DERIVATION OF A ONE-WAY RADIATIVE TRANSFER EQUATION IN RANDOM MEDIA

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Abstract. We derive from first principles a one-way radiative transfer equation for the wave intensity resolved over directions (Wigner transform of the wave field) in random media. It is an initial value problem with excitation from a source which emits waves in a preferred, forward direction. The equation is derived in a regime with small random fluctuations of the wave speed but long distances of propagation with respect to the wavelength, so that cumulative scattering is significant. The correlation length of the medium and the scale of the support of the source are slightly larger than the wavelength, and the waves propagate in a wide cone with opening angle less than 180^o, so that the backward and evanescent waves are negligible. The scattering regime is a bridge between that of radiative transfer, where the waves propagate in all directions and the paraxial regime, where the waves propagate in a narrow angular cone. We connect the one-way radiative transport equation with the equations satisfied by the Wigner transform of the wave field in these regimes.

1. Introduction.

Light propagation in scattering media can be modeled by a boundary value problem for the radiative transfer equation [16, 25, 2]. The light intensity resolved over directions, also known as the Wigner transform of the wave field, satisfies this equation with incoming boundary conditions on the illuminated part of the boundary, and outgoing conditions on the remainder of the boundary. The problem is of interest in applications such as optical tomography, where structural variations in tissue are to be determined from measurements of scattered light [1].

The derivation of the radiative transfer equation from the wave equation is a fundamental challenge. Existing heuristic derivations from the wave equation in random media, obtained when the wavelength, the correlation length of the medium and the scale of variation of the source are of the same order, and much smaller than the propagation distance, use either multiscale asymptotic analysis [25] or diagrammatic perturbation theory [3, 26]. However, as discussed by Mandel and Wolf in their monography [19], or more recently in the tutorial [4], there is no satisfactory or rigorous derivation of the macroscopic theory of radiative transfer from the microscopic theory of wave propagation in random media, except in some special cases. Therefore, the rigorous derivation of a radiative transfer-like equation from the wave equation, beyond the special cases mentioned in these references, would be of interest for the radiative transfer community.

The radiative transfer equation poses formidable computational challenges in optical tomography, where repeated solutions of the equation are needed to solve the inverse problem with optimization [2, 1]. This is why a simplified diffusion model is often used [1], where the medium is assumed optically thick, so that light is diffusive due to very strong scattering. This leads to considerable simplification, but may produce anomalies in the reconstructed images [6]. A recent study [13] shows that in mesoscopic scattering regimes, where light penetrates to about one centimeter depth in tissue [12], scattering is forward-peaked and a simpler one-way radiative transport model can be used, where the intensity satisfies an initial value problem. The one-way radiative transfer equation is obtained in [13] from the standard radiative transfer equation by simply ignoring the intensity in the backward directions.

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Our first goal in this paper is to derive rigorously the one-way radiative transfer equation, from first principles, starting from the wave equation in random media. The second goal is to bridge between the mesoscopic scattering regime, the standard radiative transfer regime on one side, and the paraxial approximation regime on the other side. We also connect to the diffusion approximation.

To derive the one-way transfer equation we consider waves in media with small random fluctuations of the wave speed, at long propagation distances with respect to the wavelength, where cumulative scattering effects are significant. The typical size of the inhomogeneities, measured by the correlation length, and the scale of variation of the source are slightly larger than the wavelength, so that the waves propagate in an angular cone with axis along a preferred forward direction called range. We analyze the propagation in this regime using a plane wave decomposition of the field, with amplitudes that are range dependent random fields. They satisfy a system of coupled stochastic differential equations driven by the random fluctuations of the wave speed, and can be analyzed in detail with probabilistic limit theorems. Consequently, we can quantify the loss of coherence of the wave field i.e., its randomization due to scattering, and derive the radiative transfer equation satisfied by the Wigner transform of the wave amplitudes. The result extends the model proposed in [13], and defines the differential scattering cross-section and the total scattering cross-section in terms of the autocovariance of the fluctuations of the wave speed.

