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WAVE PROPAGATION IN WAVEGUIDES WITH RANDOM BOUNDARIES*

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Abstract. We give a detailed analysis of long range cumulative scattering effects from rough boundaries in waveguides. We assume small random fluctuations of the boundaries and obtain a quantitative statistical description of the wave field. The method of solution is based on coordinate changes that straighten the boundaries. The resulting problem is similar from the mathematical point of view to that of wave propagation in random waveguides with interior inhomogeneities. We quantify the net effect of scattering at the random boundaries and show how it differs from that of scattering by internal inhomogeneities.

Key words. Waveguides, random media, asymptotic analysis.

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1. Introduction

We consider acoustic waves propagating in a waveguide with axis along the range direction z. In general, the waveguide effect may be due to boundaries or the variation of the wave speed with cross-range, as described for example in [13, 10]. We consider here only the case of waves trapped by boundaries, and take for simplicity the case of two-dimensional waveguides with cross-section $\mathcal{D}$ given by a bounded interval of the cross-range x. The results extend to three-dimensional waveguides with bounded, simply connected cross-section $\mathcal{D}\subset\mathbb{R}^2$.

The pressure field $p(t,x,z)$ satisfies the wave equation

$$\left[\partial_z^2+\partial_x^2-\frac{1}{c^2(x)}\partial_t^2\right]p(t,x,z)=F(t,x,z), \tag{1.1}$$

with wave speed $c(x)$ and source excitation modeled by $F(t,x,z)$. Since the equation is linear, it suffices to consider a point-like source located at $(x_0,z=0)$ and emitting a pulse signal $f(t)$,

$$F(t,x,z)=f(t)\delta(x-x_0)\delta(z). \tag{1.2}$$

Solutions for distributed sources are easily obtained by superposing the wave fields computed here.

The boundaries of the waveguide are rough in the sense that they have small variations around the values $x=0$ and $x=X$, on a length scale comparable to the wavelength. Explicitly, we let

$$B(z)\le x\le T(z), \quad \text{where} \quad |B(z)|\ll X, \ \ |T(z)-X|\ll X, \tag{1.3}$$

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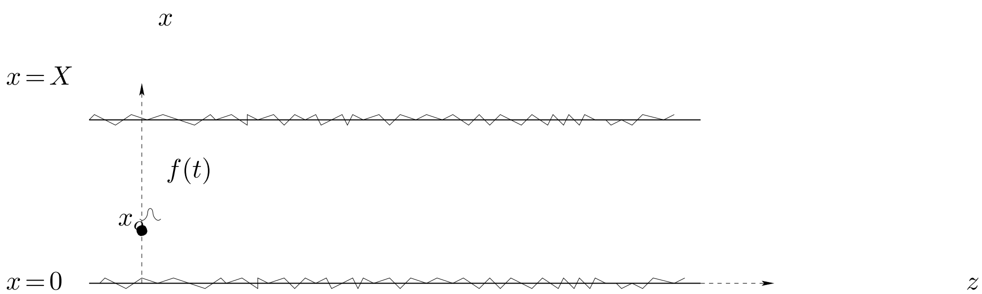

FIG. 1.1. *Schematic of the problem setup. The source is at coordinates* $(x=x_0, z=0)$ *and emits a pulse* $f(t)$ *in a waveguide with fluctuating boundaries around the values* $x=0$ *and* $x=X$.

and take either Dirichlet boundary conditions

$$p(t,x,z)=0, \quad \text{for } x=B(z) \text{ and } x=T(z), \tag{1.4}$$

or mixed, Dirichlet and Neumann conditions

$$p(t,x=B(z),z)=0, \quad \frac{\partial}{\partial n}p(t,x=T(z),z)=0, \tag{1.5}$$

where n is the unit normal to the boundary $x=T(z)$. We refer to figure 1.1 for an illustration of the setup and the system of coordinates.

The goal of the paper is to quantify the long range effect of scattering at the rough boundaries. More explicitly, to characterize in detail the statistics of the random field $p(t,x,z)$. This is useful in sensor array imaging, for designing robust source or target localization methods, as shown recently in [3] in waveguides with internal inhomogeneities. Examples of other applications are in long range secure communications and time reversal in shallow water or in tunnels [8, 14].

The paper is organized as follows. We begin in Section 2 with the case of ideal waveguides, with straight boundaries $B(z)=0$ and $T(z)=X$, where energy propagates via guided modes that do not interact with each other. Rough, randomly perturbed boundaries are introduced in Section 3. The wave speed is assumed to be known and to depend only on the cross-range. Randomly perturbed wave speeds due to internal inhomogeneities are considered in detail in [13, 12, 4, 8, 5]. Our approach in Section 3 uses changes of coordinates that straighten the randomly perturbed boundaries. We carry out the analysis in detail for the case of Dirichlet boundary conditions (1.4) in sections 3 and 4, and discuss the results in Section 5. The extension to the mixed boundary conditions (1.5) is presented in Section 6. We end in Section 7 with a summary.

Our approach, based on changes of coordinates that straighten the boundary, leads to a transformed problem that is similar from the mathematical point of view to that in waveguides with interior inhomogeneities, so we can use the techniques from [13, 12, 4, 8, 5] to obtain the long range statistical characterization of the wave field in Section 4. However, the cumulative scattering effects of rough boundaries are different from those of internal inhomogeneities, as described in Section 5. We quantify these effects by estimating in a high frequency regime three important, mode dependent length scales: the scattering mean free path, which is the distance over which the modes lose coherence; the transport mean free path, which is the distance over which the waves forget the initial direction, and the equipartition distance, over which the energy is uniformly distributed among the modes, independently of the initial conditions at the source. We show that the random boundaries affect most strongly the high order modes, which lose coherence rapidly, that is they have a short scattering mean free path. Furthermore, these modes do not exchange efficiently energy with the other modes, so they have a longer transport mean free path. The lower order modes can travel much longer distances before they lose their coherence and remarkably, their scattering mean free path is similar to the transport mean free path and to the equipartition distance. That is to say, in waveguides with random boundaries, when the waves travel distances that exceed the scattering mean free path of the low order modes, not only all the modes are incoherent, but also the energy is uniformly distributed among them. At such distances the wave field has lost all information about the cross-range location of the source in the waveguide. These results can be contrasted with the situation with waveguides with interior random inhomogeneities, in which the main mechanism for the loss of coherence of the fields is the exchange of energy between neighboring modes [13, 12, 4, 8, 5], so the scattering mean free paths and the transport mean free paths are similar for all the modes. The low order modes lose coherence much faster than in waveguides with random boundaries, and the equipartition distance is longer than the scattering mean free path of these modes.

2. Ideal waveguides

Ideal waveguides have straight boundaries $x=0$ and $x=X$. Using separation of variables, we write the wave field as a superposition of waveguide modes. A waveguide mode is a monochromatic wave $P(t,x,z)=\widehat{P}(\omega,x,z)e^{-i\omega t}$ with frequency ω, where $\widehat{P}(\omega,x,z)$ satisfies the Helmholtz equation

$$\left[\partial_z^2+\partial_x^2+\omega^2/c^2(x)\right]\widehat{P}(\omega,x,z)=0, \quad z\in\mathbb{R},\ x\in(0,X), \tag{2.1}$$

and either Dirichlet or mixed, Dirichlet and Neumann homogeneous boundary conditions. The operator $\partial_x^2+\omega^2/c^2(x)$ with either of these conditions is self-adjoint in $L^2(0,X)$, and its spectrum consists of an infinite number of discrete eigenvalues $\{\lambda_j(\omega)\}_{j\geq 1}$, assumed sorted in descending order. There is a finite number $N(\omega)$ of positive eigenvalues and an infinite number of negative eigenvalues. The eigenfunctions $\phi_j(\omega,x)$ are real and form an orthonormal set

$$\int_0^X dx\,\phi_j(\omega,x)\phi_l(\omega,x)=\delta_{jl}, \quad j,l\geq 1, \tag{2.2}$$

where δ_{jl} is the Kronecker delta symbol.

For example, in homogeneous waveguides with $c(x)=c_o$, and for the Dirichlet boundary conditions, the eigenfunctions and eigenvalues are

$$\phi_j(x)=\sqrt{\frac{2}{X}}\sin\left(\frac{\pi j x}{X}\right), \qquad \lambda_j(\omega)=\left(\frac{\pi}{X}\right)^2\left[(kX/\pi)^2-j^2\right], \qquad j=1,2,\ldots, \tag{2.3}$$

and the number of propagating modes is $N(\omega)=\lfloor kX/\pi\rfloor$, where $\lfloor y\rfloor$ is the integer part of y and $k=\omega/c_o$ is the wavenumber.

To simplify the analysis, we assume that the source emits a pulse $f(t)$ with Fourier transform

$$\widehat{f}(\omega)=\int_{-\infty}^{\infty}dt\,e^{i\omega t}f(t),$$

supported in a frequency band in which the number of positive eigenvalues is fixed, so we can set $N(\omega)=N$. We also assume that there is no zero eigenvalue, and that the eigenvalues are simple. The positive eigenvalues define the modal wavenumbers $\beta_j(\omega)=\sqrt{\lambda_j(\omega)}$ of the forward and backward propagating modes

$$\widehat{P}_j(\omega,x,z)=\phi_j(\omega,x)e^{\pm i\beta_j(\omega)z}, \quad j=1,\ldots,N.$$

The infinitely many remaining modes are evanescent

$$\widehat{P}_j(\omega,x,z)=\phi_j(\omega,x)e^{-\beta_j(\omega)|z|}, \quad j>N,$$

with wavenumber $\beta_j(\omega)=\sqrt{-\lambda_j(\omega)}$.

The wave field $p(t,x,z)$, due to the source located at $(x_0,0)$, is given by the superposition of $\widehat{P}_j(\omega,x,z)$,

$$\begin{aligned}
&p(t,x,z)\\
&=\int\frac{d\omega}{2\pi}e^{-i\omega t}\left[\sum_{j=1}^{N}\frac{\widehat{a}_{j,o}(\omega)}{\sqrt{\beta_j(\omega)}}e^{i\beta_j(\omega)z}\phi_j(\omega,x)+\sum_{j=N+1}^{\infty}\frac{\widehat{e}_{j,o}(\omega)}{\sqrt{\beta_j(\omega)}}e^{-\beta_j(\omega)z}\phi_j(\omega,x)\right]\mathbf{1}_{(0,\infty)}(z)\\
&\quad+\int\frac{d\omega}{2\pi}e^{-i\omega t}\left[\sum_{j=1}^{N}\frac{\widehat{a}^{-}_{j,o}(\omega)}{\sqrt{\beta_j(\omega)}}e^{-i\beta_j(\omega)z}\phi_j(\omega,x)+\sum_{j=N+1}^{\infty}\frac{\widehat{e}^{-}_{j,o}(\omega)}{\sqrt{\beta_j(\omega)}}e^{\beta_j(\omega)z}\phi_j(\omega,x)\right]\mathbf{1}_{(-\infty,0)}(z).
\end{aligned}$$

The first term is supported at positive range, and it consists of forward going modes with amplitudes $\widehat{a}_{j,o}/\sqrt{\beta_j}$ and evanescent modes with amplitudes $\widehat{e}_{j,o}/\sqrt{\beta_j}$. The second term is supported at negative range, and it consists of backward going and evanescent modes. The modes do not interact with each other and their amplitudes

$$\begin{aligned}
\widehat{a}_{j,o}(\omega)&=\widehat{a}^{-}_{j,o}(\omega)=\frac{\widehat{f}(\omega)}{2i\sqrt{\beta_j(\omega)}}\phi_j(\omega,x_0), \quad j=1,\ldots,N,\\
\widehat{e}_{j,o}(\omega)&=\widehat{e}^{-}_{j,o}(\omega)=-\frac{\widehat{f}(\omega)}{2\sqrt{\beta_j(\omega)}}\phi_j(\omega,x_0), \quad j>N
\end{aligned} \tag{2.4}$$

are determined by the source excitation (1.2), which gives the jump conditions at $z=0$,

$$\begin{aligned}
&\widehat{p}(\omega,x,z=0^+)-\widehat{p}(\omega,x,z=0^-)=0,\\
&\partial_z\widehat{p}(\omega,x,z=0^+)-\partial_z\widehat{p}(\omega,x,z=0^-)=\widehat{f}(\omega)\delta(x-x_0).
\end{aligned} \tag{2.5}$$

We show next how to use the solution in the ideal waveguides as a reference for defining the wave field in the case of randomly perturbed boundaries.

3. Waveguides with randomly perturbed boundaries

We consider a randomly perturbed section of an ideal waveguide, over the range interval $z\in[0,L/\varepsilon^2]$. There are no perturbations for $z<0$ and $z>L/\varepsilon^2$. The domain of the perturbed section is denoted by

$$\Omega^\varepsilon=\left\{(x,z)\in\mathbb{R}^2,\ B(z)\le x\le T(z),\ 0<z<L/\varepsilon^2\right\}, \tag{3.1}$$

where

$$B(z)=\varepsilon X\mu(z), \qquad T(z)=X[1+\varepsilon\nu(z)], \qquad \varepsilon\ll 1. \tag{3.2}$$

Here ν and μ are independent, zero-mean, stationary, and ergodic random processes in z, with covariance function

$$\mathcal{R}_\nu(z) = \mathbb{E}[\nu(z+s)\nu(s)] \quad \text{and} \quad \mathcal{R}_\mu(z) = \mathbb{E}[\mu(z+s)\mu(s)]. \tag{3.3}$$

We assume that $\nu(z)$ and $\mu(z)$ are bounded, at least twice differentiable with almost sure bounded derivatives, and have enough decorrelation[1]. The covariance functions are normalized so that $\mathcal{R}_\nu(0)$ and $\mathcal{R}_\mu(0)$ are of order one, and the magnitude of the fluctuations is scaled by the small, dimensionless parameter ε.

That the random fluctuations are confined to the range interval $z \in (0, L/\varepsilon^2)$, with L an order one length scale, can be motivated as follows: By the hyperbolicity of the wave equation, we know that if we observe $p(t,x,z)$ over a finite time window $t \in (0, T^\varepsilon)$, the wave field is affected only by the medium within a finite range L^ε from the source, directly proportional to the observation time T^ε. We wish to choose T^ε large enough in order to capture the cumulative long range effects of scattering from the randomly perturbed boundaries. It turns out that these effects become significant over time scales of order $1/\varepsilon^2$, so we take $L^\varepsilon = L/\varepsilon^2$. Furthermore, we are interested in the wave field to the right of the source, at positive range. We will see that the backscattered field is small and can be neglected when the conditions of the forward scattering approximation are satisfied (see subsection 4.3). Thus, the medium on the left of the source has negligible influence on $p(t,x,z)$ for $z > 0$, and we may suppose that the boundaries are unperturbed at negative range. The analysis can be carried out when the conditions of the forward scattering approximation are not satisfied, at considerable complication of the calculations, as was done in [9] for waveguides with internal inhomogeneities.

We assume here and in sections 4 and 5 the Dirichlet boundary conditions (1.4). The extensions to the mixed boundary conditions (1.5) are presented in Section 6. The main result of this section is a closed system of random differential equations for the propagating waveguide modes, which describes the cumulative effect of scattering of the wave field by the random boundaries. We derive it in the following subsections and we analyze its solution in the long range limit in Section 4.

