

$$\sigma \times \tau = (\sigma, \tau)_*(i_p \times i_q) \in C_n(X \times Y)$$

where $(\sigma, \tau) : \Delta^p \times \Delta^q \to X \times Y$ is the product map[9].

It follows from construction that this is functorial. We still need to check the Leibniz rule. So:

$$\begin{aligned} d(\sigma \times \tau) &= d((\sigma, \tau)_*(i_p \times i_q)) \\ &= (\sigma, \tau)_*(d(i_p \times i_q)) \\ &\overset{(2)}{=} (\sigma, \tau)_*(di_p \times i_q + (-1)^p i_p \times di_q) \\ &= (\sigma, \tau)_*(di_p \times i_q) + (-1)^p (\sigma, \tau)_*(i_p \times di_q) \\ &\overset{i}{=} \sigma_*(di_p) \times \tau_*(i_q) + (-1)^p \sigma_*(i_p) \times \tau_*(di_q) \\ &= d(\sigma) \times \tau + (-1)^p \sigma \times \tau. \end{aligned}$$

where the step i comes from induction. □

It remains to prove the theorem! □

4 Cohomology

[Missing a lecture here.]

Recall: we have a functor $\mathbf{Ch}(\mathbf{Ab}) \to \mathbf{CoCh}(\mathbf{Ab})$ taking $K_\bullet \mapsto (K_\bullet)^\vee = \mathrm{Hom}(K_\bullet, \mathbb{Z})$. For a space X, we have $C^*(X) = \mathrm{Hom}(C_*(X), \mathbb{Z})$ and $H^i(X) = H^i(C^*(X))$.

More generally, for A an abelian group, $C^*(X; A) = \mathrm{Hom}(C_*(X, \mathbb{Z}), A)$ and $H^i(X; A) = H^i(C^*(X; A))$.

Remark. If A is a ring, then $C^*(X; A) = \mathrm{Hom}_{\mathbb{Z}}(C_*(X; \mathbb{Z}), A) \simeq \mathrm{Hom}_A(C_*(X; \mathbb{Z}) \otimes A, A) \simeq \mathrm{Hom}_A(C_*(X; A), A)$ where the second step comes from Hom-Tensor adjunction, and the third since $C_n(X) \otimes A \simeq C_n(X; A)$. This is most useful when A is a field, so that the right-hand side is computed using linear algebra.

4.1 Ext and universal coefficient theorems

First we consider the case of k a field. Take

$$\begin{gathered} \mathbf{Ch}(k) \to \mathbf{CoCh}(k) \\ M_\bullet \mapsto (M_\bullet)^\vee. \end{gathered}$$

Observe now that $\mathrm{Hom}_k(-, k)$ is an exact functor on k-vector spaces. It follows that $H_i(M_\bullet^\vee) \simeq \mathrm{Hom}_k(H_i(M_\bullet), k)$.

Corollary 4.1. *For X a space, $H^i(X; k) \simeq Hom_k(H_i(X; k), k)$.*

This becomes significantly harder when coefficients are not taken in a field; we want to find an analogue over $\mathbb{Z}$.

[9]This argument is known as the "acyclic models argument". Acyclicity records the fact that we used $Z_n = B_n$ in the proof here.

Definition 4.2. For $M, N \in \mathbf{Ab}$, choose a SES

$$0 \to K \to P \to M \to 0$$

where K and P are free. We define $\mathrm{Ext}(M, N) = \mathrm{coker}(\mathrm{Hom}(P, N) \to \mathrm{Hom}(K, N))$.

Remark. As we saw with Tor, this construction does not depend on the choice of resolution.

Example. Consider $M = \mathbb{Z}$. We see that in this case, $\mathrm{Ext}(M, -) = 0$, since $\mathbb{Z}$ has a trivial free resolution (in the notation of the above definition, $K = 0$, $P = \mathbb{Z}$).

Example. $M = \mathbb{Z}/k$. We claim that $\mathrm{Ext}(M, N) \simeq N/kN$. To see this, consider the resolution

$$0 \to \mathbb{Z} \xrightarrow{\cdot k} \mathbb{Z} \to \mathbb{Z}/k \to 0$$

and the claim follows.

Remark. We record the following properties of Ext.