Once we derive the one-way transfer equation we show that it is equivalent to the standard radiative transfer equation [25] in regimes with negligible backscattering. We also connect to the diffusion approximation theory, by considering the high-frequency limit of the equation. Transport in the paraxial approximation, which applies to waves propagating in a narrow angle cone along the range axis, is analyzed in [10], using the Itô-Schrödinger model of wave propagation. Here we rediscover the results starting from the one-way radiative transfer equation, in the high-frequency limit and for a large support of the source.

The paper is organized as follows: We begin in section 2 with the model of the random medium and the formulation of the problem. The main results are stated in section 3. We describe the mean wave field and the randomization of the components of the wave quantified by the scattering mean free paths. We also state the one-way radiative transfer equation. The connection to the equation in [13] is in subsection 3.1, and to the standard radiative transfer theory in subsection 3.2. The connection to the paraxial approximation is in subsection 3.3. The derivation of the results is in section 4. We begin with the scaling regime in subsection 4.1, and then give the wave decomposition in subsection 4.2. The probabilistic limit of the wave amplitudes is studied in subsection 4.3 and appendix A. We use it to describe the evolution of the mean field in subsection 4.4 and to derive the one-way radiative transfer equation for the Wigner transform in subsection 4.5. The high-frequency limit which leads to either the diffusion approximation or the paraxial approximation is studied in section 5. We end with a summary in section 6.

2. Formulation of the problem. The time-harmonic field $u(\vec{\boldsymbol{x}})$ satisfies the wave equation:

$$\frac{\omega^2}{c^2(\vec{\boldsymbol{x}})}u(\vec{\boldsymbol{x}}) + \Delta_{\vec{\boldsymbol{x}}}u(\vec{\boldsymbol{x}}) = -F\Big(\frac{\boldsymbol{x}}{X}\Big)\delta(z), \tag{2.1}$$

for $\vec{\boldsymbol{x}} = (\boldsymbol{x}, z) \in \mathbb{R}^{d+1}$ and frequency $\omega \in \mathbb{R}$. Here $\Delta_{\vec{\boldsymbol{x}}}$ is the Laplacian operator in $\mathbb{R}^{d+1}$ and since the frequency is constant, we suppress ω from the arguments of u and

F. The excitation is due to a localized source F which emits waves in the direction z, called range. The function F depends on the dimensionless vector $\boldsymbol{r} \in \mathbb{R}^d$, and its magnitude is negligible for $|\boldsymbol{r}| > O(1)$, so that X scales the spatial support of the source.

The waves propagate in a linear medium with speed of propagation $c(\vec{\boldsymbol{x}})$ satisfying

$$\frac{1}{c^2(\vec{\boldsymbol{x}})} = \frac{1}{c_o^2}\Big[1 + 1_{(0,L)}(z)\,\alpha\nu\Big(\frac{\vec{\boldsymbol{x}}}{\ell}\Big)\Big]. \tag{2.2}$$

It is a random perturbation of the constant speed c_o, modeled by the random process ν. The perturbation extends over the range interval $z \in (0, L)$, as given by the indicator function $1_{(0,L)}(z)$. We assume that $\nu(\vec{\boldsymbol{r}})$ is a dimensionless stationary random process of dimensionless argument $\vec{\boldsymbol{r}} \in \mathbb{R}^{d+1}$, with zero mean $\mathbb{E}\big[\nu(\vec{\boldsymbol{r}})\big] = 0$ and autocovariance

$$\mathbb{E}\big[\nu(\vec{\boldsymbol{r}})\nu(\vec{\boldsymbol{r}}')\big] = \mathcal{R}(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'), \qquad \forall\, \vec{\boldsymbol{r}}, \vec{\boldsymbol{r}}' \in \mathbb{R}^{d+1}.$$

Moreover, ν is bounded and $\mathcal{R}$ is integrable, with Fourier transform, the power spectral density

$$\widetilde{\mathcal{R}}(\vec{\boldsymbol{q}}) = \int_{\mathbb{R}^{d+1}} \mathrm{d}\vec{\boldsymbol{r}}\, \mathcal{R}(\vec{\boldsymbol{r}}) e^{-i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}}, \tag{2.3}$$

that is either compactly supported in a ball of radius $O(1)$ in $\mathbb{R}^{d+1}$, or is negligible outside this ball. The autocovariance is normalized by

$$\int_{\mathbb{R}^{d+1}} \mathrm{d}\vec{\boldsymbol{r}}\, \mathcal{R}(\vec{\boldsymbol{r}}) = O(1), \qquad \mathcal{R}(\boldsymbol{0}) = O(1).$$

Then, the length scale ℓ is the correlation length and the positive and small dimensionless parameter α quantifies the typical amplitude (standard deviation) of the fluctuations.