3.1. Change of coordinates. We reformulate the problem in the randomly perturbed waveguide region Ω^ε by changing coordinates that straighten the boundaries,

$$x = B(z) + [T(z) - B(z)]\,\frac{\xi}{X}, \quad \xi \in [0, X]. \tag{3.4}$$

We take this coordinate change because it is simple, but we show later, in Section 4.4.2, that the result is independent of the choice of the change of coordinates. In the new coordinate system, let

$$u(t,\xi,z) = p\left(t, B(z) + [T(z) - B(z)]\,\frac{\xi}{X}, z\right), \qquad p(t,x,z) = u\left(t, \frac{(x - B(z))X}{T(z) - B(z)}, z\right). \tag{3.5}$$

We obtain using the chain rule that the Fourier transform $\widehat{u}(\omega,\xi,z)$ satisfies the equation

$$\mathcal{L}^\varepsilon \widehat{u}(\omega,\xi,z) = 0, \tag{3.6}$$

[1] Explicitly, they are φ-mixing processes with $\varphi \in L^{1/2}(\mathbb{R}^+)$, as stated in [15, 4.6.2].

for $z\in(0,L/\varepsilon^2)$ and $\xi\in(0,X)$, where

$$\mathcal{L}^\varepsilon=\partial_z^2+\frac{\left[1+[(X-\xi)B'+\xi T']^2\right]}{(T-B)^2}X^2\partial_\xi^2-\frac{2[(X-\xi)B'+\xi T']}{T-B}X\partial_{\xi z}^2$$
$$+\left\{\frac{2B'(T'-B')}{(T-B)^2}-\frac{B''}{T-B}+\frac{\xi}{X}\left[2\left(\frac{T'-B'}{T-B}\right)^2-\frac{T''-B''}{T-B}\right]\right\}X\partial_\xi$$
$$+\frac{\omega^2}{c^2\big(B+(T-B)\xi/X\big)}. \tag{3.7}$$

Here the prime stands for the z-derivative, and the boundary conditions at $\xi=0$ and X are

$$\widehat{u}(\omega,0,z)=\widehat{u}(\omega,X,z)=0. \tag{3.8}$$

Substituting Definition (3.2) of $B(z)$ and $T(z)$, and expanding the coefficients in (3.7) in series of ε, we obtain that

$$\left(\mathcal{L}_0+\varepsilon\mathcal{L}_1+\varepsilon^2\mathcal{L}_2+\cdots\right)\widehat{u}(\omega,\xi,z)=0, \tag{3.9}$$

where

$$\mathcal{L}_0=\partial_z^2+\partial_\xi^2+\omega^2/c^2(\xi) \tag{3.10}$$

is the unperturbed Helmholtz operator. The first- and second-order perturbation operators are given by

$$\mathcal{L}_1+\varepsilon\mathcal{L}_2=q^\varepsilon(\xi,z)\partial_{\xi z}^2+\mathcal{M}^\varepsilon(\omega,\xi,z), \tag{3.11}$$

with coefficient

$$q^\varepsilon(\xi,z)=-2\left[(X-\xi)\mu'(z)+\xi\nu'(z)\right]\left[1-\varepsilon(\nu(z)-\mu(z))\right], \tag{3.12}$$

and differential operator

$$\mathcal{M}^\varepsilon=-\left\{2(\nu-\mu)-3\varepsilon(\nu-\mu)^2-\varepsilon\left[(X-\xi)\mu'+\xi\nu'\right]^2\right\}\partial_\xi^2$$
$$-\left\{\left[(X-\xi)\mu''+\xi\nu''\right]\left[1-\varepsilon(\nu-\mu)\right]-2\varepsilon(\nu'-\mu')\left[(X-\xi)\mu'+\xi\nu'\right]\right\}\partial_\xi$$
$$+\omega^2\left[(X-\xi)\mu+\xi\nu\right]\partial_\xi(c^{-2})+\frac{\varepsilon\omega^2}{2}\left[(X-\xi)\mu+\xi\nu\right]^2\partial_\xi^2(c^{-2}). \tag{3.13}$$

The dots in (3.9) denote small corrections in the expansion of $\mathcal{L}^\varepsilon$. These corrections have the form of a sum of the same partial derivatives as in $\mathcal{L}_0+\varepsilon\mathcal{L}_1+\varepsilon^2\mathcal{L}_2$ but with coefficients which come from the third-order corrections of the expansions (in ε) of the rational expressions containing B, T, B', T', B'', T'', and of $c^2(B+(T-B)\xi/X)$ in (3.7). These coefficients can be bounded by $O(\varepsilon^3)$ provided $\mu(z)$ and $\nu(z)$ have almost sure bounded second-order derivatives (in z) and the deterministic velocity profile $c(x)$ has bounded third-order derivatives (in x), which we assume from now on. As a result the corrections in (3.9) will give rise to $O(\varepsilon^3)$ terms in the forthcoming equations (3.17-3.18-...) but they will be negligible as $\varepsilon\to0$ over the long range scale L/ε^2 considered here.

3.2. Wave decomposition and mode coupling. Equation (3.9) is not separable, and its solution is not a superposition of independent waveguide modes, as was the case in ideal waveguides. However, we have a perturbation problem, and we can use the completeness of the set of eigenfunctions $\{\phi_j(\omega,\xi)\}_{j\geq 1}$ in the ideal waveguide to decompose $\widehat{u}$ in its propagating and evanescent components,

$$\widehat{u}(\omega,\xi,z) = \sum_{j=1}^{N} \phi_j(\omega,\xi)\widehat{u}_j(\omega,z) + \sum_{j=N+1}^{\infty} \phi_j(\omega,\xi)\widehat{v}_j(\omega,z). \tag{3.14}$$

The propagating components $\widehat{u}_j(\omega,z)$ are decomposed further in the forward and backward going parts, with amplitudes $\widehat{a}_j(\omega,z)$ and $\widehat{b}_j(\omega,z)$ defined by

$$\widehat{u}_j(\omega,z) = \frac{1}{\sqrt{\beta_j(\omega)}}\left(\widehat{a}_j(\omega,z)e^{i\beta_j(\omega)z} + \widehat{b}_j(\omega,z)e^{-i\beta_j(\omega)z}\right), \quad j=1,\dots,N, \tag{3.15}$$

$$\partial_z\widehat{u}_j(\omega,z) = i\sqrt{\beta_j(\omega)}\left(\widehat{a}_j(\omega,z)e^{i\beta_j(\omega)z} - \widehat{b}_j(\omega,z)e^{-i\beta_j(\omega)z}\right), \quad j=1,\dots,N. \tag{3.16}$$

This choice is motivated by the behavior of the solution in ideal waveguides, where the amplitudes $(\widehat{a}_j,\widehat{b}_j)$ are independent of range and completely determined by the source excitation. The expression (3.14) of the wave field is similar to that in ideal waveguides, except that we have both forward and backward going modes, in addition to the evanescent modes, and the amplitudes of the modes are random functions of z.

The modes are coupled due to scattering at the random boundaries, as described by the following system of random differential equations obtained by substituting (3.14) in (3.9), and using the orthogonality relation (2.2) of the eigenfunctions:

$$\begin{aligned}\partial_z\widehat{a}_j &= i\varepsilon\sum_{l=1}^{N}\left[C^{\varepsilon}_{jl}\widehat{a}_l e^{i(\beta_l-\beta_j)z} + \overline{C^{\varepsilon}_{jl}}\widehat{b}_l e^{-i(\beta_l+\beta_j)z}\right] \\ &\quad + \frac{i\varepsilon}{2\sqrt{\beta_j}}\sum_{l=N+1}^{\infty} e^{-i\beta_j z}\left(Q^{\varepsilon}_{jl}\partial_z\widehat{v}_l + M^{\varepsilon}_{jl}\widehat{v}_l\right) + O(\varepsilon^3),\end{aligned} \tag{3.17}$$

$$\begin{aligned}\partial_z\widehat{b}_j &= -i\varepsilon\sum_{l=1}^{N}\left[C^{\varepsilon}_{jl}\widehat{a}_l e^{i(\beta_l+\beta_j)z} + \overline{C^{\varepsilon}_{jl}}\widehat{b}_l e^{-i(\beta_l-\beta_j)z}\right] \\ &\quad - \frac{i\varepsilon}{2\sqrt{\beta_j}}\sum_{l=N+1}^{\infty} e^{i\beta_j z}\left(Q^{\varepsilon}_{jl}\partial_z\widehat{v}_l + M^{\varepsilon}_{jl}\widehat{v}_l\right) + O(\varepsilon^3).\end{aligned} \tag{3.18}$$

The bar denotes complex conjugation, and the coefficients are defined below. Note that, from now on, we do not write explicitly the ω-dependence in the equations, except in the statements of the propositions and theorems.

The forward going amplitudes are determined at $z=0$ by the source excitation (recall (2.4))

$$\widehat{a}_j(0) = \widehat{a}_{j,o}, \quad j=1,\dots,N, \tag{3.19}$$

and we have

$$\widehat{b}_j\left(\frac{L}{\varepsilon^2}\right) = 0, \quad j=1,\dots,N, \tag{3.20}$$

because there is no incoming wave at the end of the domain. The equations for the amplitudes of the evanescent modes indexed by $j > N$ are

$$\begin{aligned}&\left(\partial_z^2 - \beta_j^2\right)\widehat{v}_j\\&= -\varepsilon \sum_{l=1}^{N} 2\sqrt{\beta_j}\left[C_{jl}^{\varepsilon}\widehat{a}_l e^{i\beta_l z} + \overline{C_{jl}^{\varepsilon}}\widehat{b}_l e^{-i\beta_l z}\right] - \varepsilon \sum_{l=N+1}^{\infty}\left(Q_{jl}^{\varepsilon}\partial_z \widehat{v}_l + M_{jl}^{\varepsilon}\widehat{v}_l\right) + O(\varepsilon^3),\end{aligned} \tag{3.21}$$

and we complement them with the decay condition at infinity

$$\lim_{z\to\pm\infty} \widehat{v}_j(z) = 0, \quad j > N. \tag{3.22}$$

The coefficients

$$C_{jl}^{\varepsilon}(z) = C_{jl}^{(1)}(z) + \varepsilon C_{jl}^{(2)}(z), \quad \text{for } j \geq 1 \text{ and } l = 1,\ldots,N \tag{3.23}$$

are defined by

$$C_{jl}^{(1)}(z) = \frac{1}{2\sqrt{\beta_j\beta_l}}\int_0^X d\xi \phi_j(\xi)\mathcal{A}_l(\xi,z)\phi_l(\xi), \tag{3.24}$$

$$C_{jl}^{(2)}(z) = \frac{1}{2\sqrt{\beta_j\beta_l}}\int_0^X d\xi \phi_j(\xi)\mathcal{B}_l(\xi,z)\phi_l(\xi), \tag{3.25}$$

in terms of the linear differential operators

$$\begin{aligned}\mathcal{A}_l(\xi,z) = &-2(\nu-\mu)\partial_\xi^2 - 2i\beta_l\left[(X-\xi)\mu' + \xi\nu'\right]\partial_\xi - \left[(X-\xi)\mu'' + \xi\nu''\right]\partial_\xi\\&+\omega^2\left[(X-\xi)\mu + \xi\nu\right]\partial_\xi(c^{-2}),\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}\mathcal{B}_l(\xi,z) = &\left\{3(\nu-\mu)^2 + \left[(X-\xi)\mu' + \xi\nu'\right]^2\right\}\partial_\xi^2 + 2i\beta_l(\nu-\mu)\left[(X-\xi)\mu' + \xi\nu'\right]\partial_\xi\\&+\left\{(\nu-\mu)\left[(X-\xi)\mu'' + \xi\nu''\right] + 2(\nu'-\mu')\left[(X-\xi)\mu' + \xi\nu'\right]\right\}\partial_\xi\\&+\frac{\omega^2}{2}\left[(X-\xi)\mu + \xi\nu\right]^2\partial_\xi^2(c^{-2}).\end{aligned} \tag{3.27}$$

We also let, for $j \geq 1$ and $l > N$,

$$\begin{aligned}Q_{jl}^{\varepsilon}(z) &= \int_0^X d\xi q^{\varepsilon}(\xi,z)\phi_j(\xi)\partial_\xi\phi_l(\xi) = Q_{jl}^{(1)}(z) + \varepsilon Q_{jl}^{(2)}(z),\\M_{jl}^{\varepsilon}(z) &= \int_0^X d\xi \phi_j(\xi)\mathcal{M}^{\varepsilon}(\xi,z)\phi_l(\xi) = M_{jl}^{(1)}(z) + \varepsilon M_{jl}^{(2)}(z).\end{aligned} \tag{3.28}$$

3.3. Analysis of the evanescent modes. We solve equations (3.21) with radiation conditions (3.22) in order to express the amplitude of the evanescent modes in terms of the amplitudes of the propagating modes. The substitution of this expression in (3.17)-(3.18) gives a closed system of equations for the amplitudes of the propagating modes, as obtained in the next section.

We begin by rewriting (3.21) in short as

$$\left(\partial_z^2 - \beta_j^2\right)\widehat{v}_j + \varepsilon\sum_{l=N+1}^{\infty}\left(Q_{jl}^{\varepsilon}\partial_z\widehat{v}_l + M_{jl}^{\varepsilon}\widehat{v}_l\right) = -\varepsilon g_j^{\varepsilon}, \qquad j > N, \tag{3.29}$$

where

$$g_j^\varepsilon(z) = g_j^{(1)}(z) + \varepsilon g_j^{(2)}(z) + O(\varepsilon^3), \qquad j > N, \tag{3.30}$$

and

$$g_j^{(r)}(z) = 2\sqrt{\beta_j}\sum_{l=1}^N \left[C_{jl}^{(r)}(z)\widehat{a}_l(z)e^{i\beta_l z} + \overline{C_{jl}^{(r)}}(z)\widehat{b}_l(z)e^{-i\beta_l z}\right], \quad r = 1,2 \text{ and } j > N. \tag{3.31}$$

Using the Green's function $G_j(z) = e^{-\beta_j|z|}/(2\beta_j)$, which satisfies

$$\partial_z^2 G_j - \beta_j^2 G_j = -\delta(z), \quad \lim_{|z|\to\infty} G_j = 0, \qquad j > N, \tag{3.32}$$

and by integrating by parts, we get

$$[(\mathbf{I} - \varepsilon\Psi)\widehat{\boldsymbol{v}}]_j(z) = \frac{\varepsilon}{2\beta_j}\int_{-\infty}^{\infty} ds\, e^{-\beta_j|s|} g_j^\varepsilon(z+s), \qquad j > N. \tag{3.33}$$

Here $\mathbf{I}$ is the identity and Ψ is the linear integral operator

$$[\Psi\widehat{\boldsymbol{v}}]_j(z) = \frac{1}{2\beta_j}\sum_{l=N+1}^\infty \int_{-\infty}^\infty ds\, e^{-\beta_j|s|}\left(M_{jl}^\varepsilon - \partial_z Q_{jl}^\varepsilon\right)(z+s)\widehat{v}_l(z+s)$$
$$+ \frac{1}{2}\sum_{l=N+1}^\infty \int_{-\infty}^\infty ds\, e^{-\beta_j|s|}\operatorname{sgn}(s) Q_{jl}^\varepsilon(z+s)\widehat{v}_l(z+s), \tag{3.34}$$

which acts on the infinite vector $\widehat{\boldsymbol{v}} = (\widehat{v}_{N+1}, \widehat{v}_{N+2}, \ldots)$ and returns an infinite vector with entries indexed by j, for $j > N$. The solvability of Equation (3.33) follows from the following lemma proved in Appendix A.

LEMMA 3.1. *Let $\mathcal{L}_N$ be the space of square summable sequences of $L^2(\mathbb{R})$ functions with linear weights, equipped with the norm*

$$\|\widehat{\boldsymbol{v}}\|_{\mathcal{L}_N} = \sqrt{\sum_{j=N+1}^\infty \left(j\|\widehat{v}_j\|_{L^2(\mathbb{R})}\right)^2}.$$

The linear operator $\Psi : \mathcal{L}_N \to \mathcal{L}_N$, defined component wise by (3.34), *is bounded.*

Thus, the inverse operator is

$$(\mathbf{I} - \varepsilon\Psi)^{-1} = \mathbf{I} + \varepsilon\Psi + \cdots,$$

and the solution of (3.33) is given by

$$\widehat{v}_j(z) = \frac{\varepsilon}{2\beta_j}\int_{-\infty}^\infty ds\, e^{-\beta_j|s|} g_j^{(1)}(z+s) + O(\varepsilon^2). \tag{3.35}$$

Using Definition (3.31) and the fact that the z derivatives of $\widehat{a}_l$ and $\widehat{b}_l$ are of order ε, we get

$$\widehat{v}_j(z) = \frac{\varepsilon}{\sqrt{\beta_j}}\sum_{l=1}^N \widehat{a}_l(z)e^{i\beta_l z}\int_{-\infty}^\infty ds\, e^{-\beta_j|s| + i\beta_l s} C_{jl}^{(1)}(z+s)$$

$$+\frac{\varepsilon}{\sqrt{\beta_j}}\sum_{l=1}^{N}\widehat{b}_l(z)e^{-i\beta_l z}\int_{-\infty}^{\infty}ds\, e^{-\beta_j|s|-i\beta_l s}\overline{C_{jl}^{(1)}(z+s)}+O(\varepsilon^2). \quad (3.36)$$

We also need

$$\widehat{w}_j(z)=\partial_z\widehat{v}_j(z), \quad (3.37)$$

which we compute by taking a z derivative in (3.29) and using the radiation condition $\widehat{w}_j(z)\to 0$ as $|z|\to\infty$. The resulting equation is similar to (3.33):

$$\left[(\mathbf{I}-\varepsilon\tilde{\Psi})\boldsymbol{w}\right]_j(z)=\frac{\varepsilon}{2}\int_{-\infty}^{\infty}ds\, e^{-\beta_j|s|}\left[\mathrm{sgn}(s)g_j^{\varepsilon}(z+s)+\sum_{l=N+1}^{\infty}M_{jl}^{\varepsilon}(z+s)\widehat{v}_l(z+s)\right], \quad (3.38)$$

where we integrated by parts and introduced the linear integral operator

$$[\tilde{\Psi}\widehat{\boldsymbol{w}}]_j(z)=\frac{1}{2}\sum_{l=N+1}^{\infty}\int_{-\infty}^{\infty}ds\, e^{-\beta_j|s|}\mathrm{sgn}(s)Q_{jl}^{\varepsilon}(z+s)\widehat{w}_l(z+s). \quad (3.39)$$

This operator is very similar to Ψ and it is bounded, as follows from the proof in Appendix A. Moreover, substituting expression (3.36) of $\widehat{v}_l$ in (3.38) we obtain, after a calculation that is similar to that in Appendix A, that the series in the index l is convergent. Therefore, the solution of (3.38) is

$$\widehat{w}_j(z)=\frac{\varepsilon}{2}\int_{-\infty}^{\infty}ds\, e^{-\beta_j|s|}\mathrm{sgn}(s)g_j^{\varepsilon}(z+s)+O(\varepsilon^2) \quad (3.40)$$

and, more explicitly,

$$\partial_z\widehat{v}_j(z)=\varepsilon\sqrt{\beta_j}\sum_{l=1}^{N}\widehat{a}_l(z)e^{i\beta_l z}\int_{-\infty}^{\infty}ds\, e^{-\beta_j|s|+i\beta_l s}\mathrm{sgn}(s)C_{jl}^{(1)}(z+s)$$
$$+\varepsilon\sqrt{\beta_j}\sum_{l=1}^{N}\widehat{b}_l(z)e^{-i\beta_l z}\int_{-\infty}^{\infty}ds\, e^{-\beta_j|s|-i\beta_l s}\mathrm{sgn}(s)\overline{C_{jl}^{(1)}(z+s)}+O(\varepsilon^2). \quad (3.41)$$