1. Ext is not symmetric. Like Hom, it is contravariant in the first factor and covariant in the second.

2. $\mathrm{Ext}(M, N)$ is functorial in both M and N.

3. $\mathrm{Ext}(\oplus_{i \in I} M_i, N) \simeq \prod_{i \in I} \mathrm{Ext}(M_i, N)$. To show this, use that $\mathrm{Hom}(\oplus A_i, B) \simeq \prod \mathrm{Hom}(A_i, B)$, and that taking products is exact.

4. $\mathrm{Ext}(M, -) = 0$ if and only if M is free. The reverse implication follows from the above example. For the forward implication, choose a resolution

$$0 \to K \to P \to M \to 0$$

   for K and P free. We will split this sequence, in particular the map $K \to P$. As $\mathrm{Ext}(M, N) = 0$, the map $\mathrm{Hom}(P, K) \to \mathrm{Hom}(K, K)$ is surjective. Therefore there exists some $g : P \to K$ such that $g\alpha = \mathrm{id}_K$. Therefore $P \simeq K \oplus M$, and M is free.

5. $\mathrm{Ext}(-, N) = 0$ if and only if N is divisible, i.e. $kN = N$ for all nonzero $k \in \mathbb{Z}_{>0}$ (the reverse direction uses Baer's criterion for injective modules).

6. For M finite (torsion), then $\mathrm{Ext}(M, \mathbb{Z})$ is non-canonically identified with M: using the classification of finitely generated abelian groups, we may reduce to the cyclic case. We checked this case above. (More naturally, $\mathrm{Ext}(M^{finite}, \mathbb{Z}) = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$, but we won't need this.)

Remark. Why is it called Ext? For $M, N \in \mathbf{Ab}$, it turns out that $\mathrm{Ext}(M, N) =$ "extensions of M by N", i.e. SESs of the form

$$0 \to N \to ? \to M \to 0$$

modulo isomorphisms (of SESs), and split sequences.

Theorem 4.3 (Universal coefficients for Hom)**.** *For $K \in \mathbf{Ch}(\mathbf{Ab})$ a complex of free abelian groups, for every $N \in \mathbf{Ab}$ and $n \in \mathbb{Z}$, we have a functorial SES*

$$0 \to \mathit{Ext}(H_{n-1}(K), A) \to H^n(\mathit{Hom}(K, A)) \to \mathit{Hom}(H_n(K), A) \to 0,$$

and moreover this is non-functorially split.

We don't prove 4.3, as its proof is similar to that of the algebraic Künneth formula.

Example. Consider $K = (\mathbb{Z} \xrightarrow{n} N)$ in degrees 1 and 0, so homology is isolated at degree 0 and $H_0(K) = \mathbb{Z}/n$. Set $M^* = \mathrm{Hom}(K_*, \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{n} \mathbb{Z}$, where degrees are flipped w/r/t K. Homology is now isolated at degree 1.

One should check: $H^1(M) \simeq \mathrm{Ext}(H_0(K), \mathbb{Z})$.

Corollary 4.4. *For X a space such that $H_{n-1}(X)$ and $H_1(X)$ are finitely generated (this will be standard in practice), write $H_{n-1}(X) = \mathbb{Z}^{\oplus r} \oplus T_{n-1}$ where T_{n-1} is torsion. Then $H^n(X) = (H_n(X)/\langle torsion \rangle) \oplus T_{n-1}$, non-canonically.*

Proof. We use the theorem to write

$$H^n(X) = H^n(C^*(X)) = H^n(\mathrm{Hom}(C_*(X), \mathbb{Z})) \overset{UCT}{\simeq} \mathrm{Hom}(H_n(X), \mathbb{Z}) \oplus \mathrm{Ext}(H_{n-1}(X)), \mathbb{Z}$$

and identifying $\mathrm{Hom}(H_n(X), \mathbb{Z}) \simeq H_n(X)/\text{torsion}$, and $\mathrm{Ext}(H_{n-1}(X), \mathbb{Z}) \simeq T_{n-1}$, we obtain the needed statement. □

Example. $H^*(S^1) = \mathbb{Z}$ for $* = 0, 1$, but the degrees are flipped w/r/t homology, as there is no torsion in $H_*(S^1)$. Similarly for S^n.