The problem is to characterize the wave field $u(\vec{\boldsymbol{x}})$ in the scaling regime

$$\lambda < \ell \sim X \ll L, \qquad \alpha \sim \big(\lambda/L\big)^{\frac{1}{2}} \ll 1. \tag{2.4}$$

Here $\lambda = 2\pi c_o/\omega$ is the wavelength and the reference length scale is L, which is of the order of the distance of propagation. We are particularly interested in the coherent (mean) field $\mathbb{E}[u(\vec{\boldsymbol{x}})]$ and the intensity resolved over directions of propagation, the mean Wigner transform of $u(\vec{\boldsymbol{x}})$. Its evolution in z is governed by the one-way radiative transfer equation that we derive.

3. Main results. Because the interaction of the waves with the random medium depends on the direction of propagation, we decompose $u(\vec{\boldsymbol{x}})$ over plane waves, using the Fourier transform with respect to the transverse coordinates $\boldsymbol{x} \in \mathbb{R}^d$ of $\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$,

$$\widehat{u}(\boldsymbol{\kappa}, z) = \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{x}\, u(\boldsymbol{x}, z) e^{-ik\boldsymbol{\kappa}\cdot\boldsymbol{x}}. \tag{3.1}$$

Here $\boldsymbol{\kappa} \in \mathbb{R}^d$ is the normalized transverse wave vector, and we suppressed the wavenumber $k = \omega/c_o$ in the argument of $\widehat{u}$. We show in section 4 that in the scaling regime (2.4), the field $\widehat{u}(\boldsymbol{\kappa}, z)$ consists of forward propagating waves with longitudinal wavenumber $k\beta(\boldsymbol{\kappa})$, where

$$\beta(\boldsymbol{\kappa}) = \sqrt{1 - |\boldsymbol{\kappa}|^2}, \quad |\boldsymbol{\kappa}| < 1. \tag{3.2}$$

The amplitudes of these waves (modes) are denoted by $a(\boldsymbol{\kappa}, z)$. They are complex-valued z-dependent random fields which model wave scattering in the random medium.

The wave field $u(\vec{\boldsymbol{x}})$ is given by the Fourier synthesis of the modes, the plane waves with wave vector $k\vec{\boldsymbol{\kappa}} = k(\boldsymbol{\kappa}, \beta(\boldsymbol{\kappa}))$,

$$u(\vec{\boldsymbol{x}}) = \int_{|\boldsymbol{\kappa}|<1} \frac{\mathrm{d}(k\boldsymbol{\kappa})}{(2\pi)^d} \frac{a(\boldsymbol{\kappa}, z)}{\beta^{\frac{1}{2}}(\boldsymbol{\kappa})} e^{ik\vec{\boldsymbol{\kappa}}\cdot\vec{\boldsymbol{x}}}, \quad \vec{\boldsymbol{x}} = (\boldsymbol{x}, z), \tag{3.3}$$

where have used the notation $\mathrm{d}(k\boldsymbol{\kappa}) = k^d \mathrm{d}\boldsymbol{\kappa}$ for the infinitesimal volume in $\mathbb{R}^d$. The mode amplitudes are normalized by the factors $\beta^{\frac{1}{2}}(\boldsymbol{\kappa})$ in order to simplify the formulae that follow [1]. In the scaling regime (2.4) the mode amplitudes form a Markov process whose statistical moments can be characterized explicitly, as explained in subsection 4.3. Here we describe the expectation of $a(\boldsymbol{\kappa}, z)$, which defines the coherent field, and its second moments, which define the mean Wigner transform of $u(\vec{\boldsymbol{x}})$.