3.4. The closed system of equations for the propagating modes. The substitution of equations (3.36) and (3.41) in (3.17) and (3.18) gives the main result of this section: a closed system of differential equations for the propagating mode amplitudes. We write it in compact form using the $2N$-vector

$$\boldsymbol{X}(z)=\begin{bmatrix}\widehat{\boldsymbol{a}}(z)\\ \widehat{\boldsymbol{b}}(z)\end{bmatrix}, \quad (3.42)$$

obtained by concatenating vectors $\widehat{\boldsymbol{a}}(z)$ and $\widehat{\boldsymbol{b}}(z)$ with components $\widehat{a}_j(z)$ and $\widehat{b}_j(z)$, for $j=1,\ldots,N$. We have

$$\partial_z\boldsymbol{X}(z)=\varepsilon\mathbf{H}(z)\boldsymbol{X}(z)+\varepsilon^2\mathbf{G}(z)\boldsymbol{X}(z)+O(\varepsilon^3), \quad (3.43)$$

with $2N\times 2N$ complex matrices given in block form by

$$\mathbf{H}ega(z)=\begin{bmatrix}\mathbf{H}^{(a)}(z) & \mathbf{H}^{(b)}(z)\\ \overline{\mathbf{H}^{(b)}(z)} & \overline{\mathbf{H}^{(a)}(z)}\end{bmatrix}, \qquad \mathbf{G}(z)=\begin{bmatrix}\mathbf{G}^{(a)}(z) & \mathbf{G}^{(b)}(z)\\ \overline{\mathbf{G}^{(b)}(z)} & \overline{\mathbf{G}^{(a)}(z)}\end{bmatrix}. \quad (3.44)$$

The entries of the blocks in $\mathbf{H}$ are

$$H^{(a)}_{jl}(z) = iC^{(1)}_{jl}(z)e^{i(\beta_l-\beta_j)z}, \qquad H^{(b)}_{jl}(z) = iC^{(1)}_{jl}(z)e^{-i(\beta_l+\beta_j)z}, \tag{3.45}$$

and the entries of the blocks in $\mathbf{G}$ are

$$\begin{aligned} G^{(a)}_{jl}(z) = & ie^{i(\beta_l-\beta_j)z}C^{(2)}_{jl}(z) + ie^{i(\beta_l-\beta_j)z}\sum_{l'=N+1}^{\infty}\frac{M^{(1)}_{jl'}(z)}{2\sqrt{\beta_j\beta_{l'}}}\int_{-\infty}^{\infty} ds\, e^{-\beta_{l'}|s|+i\beta_l s}C^{(1)}_{l'l}(z+s) \\ & + ie^{i(\beta_l-\beta_j)z}\sum_{l'=N+1}^{\infty}\frac{Q^{(1)}_{jl'}(z)}{2\sqrt{\beta_j\beta_{l'}}}\int_{-\infty}^{\infty} ds\, e^{-\beta_{l'}|s|+i\beta_l s}\beta_{l'}\,\mathrm{sgn}(s)\,C^{(1)}_{l'l}(z+s), \end{aligned} \tag{3.46}$$

$$\begin{aligned} G^{(b)}_{jl}(z) = & ie^{-i(\beta_l+\beta_j)z}C^{(2)}_{jl}(z) - ie^{-i(\beta_l+\beta_j)z}\sum_{l'=N+1}^{\infty}\frac{M^{(1)}_{jl'}(z)}{2\sqrt{\beta_j\beta_{l'}}}\int_{-\infty}^{\infty} ds\, e^{-\beta_{l'}|s|-i\beta_l s}\overline{C^{(1)}_{l'l}(z+s)} \\ & + ie^{-i(\beta_l+\beta_j)z}\sum_{l'=N+1}^{\infty}\frac{Q^{(1)}_{jl'}(z)}{2\sqrt{\beta_j\beta_{l'}}}\int_{-\infty}^{\infty} ds\, e^{-\beta_{l'}|s|-i\beta_l s}\beta_{l'}\,\mathrm{sgn}(s)\,\overline{C^{(1)}_{l'l}(z+s)}. \end{aligned} \tag{3.47}$$

The coefficients in (3.45)-(3.47) are defined in terms of the random functions $\nu(z)$, $\mu(z)$, their derivatives, and the following integrals:

$$c_{\nu,jl} = \frac{1}{2\sqrt{\beta_j\beta_l}}\int_0^X d\xi\,\phi_j(\xi)\left[-2\partial_\xi^2 + \omega^2\xi\partial_\xi c^{-2}(\xi)\right]\phi_l(\xi), \tag{3.48}$$

$$c_{\mu,jl} = \frac{1}{2\sqrt{\beta_j\beta_l}}\int_0^X d\xi\,\phi_j(\xi)\left[2\partial_\xi^2 + \omega^2(X-\xi)\partial_\xi c^{-2}(\xi)\right]\phi_l(\xi), \tag{3.49}$$

$$d_{\nu,jl} = -\frac{1}{2\sqrt{\beta_j\beta_l}}\int_0^X d\xi\,\xi\,\phi_j(\xi)\partial_\xi\phi_l(\xi), \tag{3.50}$$

$$d_{\mu,jl} = -\frac{1}{2\sqrt{\beta_j\beta_l}}\int_0^X d\xi\,(X-\xi)\,\phi_j(\xi)\partial_\xi\phi_l(\xi), \tag{3.51}$$

which satisfy the symmetry relations

$$\begin{aligned} c_{\nu,jl} &= c_{\nu,lj}, \\ c_{\mu,jl} &= c_{\mu,lj}, \\ d_{\nu,jl} + d_{\nu,lj} &= \frac{\delta_{jl}}{2\sqrt{\beta_j\beta_l}}, \\ d_{\mu,jl} + d_{\mu,lj} &= -\frac{\delta_{jl}}{2\sqrt{\beta_j\beta_l}}. \end{aligned} \tag{3.52}$$

We have from (3.24) that

$$C^{(1)}_{jl}(z) = \nu(z)c_{\nu,jl} + [\nu''(z) + 2i\beta_l\nu'(z)]\,d_{\nu,jl} + \mu(z)c_{\mu,jl} + [\mu''(z) + 2i\beta_l\mu'(z)]\,d_{\mu,jl}, \tag{3.53}$$

and from (3.28), (3.12), (3.13) that

$$\begin{aligned} \frac{Q^{(1)}_{jl'}(z)}{2\sqrt{\beta_j\beta_{l'}}} &= 2\left[\nu'(z)d_{\nu,jl'} + \mu'(z)d_{\mu,jl'}\right], \\ \frac{M^{(1)}_{jl'}(z)}{2\sqrt{\beta_j\beta_{l'}}} &= \nu(z)c_{\nu,jl'} + \mu(z)c_{\mu,jl'} + \nu''(z)d_{\nu,jl'} + \mu''(z)d_{\mu,jl'}. \end{aligned} \tag{3.54}$$

4. The long range limit

In this section we use the system (3.43) to quantify the cumulative scattering effects at the random boundaries. We begin with the long range scaling chosen so that these effects are significant. Then, we explain why the backward going amplitudes are small and can be neglected. This is the forward scattering approximation, which gives a closed system of random differential equations for the amplitudes $\{\widehat{a}_j\}_{j=1,\ldots,N}$. We use this system to derive the main result of the section, which says that the amplitudes $\{\widehat{a}_j\}_{j=1,\ldots,N}$ converge in distribution as $\varepsilon \to 0$ to a diffusion Markov process, whose generator we compute explicitly. This allows us to calculate all the statistical moments of the wave field.

4.1. Long range scaling. It is clear from (3.42) that since the right hand side is small, of order ε, there is no net effect of scattering from the boundaries over ranges of order one. If we considered ranges of order $1/\varepsilon$, the resulting equation would have an order one right hand side given by $\mathbf{H}(z/\varepsilon)\boldsymbol{X}(z/\varepsilon)$, but this becomes negligible as well for $\varepsilon \to 0$ because the expectation of $\mathbf{H}(z/\varepsilon)$ is zero [5, Chapter 6]. We need longer ranges, of order $1/\varepsilon^2$, to see the effect of scattering from the randomly perturbed boundaries.

Let then $\widehat{a}_j^\varepsilon$, $\widehat{b}_j^\varepsilon$ be the rescaled amplitudes

$$\widehat{a}_j^\varepsilon(z) = \widehat{a}_j\left(\frac{z}{\varepsilon^2}\right), \qquad \widehat{b}_j^\varepsilon(z) = \widehat{b}_j\left(\frac{z}{\varepsilon^2}\right), \quad j = 1, \ldots, N, \tag{4.1}$$

and obtain from (3.43) that $\boldsymbol{X}^\varepsilon(z) = \boldsymbol{X}(z/\varepsilon^2)$ satisfies the equation

$$\frac{d\boldsymbol{X}^\varepsilon(z)}{dz} = \frac{1}{\varepsilon}\mathbf{H}\left(\frac{z}{\varepsilon^2}\right)\boldsymbol{X}^\varepsilon(z) + \mathbf{G}\left(\frac{z}{\varepsilon^2}\right)\boldsymbol{X}^\varepsilon(z), \quad 0 < z < L, \tag{4.2}$$

with boundary conditions

$$\widehat{a}_j^\varepsilon(0) = \widehat{a}_{j,o}, \quad \widehat{b}_j^\varepsilon(L) = 0, \quad j = 1, \ldots, N. \tag{4.3}$$

We can solve this using the complex valued, random propagator matrix $\mathbf{P}^\varepsilon(z) \in \mathbb{C}^{2N\times 2N}$, which is the solution of the initial value problem

$$\frac{d\mathbf{P}^\varepsilon(z)}{dz} = \frac{1}{\varepsilon}\mathbf{H}\left(\frac{z}{\varepsilon^2}\right)\mathbf{P}^\varepsilon(z) + \mathbf{G}\left(\frac{z}{\varepsilon^2}\right)\mathbf{P}^\varepsilon(z) \quad \text{for } z > 0, \text{ and } \mathbf{P}^\varepsilon(0) = \mathbf{I}. \tag{4.4}$$

The solution is

$$\boldsymbol{X}^\varepsilon(z) = \mathbf{P}^\varepsilon(z)\begin{bmatrix} \widehat{\boldsymbol{a}}_0 \\ \widehat{\boldsymbol{b}}^\varepsilon(0) \end{bmatrix},$$

and $\widehat{\boldsymbol{b}}^\varepsilon(0)$ can be eliminated from the boundary identity

$$\begin{bmatrix} \widehat{\boldsymbol{a}}^\varepsilon(L) \\ \mathbf{0} \end{bmatrix} = \mathbf{P}^\varepsilon(L)\begin{bmatrix} \widehat{\boldsymbol{a}}_0 \\ \widehat{\boldsymbol{b}}^\varepsilon(0) \end{bmatrix}. \tag{4.5}$$

Furthermore, it follows from the symmetry relations (3.44) satisfied by the matrices $\mathbf{H}$ and $\mathbf{G}$ that the propagator has the block form

$$\mathbf{P}^\varepsilon(z) = \begin{bmatrix} \mathbf{P}^{\varepsilon,a}(z) & \mathbf{P}^{\varepsilon,b}(z) \\ \overline{\mathbf{P}^{\varepsilon,b}(z)} & \overline{\mathbf{P}^{\varepsilon,a}(z)} \end{bmatrix}, \tag{4.6}$$

where $\mathbf{P}^{\varepsilon,a}(z)$ and $\mathbf{P}^{\varepsilon,b}(z)$ are $N \times N$ complex matrices. The first block $\mathbf{P}^{\varepsilon,a}$ describes the coupling between different forward going modes, while $\mathbf{P}^{\varepsilon,b}$ describes the coupling between forward going and backward going modes.

4.2. The diffusion approximation. The limit $\mathbf{P}^\varepsilon$ as $\varepsilon \to 0$ can be obtained and identified as a multi-dimensional diffusion process, meaning that the entries of the limit matrix satisfy a system of linear stochastic equations. This follows from the application of the diffusion approximation theorem proved in [18], which applies to systems of the general form

$$\frac{d\mathcal{X}^\varepsilon(z)}{dz} = \frac{1}{\varepsilon}\mathcal{F}\left(\mathcal{X}^\varepsilon(z), \mathcal{Y}\left(\frac{z}{\varepsilon^2}\right), \frac{z}{\varepsilon^2}\right) + \mathcal{G}\left(\mathcal{X}^\varepsilon(z), \mathcal{Y}\left(\frac{z}{\varepsilon^2}\right), \frac{z}{\varepsilon^2}\right) \text{for } z>0, \text{ and } \mathcal{X}^\varepsilon(0) = \mathcal{X}_o, \quad (4.7)$$

for a vector or matrix $\mathcal{X}^\varepsilon(z)$ with real entries. The system is driven by a stationary, mean zero and mixing random process $\mathcal{Y}(z)$. The functions $\mathcal{F}(\chi, y, \tau)$ and $\mathcal{G}(\chi, y, \tau)$ are assumed at most linearly growing and smooth in χ, and the dependence in τ is periodic or almost periodic [5, Section 6.5]. The function $\mathcal{F}(\chi, y, \tau)$ must also be centered: For any fixed χ and τ, $\mathbb{E}[\mathcal{F}(\chi, \mathcal{Y}(0), \tau)] = 0$.

The diffusion approximation theorem states that as $\varepsilon \to 0$, $\mathcal{X}^\varepsilon(z)$ converges in distribution to the diffusion Markov process $\mathcal{X}(z)$ with generator $\mathcal{L}$, acting on sufficiently smooth functions $\varphi(\chi)$ as

$$\begin{aligned}\mathcal{L}\varphi(\chi) = \lim_{T\to\infty} \frac{1}{T}\int_0^T d\tau \int_0^\infty dz\, \mathbb{E}\left[\mathcal{F}(\chi, \mathcal{Y}(0), \tau) \cdot \nabla_\chi \left[\mathcal{F}(\chi, \mathcal{Y}(z), \tau) \cdot \nabla_\chi \varphi(\chi)\right]\right] \\ + \lim_{T\to\infty} \frac{1}{T}\int_0^T d\tau\, \mathbb{E}\left[\mathcal{G}(\chi, \mathcal{Y}(0), \tau) \cdot \nabla_\chi \varphi(\chi)\right]. \qquad (4.8)\end{aligned}$$

To apply it to the initial value problem (4.4) for the complex $2N \times 2N$ matrix $\mathbf{P}^\varepsilon(z)$, we let $\mathcal{X}^\varepsilon(z)$ be the matrix obtained by concatenating the absolute values and phases of the entries in $\mathbf{P}^\varepsilon(z)$. The driving random process $\mathcal{Y}$ is given by $\mu(z)$, $\nu(z)$, and their derivatives, which are stationary, mean zero, and mixing by assumption. The expression of functions $\mathcal{F}$ and $\mathcal{G}$ follows from (4.4) and the chain rule. The dependence on the fast variable $\tau = z/\varepsilon^2$ is in the arguments of the cos and sin functions, the real and imaginary parts of the complex exponentials in (3.45)-(3.47).

4.3. The forward scattering approximation. When we use the diffusion-approximation theorem in [18], we obtain that the limit entries of $\mathbf{P}^{\varepsilon,b}(z)$ are coupled to the limit entries of $\mathbf{P}^{\varepsilon,a}(z)$ through the coefficients

$$\widehat{\mathcal{R}}_\nu(\beta_j + \beta_l) = 2\int_0^\infty dz\, \mathcal{R}_\nu(z) \cos[(\beta_j + \beta_l)z],$$
$$\widehat{\mathcal{R}}_\mu(\beta_j + \beta_l) = 2\int_0^\infty dz\, \mathcal{R}_\mu(z) \cos[(\beta_j + \beta_l)z],$$

for $j, l = 1, \ldots, N$. Here $\widehat{\mathcal{R}}_\nu$ and $\widehat{\mathcal{R}}_\mu$ are the power spectral densities of the processes ν and μ, the Fourier transform of their covariance functions. They are evaluated at the sum of the wavenumbers $\beta_j + \beta_l$ because the phase factors present in the matrix $\mathbf{H}^{(b)}(z)$ are $\pm(\beta_j + \beta_l)z$. The limit entries of $\mathbf{P}^{\varepsilon,a}(z)$ are coupled to each other through the power spectral densities evaluated at the difference of the wavenumbers, $\widehat{\mathcal{R}}_\nu(\beta_j - \beta_l)$ and $\widehat{\mathcal{R}}_\mu(\beta_j - \beta_l)$, for $j, l = 1, \ldots, N$, because the phase factors in the matrix $\mathbf{H}^{(a)}(z)$ are $\pm(\beta_j - \beta_l)z$. Thus, if we assume that the power spectral densities are small at large frequencies, we may make the approximation

$$\widehat{\mathcal{R}}_\nu(\beta_j + \beta_l) \approx 0, \qquad \widehat{\mathcal{R}}_\mu(\beta_j + \beta_l) \approx 0, \quad \text{for } \ j, l = 1, \ldots, N, \qquad (4.9)$$

which implies that we can neglect coupling between the forward and backward propagating modes as $\varepsilon \to 0$. The forward going modes remain coupled to each other, because at least some combinations of the indexes j,l, for instance those with $|j-l|=1$, give non-zero coupling coefficients $\widehat{\mathcal{R}}_\nu(\beta_j-\beta_l)$ and $\widehat{\mathcal{R}}_\mu(\beta_j-\beta_l)$.