Example. $H^*(\mathbb{C}P^n) = \mathbb{Z}$ for $*$ even, and zero otherwise, by the same reasoning.

Example. $H^*(\mathbb{R}P^2) = \mathbb{Z}$ for $* = 0$, 0 for $* = 1$ (because $\mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}) = 0$ and $\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}) = 0$, applying the UCT), and $\mathbb{Z}/2$ for $* = 2$, similarly. The rest are zero.

We enumerate some properties of H^*.

1. $H^0(X) \cong \{\text{continuous maps } X \to \mathbb{Z}\}$, for X locally path-connected.
2. For X path-connected, $x \in X$, $H^1(X) \cong \mathrm{Hom}(\pi_1(X, x), \mathbb{Z})$.

Proof. We have:

We may use UCT to see this, but may also we may use bare bones. Both sides of the statement take disjoint unions to products, so we may reduce to the case where X is path-connected. We want to show that when X is path-connected, $H^0(X) \simeq \mathbb{Z}$. We have

$$\begin{aligned} H^0(X) &= \ker(C^0(X) \xrightarrow{d} C^1(X)) \\ &= \ker(\mathrm{Hom}(\oplus_{x \in X} \mathbb{Z} \cdot x, \mathbb{Z}) \to \mathrm{Hom}(\oplus_{a \in A} \mathbb{Z} \cdot a, \mathbb{Z})) \quad A = \{\text{path-components of } X\} \\ &= \ker(\mathrm{Maps}(X, \mathbb{Z}) \xrightarrow{d} \mathrm{Maps}(\text{paths in } X, \mathbb{Z})) \end{aligned}$$

where given $f : X \to \mathbb{Z}$, we have

$$(df)(\gamma) = f(\gamma(0)) - f(\gamma(1))$$

for a path γ in X. Thus $f \in \ker(d)$ if and only if $f(x) = f(y)$ for all x, y connected by a path in X. Thus $\ker(\mathrm{Maps}(X, \mathbb{Z}) \xrightarrow{d} \mathrm{Maps}(\text{paths in } X, \mathbb{Z}))$ is equal to the constant maps $X \to \mathbb{Z}$, isomorphic to $\mathbb{Z}$.

We may also use UCT to see this abstractly. We have

$$\begin{aligned} H^1(X) &= \frac{\ker(C_1(X) \to C_2(X))}{C_0(X) \to C_1(X)} \\ &= \frac{\ker(\mathrm{Maps}(\text{paths in } X) \xrightarrow{d^1} \mathrm{Maps}(\Delta\text{'s in } X, \mathbb{Z}))}{\mathrm{im}(\mathrm{Maps}(X, \mathbb{Z}) \xrightarrow{d^0} \mathrm{Maps}(\text{paths in } X, \mathbb{Z}))} \end{aligned}$$

and we have $\mathrm{im}(d^0) = \{f \in \mathrm{Maps}(\text{paths in } X, \mathbb{Z}) \mid f \text{ “only depends on endpoints"}\}$. To understand d^1, for a simplex $\sigma \subset X$ composed of edges a, b, c where c's orientation opposes the ordering, we have

$$(d^1 g)(\sigma) = g(a) - g(c) + g(b).$$

Thus $\ker(d^1) = \{g \mid g(a) + g(b) = g(c) \text{ for all } \sigma \text{ as above}\}$. We may rephrase this as,

$$\ker(d^1) = \{g \mid g \text{ is additive under composition of paths and only depends on the path up to homotopy}\}.$$

We rewrite

$$H^1(X) = \frac{\{f : \{\text{paths in } X\}/\text{homotopy} \to \mathbb{Z} \mid f \text{ additive}\}}{\{g : \{\text{paths in } X\}/\text{homotopy} \to \mathbb{Z} \mid g \text{ only depends on endpoints}\}}$$

Choose paths $\gamma_y : y \rightsquigarrow x$ for all $y \in X$.

$$H^1(X) = \frac{\{f : \{\text{loops based at } x \text{ in } X\}/\text{homotopy} \to \mathbb{Z} \mid f \text{ additive}\}}{0} = \mathrm{Hom}(\pi_1(X, x), \mathbb{Z})$$

where the fact that f is additive implies that these are homomorphisms. □

3. There exist relative cohomologies $H^*(X, A)$, obtained by dualizing homology of a pair:

   $$C^*(X, A) = \mathrm{Hom}(C_*(X, A), \mathbb{Z}), \quad H^i(X, A) = H^i(C^*(X, A)).$$

   Explicitly, this is

   $$0 \to C^n(X, A) \to C^n(X) \overset{\text{restrict}}{\longrightarrow} C^n(A)$$

   i.e., functions on n-simplices of X that vanish when restricted to A.