The mean mode amplitudes are

$$\mathbb{E}[a(\boldsymbol{\kappa}, z)] = a_o(\boldsymbol{\kappa}) \exp[\mathrm{Q}(\boldsymbol{\kappa}) z],$$

where $a_o(\boldsymbol{\kappa})$ are the amplitudes in the homogeneous medium, defined in Eq. (4.7) by the source excitation. The effect of the random medium is in the complex exponent

$$\mathrm{Q}(\boldsymbol{\kappa}) = -\frac{k^2\alpha^2\ell^{d+1}}{4} \int_{|\boldsymbol{\kappa}'|<1} \frac{\mathrm{d}(k\boldsymbol{\kappa}')}{(2\pi)^d} \frac{1}{\beta(\boldsymbol{\kappa})\beta(\boldsymbol{\kappa}')} \times \int_0^\infty \mathrm{d}\zeta \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{r}\, \mathcal{R}(\boldsymbol{r}, \zeta) e^{-ik\ell\big(\boldsymbol{\kappa}-\boldsymbol{\kappa}', \beta(\boldsymbol{\kappa})-\beta(\boldsymbol{\kappa}')\big)\cdot(\boldsymbol{r},\zeta)}. \tag{3.4}$$

Since $\mathcal{R}$ is even, the real part of $\mathrm{Q}(\boldsymbol{\kappa})$ is determined by the power spectral density $\widetilde{\mathcal{R}}$ defined in (2.3), which is non-negative by Bochner's theorem [2]. Thus $\mathrm{Re}\big[\mathrm{Q}(\boldsymbol{\kappa})\big] < 0$, and the mean amplitudes decay exponentially in z, with the decay rate

$$\frac{1}{\mathcal{S}(\boldsymbol{\kappa})} = -\mathrm{Re}\big[\mathrm{Q}(\boldsymbol{\kappa})\big]. \tag{3.5}$$

The length $\mathcal{S}(\boldsymbol{\kappa})$ is the scattering mean free path. By choosing the magnitude α of the fluctuations as in (2.4), we have $L \sim \mathcal{S}(\boldsymbol{\kappa})$, so the decay with z is significant in our scaling regime. It is the manifestation of the randomization of the wave, due to scattering in the medium.

The strength of the random fluctuations of the mode amplitudes is described by the energy density (Wigner transform)

$$\mathcal{W}(\boldsymbol{\kappa}, \boldsymbol{x}, z) = \int \frac{\mathrm{d}(k\boldsymbol{q})}{(2\pi)^d} \exp\Big[ik\boldsymbol{q}\cdot\big(\nabla\beta(\boldsymbol{\kappa})z + \boldsymbol{x}\big)\Big] \times \mathbb{E}\Big[a\big(\boldsymbol{\kappa} + \frac{\boldsymbol{q}}{2}, z\big)\overline{a\big(\boldsymbol{\kappa} - \frac{\boldsymbol{q}}{2}, z\big)}\Big], \tag{3.6}$$

[1] In particular these factors ensure that the energy fluxes of the plane wave modes through the planes $z =$ constant are $|a(\boldsymbol{\kappa}, z)|^2$.

[2] Bochner's theorem states that a function is an autocovariance function of a stationary process if and only if its Fourier transform is nonnegative [11].

where the bar denotes complex conjugate and the integral is over all $\boldsymbol{q} \in \mathbb{R}^d$ such that $|\boldsymbol{\kappa} \pm \boldsymbol{q}/2| < 1$. The Wigner transform satisfies the transport equation

$$\partial_z \mathcal{W}(\boldsymbol{\kappa}, \boldsymbol{x}, z) - \nabla\beta(\boldsymbol{\kappa}) \cdot \nabla_{\boldsymbol{x}} \mathcal{W}(\boldsymbol{\kappa}, \boldsymbol{x}, z) = \int_{|\boldsymbol{\kappa}'|<1} \frac{\mathrm{d}(k\boldsymbol{\kappa}')}{(2\pi)^d} \mathcal{Q}(\boldsymbol{\kappa}, \boldsymbol{\kappa}') \big[\mathcal{W}(\boldsymbol{\kappa}', \boldsymbol{x}, z) - \mathcal{W}(\boldsymbol{\kappa}, \boldsymbol{x}, z)\big], \tag{3.7}$$

for $z > 0$, with differential scattering cross section

$$\mathcal{Q}(\boldsymbol{\kappa}, \boldsymbol{\kappa}') = \frac{k^2 \alpha^2 \ell^{d+1}}{4\beta(\boldsymbol{\kappa})\beta(\boldsymbol{\kappa}')} \widetilde{\mathcal{R}}\Big(k\ell\big(\boldsymbol{\kappa} - \boldsymbol{\kappa}'\big), k\ell\big(\beta(\boldsymbol{\kappa}) - \beta(\boldsymbol{\kappa}')\big)\Big). \tag{3.8}$$