Because the backward going mode amplitudes satisfy the homogeneous end condition $\widehat{b}_j^\varepsilon(L)=0$, and because they are asymptotically uncoupled from $\{\widehat{a}_j^\varepsilon\}_{j=1,\dots,N}$, we can set them to zero. This is the forward scattering approximation, where the forward propagating mode amplitudes satisfy the closed system

$$\frac{d\widehat{\boldsymbol{a}}^\varepsilon}{dz} = \frac{1}{\varepsilon}\mathbf{H}^{(a)}\left(\frac{z}{\varepsilon^2}\right)\widehat{\boldsymbol{a}}^\varepsilon + \mathbf{G}^{(a)}\left(\frac{z}{\varepsilon^2}\right)\widehat{\boldsymbol{a}}^\varepsilon \quad \text{for } z>0, \ \text{ and } \widehat{a}_j^\varepsilon(z=0)=\widehat{a}_{j,o}. \tag{4.10}$$

REMARK 4.1. Note that the matrix $\mathbf{H}^{(a)}$ is not skew Hermitian, which implies that for a given ε there is no conservation of energy of the forward propagating modes over the randomly perturbed region,

$$\sum_{j=1}^N |\widehat{a}_j^\varepsilon(L)|^2 \neq \sum_{j=1}^N |\widehat{a}_{j,o}|^2.$$

This is due to the local exchange of energy between the propagating and evanescent modes. However, we will see that the energy of the forward propagating modes is conserved in the limit $\varepsilon \to 0$.

4.4. The coupled mode diffusion process. We now apply the diffusion approximation theorem to the system (4.10) and obtain, after a long calculation, that we do not include for brevity, the main result of this section.

THEOREM 4.2. *The complex mode amplitudes $\{\widehat{a}_j^\varepsilon(\omega,z)\}_{j=1,\dots,N}$ converge in distribution as $\varepsilon \to 0$ to a diffusion Markov process process $\{\widehat{a}_j(\omega,z)\}_{j=1,\dots,N}$ with generator $\mathcal{L}$ given below.*

Let us write the limit process as

$$\widehat{a}_j(\omega,z) = P_j(\omega,z)^{1/2} e^{i\theta_j(\omega,z)}, \quad j=1,\dots,N,$$

in terms of the power $|\widehat{a}_j|^2 = P_j$ and the phase θ_j. Then we can express the infinitesimal generator $\mathcal{L}$ of the limit diffusion as the sum of two operators

$$\mathcal{L} = \mathcal{L}_P + \mathcal{L}_\theta. \tag{4.11}$$

The first is a partial differential operator in the powers

$$\mathcal{L}_P = \sum_{\substack{j,l=1 \\ j\neq l}}^N \Gamma_{jl}^{(c)}(\omega)\left[P_l P_j \left(\frac{\partial}{\partial P_j} - \frac{\partial}{\partial P_l}\right)\frac{\partial}{\partial P_j} + (P_l - P_j)\frac{\partial}{\partial P_j}\right], \tag{4.12}$$

with matrix $\mathbf{\Gamma}^{(c)}(\omega)$ of coefficients that are non-negative off the diagonal, and sum to zero in the rows, i.e.

$$\Gamma_{jj}^{(c)}(\omega) = -\sum_{l\neq j} \Gamma_{jl}^{(c)}(\omega). \tag{4.13}$$

The off-diagonal entries are defined by the power spectral densities of the fluctuations ν and μ, and the derivatives of the eigenfunctions at the boundaries,

$$\Gamma_{jl}^{(c)}(\omega) = \frac{X^2}{4\beta_j(\omega)\beta_l(\omega)} \Big\{ [\partial_\xi \phi_j(\omega,X)\partial_\xi \phi_l(\omega,X)]^2 \widehat{\mathcal{R}}_\nu[\beta_j(\omega)-\beta_l(\omega)] + [\partial_\xi \phi_j(\omega,0)\partial_\xi \phi_l(\omega,0)]^2 \widehat{\mathcal{R}}_\mu[\beta_j(\omega)-\beta_l(\omega)] \Big\}, \quad j \neq l. \tag{4.14}$$

The second partial differential operator is with respect to the phases

$$\mathcal{L}_\theta = \frac{1}{4} \sum_{\substack{j,l=1\\ j\neq l}}^{N} \Gamma_{jl}^{(c)}(\omega) \left[\frac{P_j}{P_l}\frac{\partial^2}{\partial \theta_l^2} + \frac{P_l}{P_j}\frac{\partial^2}{\partial \theta_j^2} + 2\frac{\partial^2}{\partial\theta_j\partial\theta_l}\right] + \frac{1}{2}\sum_{j,l=1}^{N} \Gamma_{jl}^{(0)}(\omega)\frac{\partial^2}{\partial\theta_j\partial\theta_l} + \frac{1}{2} \sum_{\substack{j,l=1\\ j\neq l}}^{N} \Gamma_{jl}^{(s)}(\omega)\frac{\partial}{\partial\theta_j} + \sum_{j=1}^{N} \kappa_j(\omega)\frac{\partial}{\partial\theta_j}, \tag{4.15}$$

with nonnegative coefficients

$$\Gamma_{jl}^{(0)}(\omega) = \frac{X^2}{4\beta_j(\omega)\beta_l(\omega)} \Big\{ [\partial_\xi \phi_j(\omega,X)\partial_\xi \phi_l(\omega,X)]^2 \widehat{\mathcal{R}}_\nu(0) + [\partial_\xi \phi_j(\omega,0)\partial_\xi \phi_l(\omega,0)]^2 \widehat{\mathcal{R}}_\mu(0) \Big\} \tag{4.16}$$

and

$$\Gamma_{jl}^{(s)}(\omega) = \frac{X^2}{4\beta_j(\omega)\beta_l(\omega)} \left\{ [\partial_\xi \phi_j(\omega,X)\partial_\xi \phi_l(\omega,X)]^2 \gamma_{\nu,jl}(\omega) + [\partial_\xi \phi_j(\omega,0)\partial_\xi \phi_l(\omega,0)]^2 \gamma_{\mu,jl}(\omega) \right\}, \tag{4.17}$$

for $j \neq l$, where

$$\gamma_{\nu,jl}(\omega) = 2\int_0^\infty dz \sin[(\beta_j(\omega)-\beta_l(\omega))z]\,\mathcal{R}_\nu(z), \tag{4.18}$$

$$\gamma_{\mu,jl}(\omega) = 2\int_0^\infty dz \sin[(\beta_j(\omega)-\beta_l(\omega))z]\,\mathcal{R}_\mu(z). \tag{4.19}$$

The diagonal part of $\Gamma^{(s)}(\omega)$ is defined by

$$\Gamma_{jj}^{(s)}(\omega) = -\sum_{l\neq j} \Gamma_{jl}^{(s)}(\omega). \tag{4.20}$$

All the terms in the generator except for the last one in (4.15) are due to the direct coupling of the propagating modes. The coefficient κ_j in the last term is

$$\kappa_j(\omega) = \kappa_j^{(a)}(\omega) + \kappa_j^{(e)}(\omega), \tag{4.21}$$

with the first part due to the direct coupling of the propagating modes and given by

$$\kappa_j^{(a)} = \mathcal{R}_\nu(0) \Bigg\{ \int_0^X d\xi \left[\frac{\omega^2}{4\beta_j}\xi^2 \phi_j^2 \partial_\xi^2 c^{-2} - \frac{3}{2\beta_j}(\partial_\xi \phi_j)^2 \right] + \sum_{l\neq j, l=1}^{N} (\beta_l+\beta_j)\left[d_{\nu,jl}^2(\beta_l^2-\beta_j^2) + 2 d_{\nu,jl} c_{\nu,jl}\right] \Bigg\}$$

$$-\mathcal{R}_\nu''(0)\left\{\frac{1}{4\beta_j}-\frac{1}{2\beta_j}\int_0^X d\xi\,\xi^2(\partial_\xi\phi_j)^2+\sum_{l\neq j,l=1}^{N}(\beta_l-\beta_j)d_{\nu,jl}^2\right\} \;+\; \mu \text{ terms}, \quad (4.22)$$

with the abbreviation "μ terms" for the similar contribution of the μ process. The coupling via the evanescent modes determines the second term in (4.21), and it is given by

$$\kappa_j^{(e)}=\sum_{l=N+1}^{\infty}\frac{X^2\left[\partial_\xi\phi_j(X)\partial_\xi\phi_l(X)\right]^2}{2\beta_j\beta_l(\beta_j^2+\beta_l^2)^2}\int_0^\infty ds\,e^{-\beta_l s}\mathcal{R}_\nu''(s)\left[(\beta_l^2-\beta_j^2)\cos(\beta_j s)-2\beta_j\beta_l\sin(\beta_j s)\right]$$
$$+\sum_{l=N+1}^{\infty}2\beta_l\left[-d_{\nu,lj}^2\mathcal{R}_\nu''(0)+\frac{c_{\nu,lj}^2}{\beta_j^2+\beta_l^2}\mathcal{R}_\nu(0)\right]+\mu \text{ terms}. \quad (4.23)$$

4.4.1. Discussion. We now describe some properties of the diffusion process $\widehat{\boldsymbol{a}}$:

1. Note that the coefficients of the partial derivatives in P_j of the infinitesimal generator $\mathcal{L}$ depend only on $\{P_l\}_{l=1,\ldots,N}$. This means that the mode powers $\{|\widehat{a}_j^\varepsilon(\omega,z)|^2\}_{j=1,\ldots,N}$ converge in distribution as $\varepsilon\to 0$ to the diffusion Markov process $\{|\widehat{a}_j(\omega,z)|^2=P_j(\omega,z)\}_{j=1,\ldots,N}$, with generator $\mathcal{L}_P$.
2. As we remarked before, the evanescent modes influence only the coefficient $\kappa_j(\omega)$ which appears in $\mathcal{L}_\theta$ but not in $\mathcal{L}_P$. This means that the evanescent modes do not change the energies of the propagating modes in the limit $\varepsilon\to 0$. They also do not affect the coupling of the modes of the limit process, because κ_j is in the diagonal part of (4.15). The only effect of the evanescent modes is a net dispersion (frequency-dependent phase modulation) for each propagating mode.
3. The generator $\mathcal{L}$ can also be written in the equivalent form [5, Section 20.3]

$$\mathcal{L}=\frac{1}{4}\sum_{\substack{j,l=1\\ j\neq l}}\Gamma_{jl}^{(c)}(\omega)\left(A_{jl}\overline{A_{jl}}+\overline{A_{jl}}A_{jl}\right)+\frac{1}{2}\sum_{j,l=1}^{N}\Gamma_{jl}^{(0)}(\omega)A_{jj}\overline{A_{ll}}$$
$$+\frac{i}{4}\sum_{\substack{j,l=1\\ j\neq l}}\Gamma_{jl}^{(s)}(\omega)(A_{jj}-A_{ll})+i\sum_{j=1}^{N}\kappa_j(\omega)A_{jj}, \quad (4.24)$$

in terms of the differential operators

$$A_{jl}=\widehat{a}_j\frac{\partial}{\partial\widehat{a}_l}-\overline{\widehat{a}_l}\frac{\partial}{\partial\overline{\widehat{a}_j}}=-\overline{A_{lj}}. \quad (4.25)$$

Here the complex derivatives are defined in the standard way: if $z=x+iy$, then $\partial_z=(1/2)(\partial_x-i\partial_y)$ and $\partial_{\overline{z}}=(1/2)(\partial_x+i\partial_y)$.
4. The coefficients of the second derivatives in (4.24) are homogeneous of degree two, while the coefficients of the first derivatives are homogeneous of degree one. This implies that we can write closed ordinary differential equations in the limit $\varepsilon\to 0$ for the moments of any order of $\{\widehat{a}_j^\varepsilon\}_{j=1,\ldots,N}$.

5. Because

$$\mathcal{L}\left(\sum_{l=1}^{N}|\widehat{a}_l|^2\right)=0, \tag{4.26}$$

we have conservation of energy of the limit diffusion process. More explicitly, the process is supported on the sphere in $\mathbb{C}^N$ with center at zero and radius R_o determined by the initial condition

$$R_o^2=\sum_{l=1}^{N}|\widehat{a}_{l,o}(\omega)|^2.$$

Since $\mathcal{L}$ is not self-adjoint on the sphere, the process is not reversible. But the uniform measure on the sphere is invariant, and the generator is strongly elliptic. From the theory of irreducible Markov processes with compact state space, we know that the process is ergodic and thus $\widehat{\boldsymbol{a}}(z)$ converges for large z to the uniform distribution over the sphere of radius R_o. This can be used to compute the limit distribution of the mode powers $(|\widehat{a}_j|^2)_{j=1,\dots,N}$ for large z, which is the uniform distribution over the set

$$\mathcal{H}_N=\left\{\{P_j\}_{j=1,\dots,N}, P_j\geq 0, \sum_{j=1}^{N}P_j=R_o^2\right\}. \tag{4.27}$$

We carry out a more detailed analysis that is valid for any z in the next section.

4.4.2. Independence of the change of coordinates that flatten the boundaries. The coefficients (4.14), (4.16), and (4.17) of the generator $\mathcal{L}$ have simple expressions and are determined only by the covariance functions of the fluctuations $\nu(z)$ and $\mu(z)$ and the boundary values of the derivatives of the eigenfunctions $\phi_j(\omega,\xi)$ in the unperturbed waveguide. The dispersion coefficient κ_j has a more complicated expression (4.21)-(4.23), which involves integrals of products of the eigenfunctions and their derivatives with powers of ξ or $X-\xi$. These factors in ξ are present in our change of coordinates

$$\ell^\varepsilon(z,\xi)=B(z)+[T(z)-B(z)]\frac{\xi}{X}=\xi+\varepsilon\left[(X-\xi)\mu(z)+\xi\nu(z)\right], \tag{4.28}$$

so it is natural to ask if the generator $\mathcal{L}$ depends on the change of coordinates. We show here that this is not the case.

Let $F^\varepsilon(z,\xi)\in C^1([0,\infty)\times[0,X])$ be a general change of coordinates satisfying

$$F^\varepsilon(z,\xi)=\begin{cases} X(1+\varepsilon\nu(z)) & \text{for } \xi=X,\\ \varepsilon X\mu(z) & \text{for } \xi=0\end{cases} \tag{4.29}$$

for each $\varepsilon>0$, and converging uniformly to the identity mapping as $\varepsilon\to 0$,

$$\sup_{z\geq 0}\sup_{\xi\in[0,X]}|F^\varepsilon(z,\xi)-\xi|=O(\varepsilon), \qquad \sup_{z\geq 0}\sup_{\xi\in[0,X]}|\partial_z F^\varepsilon(z,\xi)|=O(\varepsilon). \tag{4.30}$$

Note that (4.30) is not restrictive in our context since $(\mu(z),\nu(z))$ and their derivatives are uniformly bounded. Define the wavefield

$$\widehat{w}(\omega,\xi,z)=\widehat{p}(\omega,F^\varepsilon(z,\xi),z), \tag{4.31}$$

and decompose it into the waveguide modes, as we did for $\widehat{u}(\omega,\xi,z)=\widehat{p}(\omega,\ell^\varepsilon(z,\xi),z)$. We have the following result proved in Appendix B.

THEOREM 4.3. *The amplitudes of the propagating modes of the wave field* (4.31) *converge in distribution as $\varepsilon\to 0$ to the same limit diffusion as in Theorem 4.2.*

4.4.3. The loss of coherence of the wave field. From Theorem 4.2 and the expression (4.24) of the generator we get by direct calculation the following result for the mean mode amplitudes.

PROPOSITION 4.4. *As $\varepsilon\to 0$, $\mathbb{E}[\widehat{a}_j^\varepsilon(\omega,z)]$ converges to the expectation of the limit diffusion $\widehat{a}_j(\omega,z)$, given by*

$$\mathbb{E}[\widehat{a}_j(\omega,z)]=\widehat{a}_{j,o}(\omega)\exp\left\{\left[\frac{\Gamma_{jj}^{(c)}(\omega)-\Gamma_{jj}^{(0)}(\omega)}{2}\right]z+i\left[\frac{\Gamma_{jj}^{(s)}(\omega)}{2}+\kappa_j(\omega)\right]z\right\}. \tag{4.32}$$

As we remarked before, $\Gamma_{jj}^{(c)}-\Gamma_{jj}^{(0)}$ is negative, so the mean mode amplitudes decay exponentially with the range z. Furthermore, we see from (4.14) and (4.16) that $\Gamma_{jj}^{(c)}-\Gamma_{jj}^{(0)}$ is the sum of terms proportional to $(\partial_\xi\phi_j(X))^2/\beta_j$ and $(\partial_\xi\phi_j(0))^2/\beta_j$. These terms increase with j, and they can be very large when $j\sim N$. Thus, the mean amplitudes of the high order modes decay faster in z than the ones of the low order modes. We return to this point in Section 5, where we estimate the net attenuation of the wave field in the high frequency regime $N\gg 1$.