Proposition 4.5. *There is a long exact sequence of a pair*

$$\cdots \longrightarrow H^i(X, A) \longrightarrow H^i(X) \longrightarrow H^i(A) \longrightarrow H^{i+1}(X, A) \longrightarrow H^{i+1}(X) \longrightarrow H^{i+1}(A) \longrightarrow \cdots.$$

Proof. There is a SES

$$0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$$

which is termwise split. Thus $\mathrm{Hom}(-, \mathbb{Z})$ produces a SES

$$0 \to C^*(X, A) \to C^*(X) \to C^*(A) \to 0$$

and now take the LES of H^*. □

Remark. There is a UCT for relative cohomology as well.

4. $C^*(-)$ and $H^*(-)$ are contravariant functors. For $f : X \to Y$, we write f^* for the induced map.

5. Homotopy invariance: for homotopic $f, g : X \to Y$ the maps $f^*, g^* : C^*(Y) \to C^*(X)$ are homotopic, hence $f^* = g^*$ on homology.

6. There is excision in cohomology.

7. Additionally, we have: CW cohomology for Δ-complexes, the LES in CW cohomology, the Mayer-Vietoris sequence, the LES of a good pair, the Eilenberg-Steenrod axioms, and more for cohomology. All proofs follow either from dualizing the steps of the proofs for homology, or by carrying through an entirely similar proof.

4.2 Cup products

For any space X, we have the diagonal map $\Delta : X \to X \times X$. Functoriality implies that there is $\Delta_* : H_*(X) \to H_*(X \times X)$ and the RHS is "roughly" $H_*(X) \otimes H_*(X)$. This endows $H_*(X)$ with the structure of a *coalgebra*, which is (in some sense) the dual of a ring structure. As this is slightly inaccessible, we have the following dual picture in cohomology.

We will have natural maps

$$\begin{gathered} H^*(X) \otimes H^*(X) \to H^*(X \times X) \xrightarrow{\Delta^*} H^*(X) \\ (f, g) \mapsto f \times g \end{gathered}$$

whose composition is the *cup product*, denoted $(f, g) \mapsto f \cup g$. We describe the first map in the diagram. For $f \in C^k(X)$ and $g \in C^l(X)$, for $n = k + l$, have

$$(f \times g)(\sigma : \Delta^n \to X \times X) = f(\sigma|_{k-\text{dim face}}) \cdot g(\sigma|_{l-\text{dim face}})$$

Theorem 4.6. *The cup product makes $H^*(X)$ into a commutative graded ring, in the sense that $f \cup g = (-1)^{\deg(f)} g \cup f$.*

Remark. The same also holds for $H^*(X, k)$ for any commutative ring k.

We establish some notation. As always, we write $\Delta^n = [v_0, ..., v_n]$.

Definition 4.7. For X a space, the cup product is defined

$$\begin{gathered} C^k(X) \times C^l(X) \xrightarrow{\cup} C^{k+l}(X) \\ (\phi \cup \psi)(\sigma : \Delta^n \to X) = \phi(\sigma|_{[v_0,...,v_k]}) \cdot \psi(\sigma|_{[v_k,...,v_{k+l}]}) \end{gathered}$$

for $n = k + l$.

We have the following properties of cup products.

1. Cup products make sense for any ring R, applied to $C^*(X, R)$.

2. Cup products are associative:

$$\begin{array}{ccc} C^k(X) \times C^l(X) \times C^m(X) & \longrightarrow & C^{k+l}(X) \times C^m(X) \\ \downarrow & & \downarrow \\ C^k(X) \times C^{l+m}(X) & \longrightarrow & C^{k+l+m}(X) \end{array}$$

   commutes. We check this as follows:

$$\begin{aligned} (f \cup (g \cup h))(\sigma : \Delta^{k+l+m} \to X) &= f(\sigma|_{[v_0,...,v_k]}) \cdot (g \cup h)(\sigma|_{[v_k,...,v_{k+l+m}]}) \\ &= f(\sigma|_{[v_0,...,v_k]}) \cdot g(\sigma|_{[v_k,...,v_{k+l}]} \cdot h(\sigma|_{[v_{k+l},...,v_{k+l+m}]})) \end{aligned}$$

   and undoing this finishes the computation.

3. The cochain $\epsilon \in C^0(X) = \text{Hom}(C_0(X), \mathbb{Z}) = \text{Maps}(X, \mathbb{Z})$ that sends each $x \in X$ to 1, is a unit for cup products. We write $1 \in C^0(X)$.

4. The cup product is bilinear:

$$(\phi_1 + \phi_2) \cup \psi = \phi_1 \cup \psi + \phi_2 \cup \psi$$

   and similarly on the left-hand side. We check this as follows:

$$\begin{aligned} ((\phi_1 + \phi_2) \cup \psi)(\sigma : \Delta^n \to X) &= (\phi_1 + \phi_2)(\sigma|_{[v_0,...,v_k]}) \cdot \psi(\sigma|_{[v_0,...,v_{k+l}]}) \\ &= (\phi_1(\sigma|_{[v_0,...,v_k]}) + \phi_2(\sigma|_{[v_0,...,v_k]})) \cdot \psi(\sigma|_{[v_k,...,v_{k+l}]}) \\ &= (\phi_1 \cup \psi + \phi_2 \cup \psi)(\sigma). \end{aligned}$$

Corollary 4.8. *$C^*(X)$ is a graded ring. So: $C^*(X) \otimes C^*(X) \xrightarrow{\cup} C^*(X)$ is a map of graded abelian groups.*

5. For $\phi \in C^k(X)$, $\psi \in C^l(X)$,

$$d(\phi \cup \psi) = d\phi \cup \psi + (-1)^{\deg(\phi)} \phi \cup d\psi$$

where the order of operations has $\cup$ first, then $+$. This implies that $C^*(X) \otimes C^*(X) \xrightarrow{\cup} C^*(X)$ is a map of chain complexes.

Proof. Observe:

$$\begin{aligned}(d\phi \cup \psi)(\sigma : \Delta^{k+l+1} \to X) &= (d\phi)(\sigma|_{[v_0,\ldots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\ldots,v_{k+l+1}]}) \\ &= \phi(d\sigma|_{[v_0,\ldots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\ldots,v_{k+l+1}]}) \\ &= \sum_{i=1}^{k+1} (-1)^i \phi(\sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\ldots,v_{k+l+1}]}).\end{aligned}$$

Similarly:

$$((-1)^k \phi \cup d\psi) = \sum_{i=k}^{k+l+1} (-1)^i \phi(\sigma|_{[v_0,\ldots,v_k]}) \cdot \psi(\sigma|_{[v_k,\ldots,\hat{v}_i,\ldots,v_{k+l+1}]}).$$

Now, we see that the last term of the first expression cancels the first term of the second expression. One can check that the rest adds up to $(\phi \cup \psi)(d\sigma)$, as required. □

Corollary 4.9. *Cup products pass to cohomology.*

Proof. The formula for $d(\phi \cup \psi)$ implies that the cup product of two cocycles is a cocycle. In fact $Z^*(X)$ contains 1, and is a graded subring of $C^*(X)$. Similarly, the cup product of a cocycle with a coboundary is a coboundary. $B^*(X)$ forms a graded ideal in $Z^*(X)$. These two formally imply the corollary, and that $H^*(X) = Z^*(X)/B^*(X)$ is a graded ring. □

Example. $H^*(S^n) = \mathbb{Z}[x]/(x^2)$, where $n > 0$ and $\deg(x) = n$.

6. Cup products are functorial. Given a map $f : X \to Y$ and $\phi, \psi \in C^*(Y)$, $f^*(\phi \cup \psi) = f^*(\phi) \cup f^*(\psi)$; it follows that the morphism $H^*(Y) \xrightarrow{f^*} H^*(X)$ is a map of graded rings.