That the mean field decays exponentially with range implies that the wave field loses its coherence, and energy is transferred to its incoherent part — the fluctuations. The incoherent part of the amplitude of the j-th mode is $\widehat{a}_j^\varepsilon-\mathbb{E}[\widehat{a}_j^\varepsilon]$, and its intensity is given by the variance $\mathbb{E}[|\widehat{a}_j^\varepsilon|^2]-\left|\mathbb{E}[\widehat{a}_j^\varepsilon]\right|^2$. The mode is incoherent if its mean amplitude is dominated by the fluctuations, that is if

$$\left[\mathbb{E}[|\widehat{a}_j^\varepsilon|^2]-\left|\mathbb{E}[\widehat{a}_j^\varepsilon]\right|^2\right]^{1/2}\gg\left|\mathbb{E}[\widehat{a}_j^\varepsilon]\right|.$$

We know that the right hand side converges to (4.32) as $\varepsilon\to 0$. We calculate next the limit of the mean powers $\mathbb{E}[|\widehat{a}_j^\varepsilon|^2]$.

4.4.4. Coupled power equations and equipartition of energy. As we remarked in Section 4.4.1, the mode powers $|\widehat{a}_j^\varepsilon(\omega,z)|^2$, for $j=1,\ldots,N$, converge in distribution as $\varepsilon\to 0$ to the diffusion Markov process $(P_j(\omega,z))_{j=1,\ldots,N}$ supported in the set (4.27), and with infinitesimal generator $\mathcal{L}_P$. We use this result to calculate the limit of the mean mode powers

$$P_j^{(1)}(\omega,z)=\mathbb{E}[P_j(\omega,z)]=\lim_{\varepsilon\to 0}\mathbb{E}[|\widehat{a}_j^\varepsilon(\omega,z)|^2].$$

PROPOSITION 4.5. *As $\varepsilon\to 0$, $\mathbb{E}[|\widehat{a}_j^\varepsilon(\omega,z)|^2]$ converges to $P_j^{(1)}(\omega,z)$, the solution of the coupled linear system*

$$\frac{dP_j^{(1)}}{dz}=\sum_{j=1}^N\Gamma_{jn}^{(c)}(\omega)\left(P_n^{(1)}-P_j^{(1)}\right),\quad z>0, \tag{4.33}$$

with initial condition $P_j^{(1)}(\omega,z=0)=|\widehat{a}_{j,o}(\omega)|^2$, for $j=1,\ldots,N$.

Matrix $\boldsymbol{\Gamma}^{(c)}(\omega)$ is symmetric, with rows summing to zero, by definition. Thus, we can rewrite (4.33) in vector-matrix form

$$\frac{d\boldsymbol{P}^{(1)}}{dz} = \boldsymbol{\Gamma}^{(c)}(\omega)\boldsymbol{P}^{(1)}, \quad z > 0, \text{ and } \boldsymbol{P}^{(1)}(\omega,0) = \boldsymbol{P}_o^{(1)}(\omega), \tag{4.34}$$

with $\boldsymbol{P}^{(1)}(\omega,z) = \left(P_1^{(1)}(\omega,z),\dots,P_n^{(1)}(\omega,z)\right)^T$ and $\boldsymbol{P}_o^{(1)}(\omega)$ the vector with components $|\widehat{a}_{j,o}(\omega)|^2$, for $j = 1,\dots,N$. The solution is given by the matrix exponential

$$\boldsymbol{P}^{(1)}(\omega,z) = \exp\left[\boldsymbol{\Gamma}^{(c)}(\omega)z\right]\boldsymbol{P}_o^{(1)}(\omega). \tag{4.35}$$

We know from (4.14) that the off-diagonal entries in $\boldsymbol{\Gamma}^{(c)}(\omega)$ are nonnegative. If we assume that they are positive, which is equivalent to asking that the power spectral densities of ν and μ do not vanish at the arguments $\beta_j - \beta_l$, for all $j,l = 1,\dots,N$, we can apply the Perron-Frobenius theorem to conclude that zero is a simple eigenvalue of $\boldsymbol{\Gamma}^{(c)}(\omega)$, and that all the other eigenvalues are negative,

$$\Lambda_N(\omega) \le \cdots \le \Lambda_2(\omega) < 0.$$

This shows that as the range z grows, the vector $\boldsymbol{P}^{(1)}(z)$ tends to the null space of $\boldsymbol{\Gamma}^{(c)}$, the span of the vector $(1,\dots,1)^T$. That is to say, the mode powers converge to the uniform distribution in the set (4.27) at exponential rate

$$\sup_{j=1,\dots,N}\left|P_j^{(1)}(\omega,z) - \frac{R_o^2(\omega)}{N}\right| \le Ce^{-|\Lambda_2(\omega)|z}. \tag{4.36}$$

As $z \to \infty$, we have equipartition of energy among the propagating modes.

4.4.5. Fluctuations of the mode powers. To estimate the fluctuations of the mode powers, we use again Theorem 4.2 to compute the fourth order moments of the mode amplitudes:

$$P_{jl}^{(2)}(\omega,z) = \lim_{\varepsilon\to 0}\mathbb{E}\left[|\widehat{a}_j^\varepsilon(\omega,z)|^2|\widehat{a}_l^\varepsilon(\omega,z)|^2\right] = \mathbb{E}[P_j(\omega,z)P_l(\omega,z)].$$

Using the generator $\mathcal{L}_P$, we get the following coupled system of ordinary differential equations for limit moments:

$$\begin{aligned}
\frac{dP_{jj}^{(2)}}{dz} &= \sum_{\substack{n=1\\ n\neq j}}^{N}\Gamma_{jn}^{(c)}(\omega)\left(4P_{jn}^{(2)} - 2P_{jj}^{(2)}\right),\\
\frac{dP_{jl}^{(2)}}{dz} &= -2\Gamma_{jl}^{(c)}(\omega)P_{jl}^{(2)} + \sum_{n=1}^{N}\Gamma_{ln}^{(c)}(\omega)\left(P_{jn}^{(2)} - P_{jl}^{(2)}\right)\\
&\quad + \sum_{n=1}^{N}\Gamma_{jn}^{(c)}(\omega)\left(P_{ln}^{(2)} - P_{jl}^{(2)}\right), \quad j \neq l, z > 0,
\end{aligned} \tag{4.37}$$

with initial conditions

$$P_{jl}^{(2)}(\omega,0) = |\widehat{a}_{j,o}(\omega)|^2|\widehat{a}_{l,o}(\omega)|^2. \tag{4.38}$$

The solution of this system can be written again in terms of the exponential of the evolution matrix.

It is straightforward to check that the function $1+\delta_{jl}$ is a stationary solution of (4.37). Using the positivity of $\mathbf{\Gamma}^{(c)}_{jl}(\omega)$ for $j\neq l$, we conclude that this stationary solution is asymptotically stable, meaning that the solution $P^{(2)}_{jl}(\omega,z)$ converges as $z\to\infty$ to

$$P^{(2)}_{jl}(\omega,z) \overset{z\to\infty}{\longrightarrow} \begin{cases} \dfrac{1}{N(N+1)} R_o^4(\omega) & \text{if } j\neq l, \\ \dfrac{2}{N(N+1)} R_o^4(\omega) & \text{if } j=l, \end{cases}$$

where $R_o^2(\omega)=\sum_{j=1}^N |\widehat{a}_{j,o}(\omega)|^2$. This implies that the correlation of $P_j(\omega,z)$ and $P_l(\omega,z)$ converges to $-1/(N-1)$ if $j\neq l$ and to $(N-1)/(N+1)$ if $j=l$ as $z\to\infty$. We see from the $j\neq l$ result that if, in addition, the number of modes N becomes large, then the mode powers become uncorrelated. The $j=l$ result shows that, whatever the number of modes N, the mode powers P_j are not statistically stable quantities in the limit $z\to\infty$, since

$$\frac{\mathrm{Var}(P_j(\omega,z))}{\mathbb{E}[P_j(\omega,z)]^2} \overset{z\to\infty}{\longrightarrow} \frac{N-1}{N+1}.$$

5. Estimation of net diffusion

To illustrate the random boundary cumulative scattering effect over long ranges, we quantify in this section the diffusion coefficients $\Gamma^{(c)}_{jl}$ and $\Gamma^{(0)}_{jl}$ in the generator $\mathcal{L}$ of the limit process. In particular, we calculate the mode-dependent net attenuation rate

$$\mathcal{K}_j(\omega)=\frac{\Gamma^{(0)}_{jj}(\omega)-\Gamma^{(c)}_{jj}(\omega)}{2}, \tag{5.1}$$

which determines the coherent (mean) amplitudes as shown in (4.32). The attenuation rate gives the range scale over which the j-th mode becomes essentially incoherent, because equations (4.32) and (4.35) give

$$\frac{|\mathbb{E}[\widehat{a}_j(\omega,z)]|}{\sqrt{\mathbb{E}\left[|\widehat{a}_j(\omega,z)|^2\right]-|\mathbb{E}[\widehat{a}_j(\omega,z)]|^2}} \ll 1 \qquad \text{if } z\gg \mathcal{K}_j^{-1}.$$

The reciprocal of the attenuation rate can therefore be interpreted as a scattering mean free path. The scattering mean free path is classically defined as the propagation distance beyond which the wave loses its coherence [20]. Here it is mode-dependent.

Note that the attenuation rate $\mathcal{K}_j(\omega)$ is the sum of two terms. The first one involves the phase diffusion coefficient $\Gamma^{(0)}_{jj}$ in the generator $\mathcal{L}_\theta$, and determines the range scale over which the cumulative random phase of the amplitude $\widehat{a}_j$ becomes significant, thus giving exponential damping of the expected field $\mathbb{E}[\widehat{a}_j]$. The second term is the mode-dependent energy exchange rate

$$\mathcal{J}_j(\omega)=-\frac{\Gamma^{(c)}_{jj}(\omega)}{2}, \tag{5.2}$$

given by the power diffusion coefficients in the generator $\mathcal{L}_P$. Each waveguide mode can be associated with a direction of incidence at the unperturbed boundary, and energy is exchanged between modes when they scatter, because of the fluctuation of the angles of incidence at the random boundaries. We can interpret the reciprocal of the energy exchange rate as a transport mean free path, which is classically defined as the distance beyond which the wave forgets its initial direction [20].

The third important length scale is the equipartition distance $1/|\Lambda_2(\omega)|$, defined in terms of the second largest eigenvalue of the matrix $\mathbf{\Gamma}^{(c)}(\omega)$. It is the distance over which the energy becomes uniformly distributed over the modes, independently of the initial excitation at the source, as shown in Equation (4.36).

5.1. Estimates for a waveguide with constant wave speed. To give sharp estimates of $\mathcal{K}_j$ and $\mathcal{J}_j$ for $j=1,\ldots,N$, we assume in this section a waveguide with constant wave speed $c(\xi)=c_o$ and a high frequency regime $N\gg 1$. Note from (4.13) that the magnitude of $\Gamma^{(c)}_{jj}$ depends on the rate of decay of the power spectral densities $\widehat{\mathcal{R}}_\nu(\beta)$ and $\widehat{\mathcal{R}}_\mu(\beta)$ with respect to the argument β. We already made the assumption (4.9) on the decay of the power spectral densities in order to justify the forward scattering approximation. In particular, we assumed that $\widehat{\mathcal{R}}_\nu(\beta)\simeq\widehat{\mathcal{R}}_\mu(\beta)\simeq 0$ for all $\beta\geq 2\beta_N$. Thus, for a given mode index j, we expect large terms in the sum in (4.13) for indices l satisfying

$$|\beta_j-\beta_l|\lesssim 2\beta_N=\frac{2\pi}{X}\sqrt{2\alpha N}, \tag{5.3}$$

where we used the definition

$$\beta_j=\frac{\pi}{X}\sqrt{(N+\alpha)^2-j^2},\quad j=1,\ldots,N,\quad \text{and}\quad \frac{kX}{\pi}=N+\alpha,\quad \text{for}\quad \alpha\in(0,1). \tag{5.4}$$

Still, it is difficult to get a precise estimate of $\Gamma^{(c)}_{jj}$ given by (4.13), unless we make further assumptions on $\mathcal{R}_\nu$ and $\mathcal{R}_\mu$. For the calculations in this section we take the Gaussian covariance functions

$$\mathcal{R}_\nu(z)=\exp\left(-\frac{z^2}{2\ell_\nu^2}\right)\quad \text{and}\quad \mathcal{R}_\mu(z)=\exp\left(-\frac{z^2}{2\ell_\mu^2}\right), \tag{5.5}$$

and we take for convenience equal correlation lengths $\ell_\nu=\ell_\mu=\ell$. The power spectral densities are

$$\widehat{\mathcal{R}}_\nu(\beta)=\widehat{\mathcal{R}}_\mu(\beta)=\sqrt{2\pi}\,\ell\exp\left(-\frac{\beta^2\ell^2}{2}\right), \tag{5.6}$$

and they are negligible for $\beta\geq 3/\ell$. Since $N=\lfloor kX/\pi\rfloor$, we see that (5.3) becomes

$$|\beta_j-\beta_l|\leq\frac{3}{\ell}\lesssim\frac{2\pi}{X}\sqrt{2\alpha N}\quad \text{or, equivalently,}\quad k\ell\gtrsim\frac{3}{2\sqrt{2\alpha}}\sqrt{N}\gg 1. \tag{5.7}$$

Thus, Assumption (4.9) amounts to having correlation lengths that are larger than the wavelength. The attenuation and exchange energy rates (5.1) and (5.2) are estimated in detailed in Appendix C. We summarize the results in the following proposition, in the case[2]

$$\sqrt{N}\lesssim k\ell\ll N. \tag{5.8}$$

[2] The case $k\ell\gtrsim N$ is also discussed in Appendix C.

PROPOSITION 5.1. *The attenuation rate $\mathcal{K}_j(\omega)$ increases monotonically with the mode index j. The energy exchange rate $\mathcal{J}_j(\omega)$ increases monotonically with the mode index j up to the high modes of order N where it can decay if $k\ell \gg \sqrt{N}$. For the low order modes we have*

$$\mathcal{J}_j(\omega)X \approx \mathcal{K}_j(\omega)X \sim (k\ell)^{-1/2}, \quad j \sim 1. \tag{5.9}$$

For the intermediate modes we have

$$\mathcal{J}_j(\omega)X \approx \mathcal{K}_j(\omega)X \sim N^2 \frac{(j/N)^3}{\sqrt{1-(j/N)^2}}, \quad 1 \ll j \ll N. \tag{5.10}$$

For the high order modes we have

$$\mathcal{J}_j(\omega)X \sim \frac{N^3}{k\ell}, \qquad \mathcal{K}_j(\omega)X \sim k\ell N^2, \quad j \sim N, \tag{5.11}$$

for $k\ell \sim \sqrt{N}$, but when $k\ell \gg \sqrt{N}$,

$$\mathcal{J}_j(\omega)X \ll \mathcal{K}_j(\omega)X \sim k\ell N^2, \quad j \sim N. \tag{5.12}$$

The results summarized in Proposition 5.1 show that scattering from the random boundaries has a much stronger effect on the high order modes than the low order ones. This is intuitive, because the modes with large index bounce more often from the boundaries. The damping rate $\mathcal{K}_j$ is very large, of order $N^2 k\ell$ for $j \sim N$, which means that the amplitudes of these modes become incoherent quickly, over scaled[3] ranges $z \sim XN^{-2}(k\ell)^{-1} \ll X$. The modes with index $j \sim 1$ keep their coherence over ranges $z = O(X)$, because their mean amplitudes are essentially undamped, that is $\mathcal{K}_j X \ll 1$ for $j \sim 1$. However, the modes lose their coherence eventually because the damping becomes visible at longer ranges, that is $z > X(k\ell)^{1/2}$.

Note that the scattering mean free paths and the transport mean free paths are approximately the same for the low and intermediate index modes, but not for the high ones. The energy exchange rate for the high order modes may be much smaller than the attenuation rate in high frequency regimes with $k\ell \gg \sqrt{N}$. These modes reach the boundary many times over a correlation length, at almost the same angle of incidence, so the exchange of energy is not efficient and it occurs only between neighboring modes. There is however a significant cumulative random phase in $\widehat{a}_j$ for $j \sim N$, given by the addition of the correlated phases gathered over the multiple scattering events. This significant phase causes the loss of coherence of the amplitudes of the high order modes, the strong damping of $\mathbb{E}[\widehat{a}_j]$.

Note also that a direct calculation[4] of the second largest eigenvalue of $\mathbf{\Gamma}^{(c)}(\omega)$ gives that

$$|\Lambda_2(\omega)| \approx |\Gamma_{11}^{(c)}(\omega)| \sim (k\ell)^{-1/2}.$$

Thus, the equipartition distance is similar to the scattering mean free path of the first mode. This mode can travel longer distances than the others before it loses its coherence, but once that happens, the waves have entered the equipartition regime, where the energy is uniformly distributed among all the modes. The waves forget the initial condition at the source.