Proof.

$$\begin{aligned}(f^*(\phi \cup \psi))(\sigma : \Delta^n \to X) &= (\phi \cup \psi)(f_*\sigma) \\ &= (\phi \cup \psi)(f\sigma : X \to Y) \\ &= \phi(f\sigma|_{[v_0,\ldots,v_k]}) \cdot \psi(f\sigma|_{[v_k,\ldots,v_n]}) \\ &= (f^*\phi)(\sigma|_{[v_0,\ldots,v_k]}) \cdot (f^*psi)(\sigma|_{[v_k,\ldots,v_n]}) \\ &= (f^*\phi \cup f^*\psi)(\sigma).\end{aligned}$$

□

7. Cup products are *graded commutative* up to homotopy. That is, $\phi \cup \psi = (-1)^{\deg \phi} \psi \cup \phi$ on $H^*(X)$.

We do not prove this, as the argument is too elaborate and combinatorial. See Hatcher.

Example. $X = S^2 \vee S^4$. We claim that $H^*(X) = \mathbb{Z}[x,y]/(x^2, y^2, xy)$, where $\deg(x) = 2$, $\deg(y) = 4$.

We have

$$H^*(S^2 \vee S^4) = \begin{cases} \mathbb{Z} & * = 0,2,4 \\ 0 & \text{else,} \end{cases}$$

via the UCT, and after noting that the homology of X has no torsion. We have

$$\begin{aligned} a &: X \to S^2 \quad \text{“collapse } S^4\text{"} \\ b &: X \to S^4 \quad \text{“collapse } S^2\text{"}. \end{aligned}$$

Which induce

$$\begin{aligned} a^* : H^*(S^2) &= \underbrace{\mathbb{Z}[x]/(x^2)}_{\deg(x)=2} \to H^*(X) \quad \text{with image in degree 2,} \\ b^* : H^*(S^4) &= \underbrace{\mathbb{Z}[y]/(y^2)}_{\deg(y)=4} \to H^*(X) \quad \text{with image in degree 4.} \end{aligned}$$

As $\otimes$-products form the coproducts in the category of commutative rings, we obtain

$$\begin{aligned} H^*(S^2) \otimes H^*(S^4) &\to H^*(X) \\ u \otimes v &\mapsto a^*(u) \otimes b^*(v). \end{aligned}$$

This map is, by construction, surjective. We have now a map

$$\mathbb{Z}[x]/(x^2) \otimes \mathbb{Z}[y]/(y^4) \to H^*(X).$$

Examining degrees, we see that $x \otimes y \mapsto 0$. We obtain

$$\mathbb{Z}[x,y]/(x^2, y^2, xy) \twoheadrightarrow H^*(X).$$

As this is a surjection of free abelian groups of rank 4 (forgetting multiplicative structure), it is an isomorphism. Its kernel is (isomorphic to) a free abelian group of rank zero.

We form external products to assist us with computation. For spaces X and Y, there is a natural map

$$\begin{aligned} H^k(X) \otimes H^l(Y) &\to H^{k+l}(X \times Y) \\ \phi \otimes \psi &\mapsto \mathrm{pr}_1^*(\phi) \cup \mathrm{pr}_2^*(\psi) =: \phi \times \psi \end{aligned}$$

where $\mathrm{pr}_1 : X \times Y \to X$ is the projection, and likewise for Y. Hence we get a map (which you can check):

$$H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$$

of commutative graded rings[10]

Also, for $X = Y$, we get from the diagonal map $\Delta : X \to X \times X$ we obtain

$$\Delta^*(\phi \times \psi) = \phi \cup \psi.$$

To see this:

$$\begin{aligned} &\Delta^*(\mathrm{pr}_1^*\phi \cup \mathrm{pr}_2^*\psi) \\ &\qquad = \Delta^*(\mathrm{pr}_1^*\phi) \cup \Delta^*(\mathrm{pr}_2^*\psi) \quad \Delta^* \text{ multiplicative} \\ &\qquad = (\mathrm{pr}_1 \circ \Delta)^*\phi \cup (\mathrm{pr}_2 \circ \Delta)^*\psi \quad H^* \text{ contravariant} \\ &\qquad = \phi \circ \psi. \end{aligned}$$

[10] A commutative graded ring is not the same thing as a graded commutative ring. The first is a graded ring which commutes in the sense of graded rings: it picks up a sign according to degree. The latter does not.