[3] Recall from Section 4.1 that the range is actually z/ε^2.

[4] By direct calculation we mean numerical calculation of the eigenvalue. We find that for $N \geq 20$ and for $k\ell \gtrsim \sqrt{N}$, $|\Lambda_2(\omega)| \approx |\Gamma_{11}^{(c)}(\omega)|$ with a relative error that is less than 1%.

5.2. Comparison with waveguides with internal random inhomogeneities. When we compare the results in Proposition 5.1 with those in [5, Chapter 20] for random waveguides with interior inhomogeneities but straight boundaries, we see that even though the random amplitudes of the propagating modes converge to a Markov diffusion process with the same form of the generator as (4.24), the net effects on coherence and energy exchange are different in terms of their dependence with respect to the modes.

Let us look in detail at the attenuation rate that determines the range scale over which the amplitudes of the propagating modes lose coherence. To distinguish it from (5.1), we denote the attenuation rate by $\widetilde{\mathcal{K}}_j$ and the energy exchange rate by $\widetilde{\mathcal{J}}_j$, and recall from [5, Section 20.3.1] that they are given by

$$\widetilde{\mathcal{K}}_j = \frac{k^4 \widehat{\mathcal{R}}_{jj}(0)}{8\beta_j^2} + \widetilde{\mathcal{J}}_j, \qquad \widetilde{\mathcal{J}}_j = \sum_{\substack{l=1 \\ l \neq j}}^{N} \frac{k^4}{8\beta_j \beta_l} \widehat{\mathcal{R}}_{jl}(\beta_j - \beta_l). \tag{5.13}$$

Here $\widehat{\mathcal{R}}_{jl}(z)$ is the Fourier transform (power spectral density) of the covariance function $\mathcal{R}_{jl}(z)$ of the stationary random processes

$$C_{jl}(z) = \int_0^X dx\, \phi_j(x) \phi_l(x) \nu(x,z),$$

the projection on the eigenfunctions of the random fluctuations $\nu(x,z)$ of the wave speed.

For our comparison we assume isotropic, stationary fluctuations with mean zero and Gaussian covariance function

$$\mathcal{R}(x,z) = \mathbb{E}[\nu(x,z)\nu(0,0)] = e^{-\frac{x^2+z^2}{2\ell^2}},$$

so the power spectral densities are

$$\widehat{\mathcal{R}}_{jl}(\beta) \approx \frac{\pi \ell^2}{X} e^{-\frac{(k\ell)^2}{2}\left(\frac{X\beta}{\pi N}\right)^2} \left[e^{-\frac{(k\ell)^2}{2}\left(\frac{j}{N} - \frac{l}{N}\right)^2} + e^{-\frac{(k\ell)^2}{2}\left(\frac{j}{N} + \frac{l}{N}\right)^2} + \delta_{jl} \right]. \tag{5.14}$$

Thus, (5.13) becomes

$$\begin{aligned} \widetilde{\mathcal{K}}_j &= \frac{\pi (k\ell)^2}{8X} \frac{2 + e^{-2(k\ell)^2 (j/N)^2}}{(1+\alpha/N)^2 - (j/N)^2} + \widetilde{\mathcal{J}}_j, \\ \widetilde{\mathcal{J}}_j &= \frac{\pi (k\ell)^2}{8X} \sum_{\substack{l=1 \\ l \neq j}}^{N} \left\{ \frac{e^{-\frac{(k\ell)^2}{2}\left[\sqrt{(1+\alpha/N)^2 - (j/N)^2} - \sqrt{(1+\alpha/N)^2 - (l/N)^2}\right]^2}}{\sqrt{\left[(1+\alpha/N)^2 - (j/N)^2\right]\left[(1+\alpha/N)^2 - (l/N)^2\right]}} \right. \\ &\qquad \left. \times \left[e^{-\frac{(k\ell)^2}{2}\left(\frac{j}{N} - \frac{l}{N}\right)^2} + e^{-\frac{(k\ell)^2}{2}\left(\frac{j}{N} + \frac{l}{N}\right)^2} \right] \right\}, \end{aligned}$$

and their estimates can be obtained using the same techniques as in Appendix C. We give here the results when $k\ell$ satisfies (5.8). For the low order modes we have

$$\widetilde{\mathcal{K}}_j X \approx \frac{\pi (k\ell)^2}{8} \left[2 + e^{-2(k\ell)^2/N^2} + \frac{N\sqrt{\pi/2}}{k\ell} \right] \sim \left[(k\ell)^2 + N k\ell \right] \sim N k\ell \gtrsim N^{3/2}, \quad j \sim 1,$$

$$\widetilde{\mathcal{J}}_j X \approx \frac{\pi(k\ell)^2}{8} \frac{N\sqrt{\pi/2}}{k\ell} \sim N k\ell \gtrsim N^{3/2}, \quad j \sim 1,$$

and for the high order modes we have

$$\widetilde{\mathcal{K}}_j X \approx \frac{\pi N(k\ell)^2}{8\alpha} \left[1 + \frac{\sqrt{\pi}N}{2\sqrt{2}k\ell}\right] = \left[N(k\ell)^2 + N^2 k\ell\right] \sim N^2 k\ell \gtrsim N^{5/2}, \quad j \sim N,$$
$$\widetilde{\mathcal{J}}_j X \approx \frac{\pi N(k\ell)^2}{8\alpha} \frac{\sqrt{\pi}N}{2\sqrt{2}k\ell} = N^2 k\ell \gtrsim N^{5/2}, \quad j \sim N.$$

Thus, we see that in waveguides with internal random inhomogeneities the low order modes lose coherence much faster than in waveguides with random boundaries. Explicitly, coherence is lost over scaled ranges

$$z \lesssim X N^{-3/2} \ll X.$$

The high order modes, with index $j \sim N$, lose coherence over the range scale

$$z \lesssim X N^{-5/2} \ll X.$$

Moreover, the main mechanism for the loss of coherence is the exchange of energy between neighboring modes. That is to say, the transport mean free path is equivalent to the scattering mean free path for all the modes in random waveguides with interior inhomogeneities. Finally, direct (numerical) calculation shows that

$$O\left((k\ell)^{-2}\right) \le \frac{|\Lambda_2|}{|\widetilde{\mathcal{J}}_1|} \le O\left((k\ell)^{-3/2}\right),$$

so the equipartition distance is larger by a factor of at least $O\left(N^{3/4}\right)$ than the scattering or transport mean free path.

6. Mixed boundary conditions

Up to now we have described in detail the wave field in waveguides with random boundaries and Dirichlet boundary conditions (1.4). In this section we extend the results to the case of mixed boundary conditions (1.5), with Dirichlet condition at $x = B(z)$ and Neumann condition at $x = T(z)$. All permutations of Dirichlet/Neumann conditions are of course possible, and the results can be readily extended.

Similar to what we stated in Section 2, the operator $\partial_x^2 + \omega^2 c^{-2}(x)$ acting on functions in $(0, X)$, with Dirichlet boundary condition at $x = 0$ and Neumann boundary condition $x = X$, is self-adjoint in $L^2(0, X)$. Its spectrum is an infinite number of discrete eigenvalues $\lambda_j(\omega)$, for $j = 1, 2, \ldots$, and we sort them in decreasing order. There is a finite number $N(\omega)$ of positive eigenvalues and an infinite number of negative eigenvalues. We assume as in Section 2 that $N(\omega) = N$ is constant over the frequency band, and that the eigenvalues are simple. The modal wavenumbers are as before, $\beta_j(\omega) = \sqrt{|\lambda_j(\omega)|}$. The eigenfunctions $\phi_j(\omega, x)$ are real and form an orthonormal set.

For example, in the case of a constant wave speed $c(x) = c_o$, we have

$$\lambda_j = k^2 - \left[\frac{(j - 1/2)\pi}{X}\right]^2, \qquad \phi_j(x) = \sqrt{\frac{2}{X}} \sin\left(\frac{(j - 1/2)\pi x}{X}\right), \qquad j = 1, 2, \ldots, \quad (6.1)$$

and the number of propagating modes is given by $N = \left\lfloor \frac{kX}{\pi} + \frac{1}{2} \right\rfloor$.

6.1. Change of coordinates. We proceed as before and straighten the boundaries using a change of coordinates that is slightly more complicated than before, due to the Neumann condition at $x=T(z)$, where the normal is along the vector $(1,-T'(z))$. We let

$$p(t,x,z)=u\big(t,\mathcal{X}(x,z),\mathcal{Z}(x,z)\big), \tag{6.2}$$

where

$$\mathcal{X}(x,z)=X\frac{x-B(z)}{T(z)-B(z)}, \tag{6.3}$$

$$\mathcal{Z}(x,z)=z+xT'(z)+Q(z), \qquad Q(z)=-\int_0^z ds\,T(s)T''(s). \tag{6.4}$$

In the new frame we get that $\xi=\mathcal{X}(x,z)\in[0,X]$, with Dirichlet condition at $\xi=0$,

$$u(t,\xi=0,\zeta)=0. \tag{6.5}$$

For the Neumann condition at $\xi=X$ we use the chain rule, and rewrite

$$\partial_\nu p(t,x=T(z),z)=\big[\partial_x-T'(z)\partial_z\big]p(t,x=T(z),z)=0$$

as

$$\begin{aligned}&\partial_\xi u(t,\xi=X,\zeta=\mathcal{Z}(T(z),z))\big[-\partial_x\mathcal{X}+T'(z)\partial_z\mathcal{X}\big](x=T(z),z)\\ &\quad+\partial_\zeta u(t,\xi=X,\zeta=\mathcal{Z}(T(z),z))\big[-\partial_x\mathcal{Z}+T'(z)\partial_z\mathcal{Z}\big](x=T(z),z)=0.\end{aligned}$$

This is the standard Neumann condition

$$\partial_\xi u(t,\xi=X,\zeta)=0, \tag{6.6}$$

because

$$\big[-\partial_x\mathcal{Z}+T'(z)\partial_z\mathcal{Z}\big](x=T(z),z)=-T'(z)+T'(z)\big[1+T(z)T''(z)+Q'(z)\big]=0,$$

and

$$\big[-\partial_x\mathcal{X}+T'(z)\partial_z\mathcal{X}\big](x=T(z),z)=-\frac{X+[T'(z)]^2}{T(z)-B(z)}\neq 0.$$

Now, the method of solution is as before. Using that ε is small, we obtain a perturbed wave equation for $\widehat{u}$, which we expand as

$$\mathcal{L}_0\widehat{u}+\varepsilon\mathcal{L}_1\widehat{u}+\varepsilon^2\mathcal{L}_2\widehat{u}=O(\varepsilon^3), \tag{6.7}$$

with leading order operator

$$\mathcal{L}_0=\partial_\zeta^2+\partial_\xi^2+\omega^2/c^2(\xi),$$

and perturbation

$$\begin{aligned}\mathcal{L}_1=&-2(\nu-\mu)\partial_\xi^2+2(X-\xi)(\nu'-\mu')\partial_{\zeta\xi}-2X(X-\xi)\nu''\partial_\zeta^2-X(X-\xi)\nu'''\partial_\zeta\\ &-\big[X\mu''+\xi(\nu''-\mu'')\big]\partial_\xi+\omega^2(\partial_\xi c^{-2}(\xi))\big[X\mu+(\nu-\mu)\xi\big].\end{aligned} \tag{6.8}$$

6.2. Coupled amplitude equations. We proceed as in Section 3.2. We find that the complex mode amplitudes satisfy (3.17)-(3.18) with ζ instead of z, where the ζ-dependent coupling coefficients are

$$C^{\varepsilon}_{jl}(\zeta) = \varepsilon C^{(1)}_{jl}(\zeta) + \varepsilon^2 C^{(2)}_{jl}(\zeta) + O(\varepsilon^3), \tag{6.9}$$

$$\begin{aligned} C^{(1)}_{jl}(\zeta) = {} & c_{\nu,jl}\nu(\zeta) + i\beta_l d_{\nu,jl}\nu'(\zeta) + e_{\nu,jl}\nu''(\zeta) + i\beta_l f_{\nu,jl}\nu'''(\zeta) \\ & + c_{\mu,jl}\mu(\zeta) + d_{\mu,jl}\big(2i\beta_l\mu'(\zeta) + \mu''(\zeta)\big), \end{aligned} \tag{6.10}$$

with

$$c_{\nu,jl} = \frac{1}{2\sqrt{\beta_j\beta_l}}\Big[\Big(\frac{\omega^2}{c(X)^2} - \beta_l^2\Big)\phi_j(X)\phi_l(X) + (\beta_j^2 - \beta_j^2)\int_0^X d\xi\,\xi\phi_l\partial_\xi\phi_j\Big], \tag{6.11}$$

$$d_{\nu,jl} = \frac{1}{2\sqrt{\beta_j\beta_l}}\Big[2\int_0^2 d\xi\,(X-\xi)\phi_j\partial_\xi\phi_l\Big], \tag{6.12}$$

$$e_{\nu,jl} = \frac{1}{2\sqrt{\beta_j\beta_l}}\Big[-\int_0^X d\xi\,(X-\xi)\phi_j\xi\partial_\xi\phi_l + 2\beta_l^2\int_0^X d\xi(X-\xi)\phi_j\phi_l\Big], \tag{6.13}$$

$$f_{\nu,jl} = \frac{1}{2\sqrt{\beta_j\beta_l}}\Big[-\int_0^X d\xi\,(X-\xi)\phi_j\phi_l\Big], \tag{6.14}$$

and coefficients $c_{\mu,jl}$ and $d_{\mu,jl}$ defined by (3.49) and (3.51). Similar formulas hold for $C^{(2)}_{jl}(\zeta)$.

In the following we neglect for simplicity the evanescent modes, which only add a dispersive (frequency dependent phase modulation) net effect in the problem. These modes can be included in the analysis using a similar method to that in Section 3.3.

6.3. The coupled mode diffusion process. As we have done in Section 4, we study under the forward scattering approximation the long range limit of the forward propagating mode amplitudes.

First, we give a lemma which shows that the description of the wave field in the variables (x,z) or (ξ,ζ) is asymptotically equivalent.

LEMMA 6.1. *We have uniformly in x*

$$\mathcal{X}\left(x,\frac{z}{\varepsilon^2}\right) - x \overset{\varepsilon\to 0}{\longrightarrow} 0, \qquad \mathcal{Z}\left(x,\frac{z}{\varepsilon^2}\right) - \frac{z}{\varepsilon^2} - \mathbb{E}[\nu'(0)^2]z \overset{\varepsilon\to 0}{\longrightarrow} 0 \text{ in probability.}$$

Proof. The convergence of $\mathcal{X}$ to x is evident from definitions (6.3) and (3.2). Moreover, (6.4) gives

$$\mathcal{Z}\left(x,\frac{z}{\varepsilon^2}\right) - \frac{z}{\varepsilon^2} = x\varepsilon X\nu'\left(\frac{z}{\varepsilon^2}\right) - \varepsilon X^2\int_0^{\frac{z}{\varepsilon^2}}(1+\varepsilon\nu(s))\nu''(s)ds,$$

and integrating by parts and using the assumption that the fluctuations vanish at $z=0$, we get

$$\mathcal{Z}\left(x,\frac{z}{\varepsilon^2}\right) - \frac{z}{\varepsilon^2} = \varepsilon X\left[(x-X)\nu'\left(\frac{z}{\varepsilon^2}\right) - \varepsilon\nu\left(\frac{z}{\varepsilon^2}\right)\nu'\left(\frac{z}{\varepsilon^2}\right)\right] + \varepsilon^2\int_0^{\frac{z}{\varepsilon^2}}\left[\nu'(s)\right]^2 ds.$$

The first term of the right-hand side is of order ε and the second term converges almost surely to $\mathbb{E}[\nu'(0)^2]z$, which gives the result. ∎

The diffusion limit is similar to that in Section 4.4, and the result is as follows.

PROPOSITION 6.2. *The complex mode amplitudes $(\widehat{a}_j^\varepsilon(\omega,\zeta))_{j=1,\ldots,N}$ converge in distribution as $\varepsilon \to 0$ to a diffusion Markov process process $(\widehat{a}_j(\omega,\zeta))_{j=1,\ldots,N}$. Writing*

$$\widehat{a}_j(\omega,\zeta) = P_j(\omega,\zeta)^{1/2} e^{i\phi_j(\omega,\zeta)}, \quad j=1,\ldots,N,$$

the infinitesimal generator of the limiting diffusion process

$$\mathcal{L} = \mathcal{L}_P + \mathcal{L}_\theta$$

is of the form (4.11), but with different expressions of the coefficients given below.