Theorem 4.10. *If X and Y are CW complexes and $H^i(Y)$ is finite free for all i, then $H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$ is an isomorphism of graded rings.*

Example. For $X = S^1 \times S^1$, this is

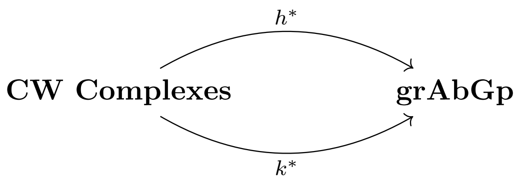
$$\begin{aligned} H^*(X) &\simeq H^*(S^1) \otimes H^*(S^1) \\ &\simeq \mathbb{Z}[x]/x^2 \otimes \mathbb{Z}[y]/y^2 \quad \deg(x) = \deg(y) = 1 \\ &\simeq \mathbb{Z}[x,y]/(x^2, y^2) \end{aligned}$$

where the adjunction occurs in the category of graded rings, i.e. $xy = -yx$. That is, this is isomorphic to $\bigwedge_{\mathbb{Z}}^*(\mathbb{Z} \cdot x \oplus \mathbb{Z} \cdot y)$.

Example. More generally, $H^*((S^1)^n) = \bigwedge_{\mathbb{Z}}^*(\mathbb{Z}^{\oplus n})$.

All of the details of the following proof sketch can be found in Hatcher.

Proof sketch of 4.10. We hold Y fixed and define 2 functors:



defined as

$$h^*(X) = H^*(X) \otimes H^*(Y), \qquad k^*(X) = H^*(X \times Y).$$

We have a natural transformation

$$\eta(-) : h^*(-) \to k^*(-).$$

Observe the following.

1. $\eta(\mathrm{pt})$ is an isomorphism.

2. Both $h^*(-)$ and $k^*(-)$ make sense for pairs (X, A) and satisfy: homotopy invariance, excision, LES of a pair (here we use that tensoring preserves long exact sequences when they are free), additivity (take disjoint unions to products; this uses that $H^*(Y)$ is finitely generated[11]).

Proposition 4.11. *Any $\eta : F^* \to G^*$ of functors on CW pairs that satisfy the above conditions is an isomorphism, provided that it is so when evaluated on a point.*

□

We compute the cohomology rings of complex and real projective space.

Example. Consider $X = \mathbb{RP}^n$. We will show that $H^*(\mathbb{RP}^n; \mathbb{Z}/2) = (\mathbb{Z}/2)[x]/(x^{n+1})$, where $\deg(x) = 1$. It will follow from a similar argument that $H^*(X) = \mathbb{Z}[x]/(x^{n+1})$, with $\deg(x) = 2$. For this example, we write $H^i(Y) = H^i(Y; \mathbb{Z}/2)$ for all spaces Y.

It is a corollary of the computation that there is no map $\mathbb{CP}^2 \to S^2$ inducing a nonzero map on H_2 (use the universal coefficient theorem). It follows from this that $\mathbb{CP}^n \not\simeq S^2 \vee S^4$.

For the computation, we proceed by induction on n. The base case(s) is/are trivial. We have a standard inclusion $i : \mathbb{RP}^{n-1} \hookrightarrow \mathbb{RP}^n$, which by induction induces $i^* : H^*(\mathbb{RP}^n) \to H^*(\mathbb{RP}^{n-1}) = (\mathbb{Z}/2)[x]/(x^n)$. An examination of cellular homology shows that this is a surjection. If $x \in H^1(\mathbb{RP}^n)$ is the unique nonzero element, it is enough to show that $x^n \neq 0$ (as we are working

[11] Tensor products commute with *finite* products, which are isomorphic to coproducts, as tensor products commute with all directed limits.

over a field). We have a map $(\mathbb{Z}/2)[x] \to H^*(\mathbb{RP}^n)$ taking $x^i \mapsto$ something nonzero, for each $i \leq n$. Hence this map is surjective, after comparing dimensions. We will show that $H^i(\mathbb{RP}^n) \otimes H^{n-i}(\mathbb{RP}^n) \to H^n(\mathbb{RP}^n) \simeq \mathbb{Z}/2$, given by multiplication, is nonzero, hence via linear algebra, an isomorphism[12].