The coefficients $\Gamma_{jl}^{(c)}$ in $\mathcal{L}_P$ are given by

$$\Gamma_{jl}^{(c)}(\omega) = \widehat{\mathcal{R}}_\mu(\beta_j - \beta_l) Q_{\nu,jl}^2 + \widehat{\mathcal{R}}_\mu(\beta_j - \beta_l) Q_{\mu,jl}^2 \quad \text{if } j \neq l, \tag{6.15}$$

where

$$\begin{aligned} Q_{\nu,jl} &= c_{\nu,jl} + d_{\nu,jl}\beta_l(\beta_l - \beta_j) - (\beta_l - \beta_j)^2 \left[e_{\nu,jl} + f_{\nu,jl}\beta_l(\beta_l - \beta_j) \right] \\ &= \frac{X}{2\sqrt{\beta_j \beta_l}} \left[\frac{\omega^2}{c(X)^2} - \beta_l \beta_j \right] \phi_j(X)\phi_l(X), \\ Q_{\mu,jl} &= c_{\mu,jl} + d_{\mu,jl}(\beta_l^2 - \beta_j^2) = \frac{X}{2\sqrt{\beta_j\beta_l}} \partial_\xi \phi_j(0) \partial_\xi \phi_l(0). \end{aligned} \tag{6.16}$$

The coefficients in $\mathcal{L}_\theta$ are similar,

$$\Gamma_{jl}^{(0)}(\omega) = \widehat{\mathcal{R}}_\mu(0) Q_{\nu,jl}^2 + \widehat{\mathcal{R}}_\mu(0) Q_{\mu,jl}^2 \quad \forall j,l, \tag{6.17}$$

and

$$\Gamma_{jl}^{(s)}(\omega) = \gamma_{\nu,jl} Q_{\nu,jl}^2 + \gamma_{\mu,jl} Q_{\mu,jl}^2 \quad \text{if } j \neq l, \tag{6.18}$$

with $\gamma_{\nu,jl}$ and $\gamma_{\mu,jl}$ defined by (4.18).

We find again that these effective coupling coefficients depend only on the behaviors of the mode profiles close to the boundaries. In the case of Dirichlet boundary conditions, the mode coupling coefficient $\Gamma_{jl}^{(c)}(\omega)$ depends on the value of $\partial_\xi \phi_j \partial_\xi \phi_l$ at the boundaries. In the case of Neumann boundary conditions, the mode coupling coefficient $\Gamma_{jl}^{(c)}(\omega)$ depends on the value of $\phi_j(X)\phi_l(X)$.

Given the generator, the analysis of the loss of coherence, and of the mode powers is the same as in sections 4.4.3-4.4.5.

7. Summary

In this paper we obtain a rigorous quantitative analysis of wave propagation in two-dimensional waveguides with random and stationary fluctuations of the boundaries, and either Dirichlet or Neumann boundary conditions. The fluctuations are small, of order ε, but their effect becomes significant over long ranges z/ε^2. We carry the analysis in three main steps: First, we change coordinates to straighten the boundaries and obtain a wave equation with random coefficients. Second, we decompose the wave field in propagating and evanescent modes, with random complex amplitudes satisfying a random system of coupled differential equations. We analyze the evanescent modes and show how to obtain a closed system of differential equations for the amplitudes of the propagating modes. In the third step we analyze the amplitudes of the propagating modes in the long range limit, and show that the result is independent of the particular choice of the change of the coordinates in the first step. The limit process is a Markov diffusion with coefficients in the infinitesimal generator given explicitly in terms of the covariance of the boundary fluctuations. Using this limit process, we quantify mode by mode the loss of coherence and the exchange (diffusion) of energy between modes induced by scattering at the random boundaries.

The long range diffusion limit is similar to that in random waveguides with interior inhomogeneities and straight boundaries, in the sense that the infinitesimal generators have the same form. However, the net scattering effects are very different. We quantify them explicitly in a high frequency regime, in the case of a constant wave speed, and compare the results with those in waveguides with interior random inhomogeneities. In particular, we estimate three important length scales: the scattering mean free path, the transport mean free path, and the equipartition distance. The first two give the distances over which the waves lose their coherence and forget their direction, respectively. The last is the distance over which the cumulative scattering distributes the energy uniformly among the modes, independently of the initial conditions at the source.

We obtain that in waveguides with random boundaries the lower order modes have a longer scattering mean free path, which is comparable to the transport mean free path and, remarkably, to the equipartition distance. The high order modes lose coherence rapidly, have a short scattering mean free path, and do not exchange energy efficiently with the other modes. They also have a transport mean free path that exceeds the scattering mean free path. In contrast, in waveguides with interior random inhomogeneities, all the modes lose their coherence over much shorter distances than in waveguides with random boundaries. Moreover, the main mechanism of loss of coherence is the exchange of energy with the nearby modes, so the scattering mean free paths and the transport mean free paths are similar for all the modes. Finally, the equipartition distance is much longer than the distance over which all the modes lose their coherence.

These results are useful in applications such as imaging with remote sensor arrays. Understanding how the waves lose coherence is essential in imaging, because it allows the design of robust methodologies that produce reliable, statistically stable images in noisy environments that we model mathematically with random processes. An example of a statistically stable imaging approach guided by the theory in random waveguides with internal inhomogeneities is in [3].

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Appendix A. Proof of Lemma 3.1. The proof given here relies on explicit estimates of the series in (3.34), obtained under the assumption that the background speed is constant $c(\xi) = c_o$. We rewrite (3.34) as

$$[\Psi \widehat{\boldsymbol{v}}](z) = [\Psi_1 \widehat{\boldsymbol{v}}](z) + [\Psi_2 \widehat{\boldsymbol{v}}](z) \tag{A.1}$$

with linear integral operators Ψ_1 and Ψ_2 defined componentwise by

$$\big[\Psi_1 \widehat{\boldsymbol{v}}\big]_j(z) = \sum_{l=N+1}^{\infty} \frac{1}{2\beta_j} \int_{-\infty}^{\infty} (M^\varepsilon_{jl} - \partial_z Q^\varepsilon_{jl})(z+s)\widehat{v}_l(z+s) e^{-\beta_j |s|} ds, \tag{A.2}$$

$$\big[\Psi_2 \widehat{\boldsymbol{v}}\big]_j(z) = \sum_{l=N+1}^{\infty} \frac{1}{2} \int_{-\infty}^{\infty} Q^\varepsilon_{jl}(z+s)\widehat{v}_l(z+s) e^{-\beta_j |s|} ds. \tag{A.3}$$

The coefficients have the explicit form

$$M^\varepsilon_{jl}(z) = \left\{ 2\left[\nu(z)-\mu(z)\right]\left(\frac{\pi j}{X}\right)^2 + \frac{\nu''(z)-\mu''(z)}{2}\right\}\delta_{jl} + (1-\delta_{jl})\left[\nu''(z)-\mu''(z)\right]\frac{2lj}{j^2-l^2}$$
$$-(1-\delta_{jl})\nu''(z)\frac{2lj}{j^2-l^2}\left[1-(-1)^{l+j}\right] + O(\varepsilon), \tag{A.4}$$

$$Q^\varepsilon_{jl}(z) = \left[\nu'(z)-\mu'(z)\right]\delta_{jl} + (1-\delta_{jl})\left[\nu'(z)-\mu'(z)\right]\frac{4lj}{j^2-l^2}$$
$$-(1-\delta_{jl})\nu'(z)\frac{4lj}{j^2-l^2}\left[1-(-1)^{l+j}\right] + O(\varepsilon). \tag{A.5}$$

Let $\ell^2_1(\mathbb{Z};L^2(\mathbb{R}))$ be the space of square summable sequences of $L^2(\mathbb{R})$ functions with linear weights, equipped with the norm

$$\|\boldsymbol{v}\|_{\ell^2_1} := \Big[\sum_{j\in\mathbb{Z}} \big(j\,\|v_j\|_{L^2(\mathbb{R})}\big)^2\Big]^{1/2}.$$

We prove that $\Psi : \ell^2_1(\mathbb{Z};L^2(\mathbb{R})) \to \ell^2_1(\mathbb{Z};L^2(\mathbb{R}))$ is bounded. The proof consists of three steps:

Step 1: Let T be an auxiliary operator acting on sequences $\boldsymbol{v} = \{v_l\}_{l\in\mathbb{Z}}$, defined componentwise by

$$[T\boldsymbol{v}]_j = \sum_{l\neq\pm j} \frac{j\,l}{j^2-l^2}\,v_l = \sum_{l\neq\pm j}\left(\frac{l/2}{j+l} + \frac{l/2}{j-l}\right)v_l = \frac{1}{2}\left((-l\,v_{-l})*\frac{1}{l} + (l\,v_l)*\frac{1}{l}\right)_j + \frac{1}{4}(v_{-j}-v_j).$$

This operator is essentially the sum of two discrete Hilbert transforms, satisfying the sharp estimates [11]

$$\Big\|\boldsymbol{v}*\frac{1}{l}\Big\|_{\ell^2} \leq \pi \|\boldsymbol{v}\|_{\ell^2}.$$

Therefore, the operator T is bounded as

$$\|T\boldsymbol{v}\|_{\ell^2} \leq (1/2+\pi)\sum_{j\in\mathbb{Z}}\|v_j\|_{\ell^2_1}. \tag{A.6}$$

Step 2: Let $\boldsymbol{v}(z) = \{v_l(z)\}_{l\in\mathbb{Z}}$ be a sequence of functions in $\mathbb{R}$ and define the operator

$$Q : \ell^2_1(\mathbb{Z};L^2(\mathbb{R})) \to \ell^2_1(\mathbb{Z};L^2(\mathbb{R})), \qquad [Q\boldsymbol{v}]_j(z) = [T\boldsymbol{v}]_j * e^{-\beta_j|s|}(z)\,1_{\{j>N\}}, \tag{A.7}$$

where

$$\beta_j = \sqrt{\left(\frac{\pi j}{X}\right)^2 - \left(\frac{\omega}{c_0}\right)^2} \geq \frac{j\,\pi}{X}\sqrt{1-\left(\frac{\omega X/(\pi c_0)}{N+1}\right)^2} =: j\,C_\beta, \quad \text{for } j>N. \tag{A.8}$$

Using Young's inequality

$$\|[Q\boldsymbol{v}]_j\|_{L^2(\mathbb{R})} = \|[T\boldsymbol{v}]_j * e^{-\beta_j|s|}\|_{L^2(\mathbb{R})} \le \|[T\boldsymbol{v}]_j\|_{L^2(\mathbb{R})}\|e^{-\beta_j|s|}\|_{L^1(\mathbb{R})} = \frac{2}{\beta_j}\|[T\boldsymbol{v}]_j\|_{L^2(\mathbb{R})}, \tag{A.9}$$

we obtain from (A.6)-(A.9) that $\|Q\| \le (1+2\pi)/C_\beta$, because

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\left(j\,\|[Q\boldsymbol{v}]_j\|_{L^2(\mathbb{R})}\right)^2 &\le \frac{4}{C_\beta^2}\sum_{j\in\mathbb{Z}}\|[T\boldsymbol{v}]_j\|^2_{L^2(\mathbb{R})}\\ &= \frac{4}{C_\beta^2}\int_{\mathbb{R}}\sum_{j\in\mathbb{Z}}|[T\boldsymbol{v}]_j(z)|^2dz\\ &\le \frac{4}{C_\beta^2}(1/2+\pi)^2\int_{\mathbb{R}}\sum_{j\in\mathbb{Z}}|j\,v_j(z)|^2dz\\ &= \frac{4(1/2+\pi)^2}{C_\beta^2}\sum_{j\in\mathbb{R}}\left(j\|v_j\|_{L^2(\mathbb{R})}\right)^2.\end{aligned} \tag{A.10}$$

This estimate applies to the operator Ψ_2. Indeed, let us express Ψ_2 in terms of the operator Q using (A.3) and (A.5),

$$[\Psi_2\boldsymbol{v}]_j(z) = \frac{1}{2}((\nu'-\mu')v_j) * e^{-\beta_j|s|}(z)1_{\{j>N\}} - 2[Q\mu'\,v_l]_j(z) + 2(-1)^j[Q\nu'(-1)^l\,v_l]_j(z). \tag{A.11}$$

That the sum in Ψ_2 is for $l>N$ is easily fixed by using the truncation $v_l = \widehat{v}_l\,1_{\{l>N\}}$. Thus, using estimate (A.10) for the last two terms, we obtain

$$\|\Psi_2\widehat{\boldsymbol{v}}\|_{\ell_1^2} \le \frac{5+8\pi}{C_\beta}\left(\|\mu\|_{W^{1,\infty}(\mathbb{R})} + \|\nu\|_{W^{1,\infty}(\mathbb{R})}\right)\|\widehat{\boldsymbol{v}}\|_{\ell_1^2}.$$

Step 3: It remains to show that the operator Ψ_1 is bounded. We see from (A.2), (A.4), and (A.5) that, for any $j>N$,

$$[\Psi_1\widehat{\boldsymbol{v}}]_j(z) = \frac{\pi^2 j^2}{\beta_j X^2}((\nu-\mu)\widehat{v}_j) * e^{-\beta_j|s|}(z)1_{\{j>N\}} - \frac{1}{\beta_j}[\tilde{\Psi}_2\widehat{\boldsymbol{v}}]_j(z),$$

where $\tilde{\Psi}_2$ is just like the operator Ψ_2, with the driving process (ν',μ') replaced by its derivative (ν'',μ''). Using again Young's inequality, we have

$$\|[\Psi_1\widehat{\boldsymbol{v}}]_j\|_{L^2(\mathbb{R})} \le 2\left(\frac{\pi}{XC_\beta}\right)^2\|(\nu-\mu)\widehat{v}_j\|_{L^2(\mathbb{R})} + \frac{1}{jC_\beta}\|[\tilde{\Psi}_2\widehat{\boldsymbol{v}}]_j\|_{L^2(\mathbb{R})}.$$

Now multiply by j and use the triangle inequality to obtain that Ψ_1 is bounded,

$$\|\Psi_1\widehat{\boldsymbol{v}}\|_{\ell_1^2} \le \left[\frac{2\pi^2}{C_\beta^2X^2}(\|\nu\|_{L^\infty} + \|\mu\|_{L^\infty}) + \frac{(5+8\pi)}{C_\beta^2}(\|\nu\|_{W^{2,\infty}} + \|\mu\|_{W^{2,\infty}})\right]\|\widehat{\boldsymbol{v}}\|_{\ell_1^2}.$$

Appendix B. Independence of the change of coordinates. We begin the proof of Theorem 4.3 with the observation that

$$\widehat{w}(\xi,z) = \widehat{u}\left(\ell^{\varepsilon,-1}(z,F^\varepsilon(z,\xi)),z\right),$$

where $\ell^{\varepsilon,-1}$ is the inverse of ℓ^{ε}, meaning that $\widehat{w}$ and $\widehat{u}$ are related by composition of the change of coordinate mappings. Clearly, the composition inherits the uniform convergence property

$$\sup_{z\geq 0}\sup_{\xi\in[0,X]}|\ell^{\varepsilon,-1}(z,F^{\varepsilon}(z,\xi))-\xi|=O(\varepsilon). \tag{B.1}$$

For the sake of simplicity we neglect the evanescent modes in the proof, but they can be added using the techniques described in Section 3.3. Using the propagating mode representation of $\widehat{u}(\xi,z)$,

$$\widehat{w}(\xi,z)=\sum_{l=1}^{N}\phi_l(\xi)\widehat{u}_l(z)+\sum_{l=1}^{N}\tilde{\phi}_l(\xi,z)\widehat{u}_l(z), \tag{B.2}$$

where we let

$$\begin{aligned}\tilde{\phi}_l(\xi,z)&=\phi_l\left(\ell^{\varepsilon,-1}(z,F^{\varepsilon}(z,\xi))\right)-\phi_l(\xi)\\&=\int_0^1\left(\ell^{\varepsilon,-1}(z,F^{\varepsilon}(z,\xi))-\xi\right)\partial_{\xi}\phi_l\left(s\,\ell^{\varepsilon,-1}(z,F^{\varepsilon}(z,\xi))+(1-s)\,\xi\right)ds.\end{aligned}$$

But we can also carry out the mode decomposition directly on $\widehat{w}$ and obtain

$$\widehat{w}(\xi,z)=\sum_{l=1}^{N}\phi_l(\xi)\widehat{w}_l(z), \tag{B.3}$$

because the number of propagating modes N and the eigenfunctions ϕ_j in the ideal waveguide are independent of the change of coordinates. Here $\widehat{w}_l(z)$ are the amplitudes of the propagating modes of $\widehat{w}(z)$. Equating identities (B.2) and (B.3), multiplying by $\phi_j(\xi)$ and integrating in $[0,X]$ we conclude that

$$\widehat{w}_j(z)=\widehat{u}_j(z)+\sum_{l=1}^{N}\tilde{c}_{lj}(z)\widehat{u}_l(z), \tag{B.4}$$

where we introduced the random processes

$$\tilde{c}_{lj}(z)=\int_0^X\phi_j(\xi)\int_0^1\partial_{\xi}\phi_l\left(s\,\ell^{\varepsilon,-1}(z,F^{\varepsilon}(z,\xi))+(1-s)\,\xi\right)\left(\ell^{\varepsilon,-1}(z,F^{\varepsilon}(z,\xi))-\xi\right)dsd\xi.$$

In addition, differentiating equation (B.4) in z, we have

$$\partial_z\widehat{w}_j(z)=\partial_z\widehat{u}_j(z)+\sum_{l=1}^{N}\partial_z\tilde{c}_{lj}(z)\widehat{u}_l(z)+\tilde{c}_{lj}(z)\partial_z\widehat{u}_l(z). \tag{B.5}$$

Now, let us recall from the definition of the forward and backward propagating modes that

$$i\beta_j\widehat{u}_j(z)+\partial_z\widehat{u}_j(z)=2i\sqrt{\beta_j}\,\widehat{a}_j(z)e^{i\beta_j z}.$$

We conclude from (B.4) and (B.5) that

$$\widehat{a}_j^w(z)=\widehat{a}_j(z)+\frac{1}{2}\sum_{l=1}^{N}\tilde{c}_{lj}(z)\left(\frac{\beta_j+\beta_l}{\sqrt{\beta_j\beta_j}}\,\widehat{a}_l(z)e^{-i(\beta_j-\beta_l)z}+\frac{\beta_j-\beta_l}{\sqrt{\beta_j\beta_j}}\,\widehat{b}_l(z)e^{-i(\beta_j+\beta_l)z}\right)$$

$$+\frac{i}{2}\sum_{l=1}^{N}\frac{\partial_z\tilde{c}_{lj}(z)}{\sqrt{\beta_j\beta_l}}\left(\widehat{a}_l(z)e^{-i(\beta_j-\beta_l)z}+\widehat{b}_l(z)e^{-i(\beta_j+\beta_l)z}\right), \tag{B.6}$$

where $\{\widehat{a}^w_j(z)\}_{j=1,\ldots,N}$ are the amplitudes of the forward propagating modes of $\widehat{w}(\xi,z)$. A similar equation holds for the backward propagating mode amplitudes $\{\widehat{b}^w_j(z)\}_{j=1,\ldots,N}$.

The processes $\tilde{c}_{lj}(z)$ can be bounded as (4.30):

$$\max_{1\le j,l\le N}\left\{\sup_{z\ge 0}|\tilde{c}_{lj}(z)|\right\}\le X\max_{1\le j,l\le N}\left\{\sup_{\xi\in[0,X]}|\phi_j(\xi)|\sup_{\xi\in[0,X]}|\partial_\xi\phi_l(\xi)|\right\}$$
$$\times\sup_{z\ge 0}\sup_{\xi\in[0,X]}|\ell^{\varepsilon,-1}(z,F^\varepsilon(z,\xi))-\xi|=O(\varepsilon). \tag{B.7}$$

For the processes $\partial_z\tilde{c}_{lj}(z)$ we find a similar estimate. Indeed, note that

$$\begin{aligned}&\partial_z\left[\partial_\xi\phi_l\left(s\,\ell^{\varepsilon,-1}(z,F^\varepsilon(z,\xi))+(1-s)\,\xi\right)\left(\ell^{\varepsilon,-1}(z,F^\varepsilon(z,\xi))-\xi\right)\right]\\&=-\lambda_l\,\phi_l(s\,\ell^{\varepsilon,-1}(z,F^\varepsilon(z,\xi))+(1-s)\,\xi)\,s\,\partial_z[\ell^{\varepsilon,-1}(z,F^\varepsilon(z,\xi))]\,(\ell^{\varepsilon,-1}(z,F^\varepsilon(z,\xi))-\xi)\\&\qquad+\partial_\xi\phi_l(s\,\ell^{\varepsilon,-1}(z,F^\varepsilon(z,\xi))+(1-s)\,\xi)\,\partial_z[\ell^{\varepsilon,-1}(z,F^\varepsilon(z,\xi))].\end{aligned}$$

A direct calculation shows that

$$\begin{aligned}&\partial_z\left[\ell^{\varepsilon,-1}(z,F^\varepsilon(z,\xi))\right]\\&=\partial_z\left[\frac{X(F^\varepsilon(z,\xi)-\varepsilon\mu(z))}{X(1+\varepsilon\nu(z))-\varepsilon\mu(z)}\right]\\&=X\frac{(\partial_zF^\varepsilon(z,\xi)-\varepsilon\mu'(z))(X(1+\varepsilon\nu(z))-\varepsilon\mu(z))-(F^\varepsilon(z,\xi)-\varepsilon\mu(z))\,\varepsilon\,(\nu'(z)-\mu'(z))}{(X(1+\varepsilon\nu(z))-\varepsilon\mu(z))^2}.\end{aligned}$$

Hence, using condition (4.30) for $\partial_zF^\varepsilon(z,\xi)$,

$$\sup_{z\ge 0}\sup_{\xi\in[0,X]}\left|\partial_z\left[\ell^{\varepsilon,-1}(z,F^\varepsilon(z,\xi))\right]\right|\le C(\|v\|_{W^{1,\infty}},\|\mu\|_{W^{1,\infty}})\,\varepsilon.$$

Therefore,

$$\max_{1\le j,l\le N}\left\{\sup_{z\ge 0}|\partial_z\tilde{c}_{lj}(z)|\right\}\le X\max_{1\le j,l\le N}\{\lambda_l\sup_{\xi\in[0,X]}|\phi_j(\xi)|\sup_{\xi\in[0,X]}|\phi_l(\xi)|\}\,O(\varepsilon^2)$$
$$+X\max_{1\le j,l\le N}\left\{\sup_{\xi\in[0,X]}|\phi_j(\xi)|\sup_{\xi\in[0,X]}|\partial_\xi\phi_l(\xi)|\right\}O(\varepsilon). \tag{B.8}$$

Let $\widehat{\boldsymbol{a}}^w(z)$ and $\widehat{\boldsymbol{b}}^w(z)$ be the vectors containing the forward and backward propagating mode amplitudes and define the joint process of propagating mode amplitudes $\boldsymbol{X}^w(z)=(\widehat{\boldsymbol{a}}^w(z),\widehat{\boldsymbol{b}}^w(z))^T$. Let us denote the long range scaled process by $\boldsymbol{X}^{\varepsilon,w}(z)=\boldsymbol{X}^w(z/\varepsilon^2)$. Equation (B.6) implies that

$$\boldsymbol{X}^{\varepsilon,w}(z)=\boldsymbol{X}^\varepsilon(z)+\mathbf{M}_\varepsilon\left(\mathbf{C}\left(\frac{z}{\varepsilon^2}\right),\partial_z\mathbf{C}\left(\frac{z}{\varepsilon^2}\right),\frac{z}{\varepsilon^2}\right)\boldsymbol{X}^\varepsilon(z), \tag{B.9}$$

where $\mathbf{C}(z):=(\tilde{c}_{lj}(z))_{j,l=1,\ldots,N}$ and $\partial_z\mathbf{C}(z):=(\partial_z\tilde{c}_{lj}(z))_{j,l=1,\ldots,N}$. The subscript ε in the matrix $\mathbf{M}_\varepsilon(\cdot)$ denotes the fact that this matrix depends explicitly on ε and, due to estimates (B.7) and (B.8), we have

$$\sup_{z\ge 0}\|\mathbf{M}_\varepsilon(\mathbf{C}(z),\partial_z\mathbf{C}(z),z)\|_\infty=O(\varepsilon). \tag{B.10}$$

Let us prove then, that the processes $\boldsymbol{X}^{\varepsilon,w}(z)$ and $\boldsymbol{X}^{\varepsilon}(z)$ converge in distribution to the same diffusion limit. Denote by $Q(\boldsymbol{X}_0, L)$ the $2N$-dimensional cube with center $\boldsymbol{X}_0$ and side L. The probability that $\boldsymbol{X}^{\varepsilon,w}(z)$ is in this cube can be calculated using (B.9),

$$\begin{aligned}\mathbb{P}[\boldsymbol{X}^{\varepsilon,w}(z)\in Q(\boldsymbol{X}_0,L)] &= \int_{\{\boldsymbol{x}\in Q(\boldsymbol{X}_0,L)\}} d\mathbb{P}^w\left(\boldsymbol{x},\frac{z}{\varepsilon^2}\right)\\ &= \int_{\{\boldsymbol{x}\in(\mathbf{I}+\mathbf{M}_\varepsilon(\mathbf{C},\partial_z\mathbf{C},z))^{-1}Q(\boldsymbol{x}_0,L)\}} d\mathbb{P}\left(\boldsymbol{x},\mathbf{C},\partial_z\mathbf{C},\frac{z}{\varepsilon^2}\right). \quad \text{(B.11)}\end{aligned}$$

Here $\mathbb{P}^w(\boldsymbol{x},z)$ is the probability distribution of the process $\boldsymbol{X}^w(z)$ and $\mathbb{P}(\boldsymbol{x},\mathbf{C},\partial_z\mathbf{C},z)$ is the joint probability distribution of the processes $(\boldsymbol{X}(z),\mathbf{C}(z),\partial_z\mathbf{C}(z))$. We can take the inverse of $\mathbf{I}+\mathbf{M}_\varepsilon(\mathbf{C},\partial_z\mathbf{C},z)$ by (B.10). The same estimate (B.10) also implies that for every $\delta>0$ there exists ε_0 such that for $\varepsilon\le\varepsilon_0$,

$$\{\boldsymbol{x}\in Q(\boldsymbol{x}_0,(1-\delta)L)\}\subseteq\{\boldsymbol{x}\in(\mathbf{I}+\mathbf{M}_\varepsilon(\mathbf{C},\partial_z\mathbf{C},z))^{-1}Q(\boldsymbol{x}_0,L)\}\subseteq\{\boldsymbol{x}\in Q(\boldsymbol{x}_0,(1+\delta)L)\}. \quad \text{(B.12)}$$

Denote the diffusion limits by

$$\tilde{\boldsymbol{X}}(z)=\lim_{\varepsilon\to0}\boldsymbol{X}^\varepsilon(z), \qquad \tilde{\boldsymbol{X}}^w(z)=\lim_{\varepsilon\to0}\boldsymbol{X}^{\varepsilon,w}(z).$$

We conclude from (B.11) and (B.12) that for any $\delta>0$,

$$\mathbb{P}[\tilde{\boldsymbol{X}}(z)\in Q(\boldsymbol{X}_0,(1-\delta)L)]\le\mathbb{P}[\tilde{\boldsymbol{X}}^w(z)\in Q(\boldsymbol{X}_0,L)]\le\mathbb{P}[\tilde{\boldsymbol{X}}(z)\in Q(\boldsymbol{X}_0,(1+\delta)L)].$$

Sending $\delta\to0$, we have that for any arbitrary cube $Q(\boldsymbol{x}_0,L)$,

$$\mathbb{P}[\tilde{\boldsymbol{X}}(z)\in Q(\boldsymbol{X}_0,L)]=\mathbb{P}[\tilde{\boldsymbol{X}}^w(z)\in Q(\boldsymbol{X}_0,L)].$$

This proves that the limit processes have the same distribution and therefore the same generator.

Appendix C. Proof of Proposition 5.1. Recall the expression (2.3) of the wavenumbers. The first term in (5.1) follows from (4.16):

$$\Gamma_{jj}^{(0)}=\left(\frac{\pi}{X}\right)^2\left[\widehat{\mathcal{R}}_\nu(0)+\widehat{\mathcal{R}}_\mu(0)\right]\frac{j^4}{(N+\alpha)^2-j^2}\approx\frac{(2\pi)^{3/2}}{X}\frac{k\ell}{N}\frac{j^4}{(N+\alpha)^2-j^2}. \quad \text{(C.1)}$$

It increases monotonically with j, with minimum value

$$\Gamma_{11}^{(0)}\approx\frac{(2\pi)^{3/2}}{X}\frac{k\ell}{N^3}\ll1, \quad \text{(C.2)}$$

and maximum value

$$\Gamma_{NN}^{(0)}\approx\frac{(2\pi)^{3/2}}{2\alpha X}k\ell N^2\gg1. \quad \text{(C.3)}$$

The second term in (5.1), which is in (5.2), follows from (4.13), (5.6), and (5.4),

$$-\Gamma_{jj}^{(c)}(\omega)\approx\frac{(2\pi)^{3/2}j^2}{X\sqrt{(N+\alpha)^2-j^2}}\sum_{\substack{l=1\\ l\neq j}}^{N}\frac{l^2k\ell}{N\sqrt{(N+\alpha)^2-l^2}}e^{-\frac{(k\ell)^2}{2}\left(\sqrt{1-j^2/(N+\alpha)^2}-\sqrt{1-l^2/(N+\alpha)^2}\right)^2}.$$

(C.4)

If $0<j/N<1$, then we can estimate (C.4) by using the fact that the main contribution to the sum in l comes from the terms with indices l close to j, provided that $k\ell$ is larger than $N^{1/2}$ and smaller than N. We find after the change of index $l=j+q$ that

$$-\Gamma_{jj}^{(c)}(\omega)\approx\frac{(2\pi)^{3/2}j^4k\ell}{X((N+\alpha)^2-j^2)N}\sum_{q\neq 0}e^{-\frac{(k\ell)^2}{2}\frac{j^2}{(N+\alpha)^2-j^2}\frac{q^2}{(N+\alpha)^2}}.$$

Interpreting this sum as the Riemann sum of a continuous integral, we get

$$-\Gamma_{jj}^{(c)}(\omega)\approx\frac{(2\pi)^{3/2}j^4k\ell}{X((N+\alpha)^2-j^2)}\int_{-\infty}^{\infty}e^{-\frac{(k\ell)^2}{2}\frac{j^2}{(N+\alpha)^2-j^2}s^2}ds=\frac{(2\pi)^2j^3}{X\sqrt{(N+\alpha)^2-j^2}}. \tag{C.5}$$

By comparing with (C.1) we find that the coefficient $-\Gamma_{jj}^{(c)}(\omega)$ is larger than $\Gamma_{jj}^{(0)}$ when $k\ell$ satisfies $\sqrt{N}\ll k\ell\ll N$.
To be complete, note that:
- If $k\ell\sim N$, then $-\Gamma_{jj}^{(c)}(\omega)$ is larger than $\Gamma_{jj}^{(0)}$ if and only if $j/N<(1+(k\ell/N)^2)^{-1/2}$.
- If $k\ell$ is larger than N, then the main contribution to the sum in l comes only from one or two terms with indices $l=j\pm 1$, and it becomes exponentially small in $(k\ell)^2/N^2$. In these conditions $-\Gamma_{jj}^{(c)}(\omega)$ becomes smaller than $\Gamma_{jj}^{(0)}$.

For $j\sim 1$ we can estimate (C.4) again by interpreting the sum over l as a Riemann sum approximation of an integral that we can estimate using the Laplace perturbation method. Explicitly, for $j=1$ we have

$$\begin{aligned}-\Gamma_{11}^{(c)}(\omega)&\approx\frac{(2\pi)^{3/2}}{X}\frac{1}{N}\sum_{l=2}^{N}\frac{(l/N)^2k\ell}{\sqrt{(1+\alpha/N)^2-(l/N)^2}}e^{-\frac{(k\ell)^2}{2}\left(1-\sqrt{1-(l/N)^2}\right)^2}\\&\approx\frac{(2\pi)^{3/2}k\ell}{X}\int_0^1 ds\frac{s^2}{\sqrt{1-s^2}}e^{-\frac{(k\ell)^2}{2}\left(1-\sqrt{1-s^2}\right)^2}.\end{aligned} \tag{C.6}$$

We approximate the integral with Watson's Lemma [2, Section 6.4], after changing variables $\zeta=(1-\sqrt{1-s^2})^2$ and obtaining that

$$\int_0^1 ds\frac{s^2}{\sqrt{1-s^2}}e^{-\frac{(k\ell)^2}{2}\left(1-\sqrt{1-s^2}\right)^2}\approx\int_0^1 d\zeta\varphi(\zeta)e^{-\frac{(k\ell)^2}{2}\zeta},\qquad\varphi(\zeta)=\frac{\zeta^{-1/4}}{\sqrt{2}}+O(\zeta^{1/4}).$$

Watson's Lemma gives

$$\int_0^1 ds\frac{s^2}{\sqrt{1-s^2}}e^{-\frac{(k\ell)^2}{2}\left(1-\sqrt{1-s^2}\right)^2}\approx\frac{\Gamma(3/4)2^{1/4}}{(k\ell)^{3/2}},$$

and therefore by (C.6) and (5.7),

$$-\Gamma_{11}^{(c)}(\omega)\approx\frac{(2\pi)^{3/2}\Gamma(3/4)2^{1/4}}{X(k\ell)^{1/2}}. \tag{C.7}$$

By comparing with (C.2) we find that the coefficient $-\Gamma_{11}^{(c)}(\omega)$ is larger than $\Gamma_{11}^{(0)}$.

For $j \sim N$ only the terms with $l \sim N$ contribute to the sum in (C.4). If $k\ell \sim \sqrt{N}$, then we find that

$$-\Gamma^{(c)}_{NN}(\omega) \approx \frac{(2\pi)^{3/2} N^2 k\ell}{2\sqrt{\alpha} X} \sum_{q=1}^{\infty} \frac{1}{\sqrt{\alpha+q}} e^{-\frac{(k\ell)^2}{2N}(\sqrt{q+\alpha}-\sqrt{\alpha})^2} \sim \frac{(2\pi)^{3/2} N^3}{2C(\alpha) k\ell X},$$

up to a constant $C(\alpha)$ that depends only on α. By comparing with (C.3) we can see that it is of the same order as $\Gamma^{(0)}_{NN}$. If $k\ell \gg \sqrt{N}$, then we find that

$$-\Gamma^{(c)}_{NN}(\omega) \approx \frac{(2\pi)^{3/2} N^2 k\ell}{2\sqrt{\alpha(1+\alpha)} X} e^{-\frac{(k\ell)^2}{2N}(\sqrt{1+\alpha}-\sqrt{\alpha})^2},$$

which is very small because the exponential term is exponentially small in $(k\ell)^2/N$. In these conditions $-\Gamma^{(c)}_{NN}(\omega)$ is smaller than $\Gamma^{(0)}_{NN}$.

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