We establish some geometric constructions to do this. We write

$$\mathbb{RP}^n = \frac{\{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \smallsetminus \{0\}\}}{\mathbb{R}^\times}.$$

We consider in particular the subspaces

$$P^i := \{(x_0, ..., x_i, 0, ..., 0)\} \subseteq \mathbb{RP}^n, \quad P^{n-i} = \{(0, ..., 0, x_i, ..., x_n)\} \subseteq \mathbb{RP}^n.$$

Note that $\mathbb{RP}^n = P^n$. Observe that $P^i \cap P^{n-i} = \{(0, ..., x_i, ..., 0)\} = \{p\} \subseteq \mathbb{RP}^n$. We set $U = \{(x_0, ..., x_n) \mid x_i \neq 0\} \subseteq \mathbb{RP}^n$. Observe that U is homeomorphic to $\mathbb{R}^n$, by mapping

$$(x_0, ..., x_n) \mapsto (x_0/x_i, x_1/x_i, ..., \widehat{x_i/x_i}, ..., x_n/x_i).$$

Now consider the diagram

$$\begin{array}{ccc}
H^i(\mathbb{RP}^n) \otimes H^{n-i}(\mathbb{RP}^n) & \xrightarrow{\alpha_1} & H^n(\mathbb{RP}^n) \\
a \uparrow & & \uparrow c \\
H^i(\mathbb{RP}^n, \mathbb{RP}^n \smallsetminus P^{n-i}) \otimes H^i(\mathbb{RP}^n, \mathbb{RP}^n \smallsetminus P^{n-i}) & \xrightarrow{\alpha_2} & H^n(H^i(\mathbb{RP}^n, \mathbb{RP}^n \smallsetminus \{p\}) \\
b \downarrow & & \downarrow d \\
H^i(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \mathbb{R}^{n-i}) \otimes H^{n-i}(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \mathbb{R}^i) & \xrightarrow{\alpha_3} & H^n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{p\})
\end{array}$$

each of whose horizontal arrows are cup products (i.e., multiplication), and whose downward arrows are induced by inclusions. We will check that a, b, c, d, α_3 are isomorphisms. It will then follow that α_1 is an isomorphism, as required.

The geometric input of this is the following. We claim that $P^n \smallsetminus P^{n-i}$ deformation retracts to $P^{i-1} \subseteq P^n \smallsetminus P^{n-i}$. We do not prove this.

Note now that d is an isomorphism, by the cohomological statement of excision. We use the geometric input to observe that c is an isomorphism, along with cellular homology. Similar (long) arguments apply also to a and b; see Hatcher for the brutal details. That α_3 is an isomorphism follows from an application of the Kunneth formula and the computation of the cohomology of spheres. These observations close the example.

We consider some fun exercises.

Example. Consider the standard map $\mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{CP}^n$. Show that this map has no section for $n > 1$.

Solution: Apply homology.

Example. For a cover $\tilde{X} \xrightarrow{\pi} X$ a covering space with $\tilde{X}, X$ finite CW complexes, then if $\tilde{X}$ is contractible, then π is an isomorphism.

Solution: Finiteness implies that π is finite degree. We have $\deg(\pi)\chi(X) = \chi(\tilde{X}) = 1$; it follows that $\deg(\pi) = 1$.

Example. Show that $\mathbb{RP}^4$ is not a Lie group.

Solution: $\chi(\mathbb{RP}^4) = 1$, however $\chi(G) = 0$ for any connected topological group G.

Example. Show that any $\mathbb{Z}/2$ action on $\mathbb{CP}^n$ has a fixed point, for $n \geq 1$.

Solution: Let $g \in \mathbb{Z}/2$ be nonzero. Then

$$\tau(g) = \sum_i \operatorname{tr}(g_*(H_i(\mathbb{CP}^n; \mathbb{Z}/2))) = \sum_i \operatorname{tr}(\mathrm{id}) = \chi(\mathbb{CP}^n) = n \neq 0$$

as any nonzero action of $\mathbb{Z}/2$ on $\mathbb{Z}/2$ is the identity.

[12] This is the statement of Poincaré duality for $\mathbb{RP}^n$.

END

“Don’t attribute any quotes to me."
-Bhargav Bhatt