The total scattering cross section is

$$\Sigma(\boldsymbol{\kappa}) = \int_{|\boldsymbol{\kappa}'|<1} \frac{\mathrm{d}(k\boldsymbol{\kappa}')}{(2\pi)^d} \mathcal{Q}(\boldsymbol{\kappa}, \boldsymbol{\kappa}') = -2\mathrm{Re}\big[\mathrm{Q}(\boldsymbol{\kappa})\big] = \frac{2}{\mathcal{S}(\boldsymbol{\kappa})}. \tag{3.9}$$

Equation (3.7) looks like the radiative transfer equation, except that it is an initial value problem in z, with $\mathcal{W}(\boldsymbol{\kappa}, \boldsymbol{x}, z = 0)$ given by the Wigner transform of mode amplitudes $a_o(\boldsymbol{\kappa})$ in the homogeneous medium. As we show in subsection 3.1 it is in fact a general form of the one-way radiative transfer equation introduced recently in the biomedical imaging literature [13]. We also establish in subsection 3.2 the connection between equation (3.7) and the standard radiative transfer theory: We show that Eq. (3.7) can be obtained heuristically from the standard radiative transfer equation by applying a forward scattering approximation. Such a calculation is heuristic, because the standard radiative transfer equation has no rigorous derivation [4], whereas Eq. (3.7) is derived here from first principles. The connection to the Itô-Schrödinger model is in subsection 3.3: We show that Eq. (3.7) can be reduced to the transport equation in the paraxial geometry by taking the limit of very small angles. Therefore Eq. (3.7) can be seen as a bridge between the radiative transfer and paraxial approximation regimes.

3.1. Connection with the one-way radiative transfer equation. The one-way radiative transfer equation was proposed recently in [13] for the application of diffusion optical tomography in forward-peaked scattering media. The equation is stated in [13] in two dimensions ($d + 1 = 2$),

$$\sin\theta \partial_z I + \cos\theta \partial_x I = \mu_s \int_0^\pi p(\theta - \theta')\big[I(\theta') - I(\theta)\big]\mathrm{d}\theta', \tag{3.10}$$

for $I(\theta, x, z)$ the light intensity at position (x, z) in the direction $(\cos\theta, \sin\theta)$, with $\theta \in [0, \pi]$. The coefficient μ_s is the total scattering cross section and the scattering phase function $p(\theta - \theta')$ is chosen of the Henyey-Greenstein form [15, 13],

$$p(\theta - \theta') = \frac{1}{2\pi} \frac{1 - g^2}{1 + g^2 - 2g\cos(\theta - \theta')}, \tag{3.11}$$

satisfying $\int_0^{2\pi} p(\theta) d\theta = 1$. Parameter $g \in (0, 1)$ is the anisotropy factor and it is argued that the one-way radiative transfer equation is valid when $g \sim 1$, so scattering is forward-peaked.

The light intensity I is in fact the Wigner transform $\mathcal{W}$ introduced in (3.6), with $\boldsymbol{\kappa} = \cos\theta \in (-1,1)$. Indeed, in statistically isotropic media, i.e., $\mathcal{R}(\vec{\boldsymbol{x}}) = \mathcal{R}_{\rm iso}(|\vec{\boldsymbol{x}}|)$, we obtain from (3.7) (multiplied by $\sin\theta$), using that $\beta(\boldsymbol{\kappa}) = \sin\theta$ and $\nabla\beta(\boldsymbol{\kappa}) = -\cot\theta$,

$$\sin\theta\partial_z\mathcal{W} + \cos\theta\partial_x\mathcal{W} = \frac{k^3\ell^2\alpha^2}{4} \times \int_0^\pi \mathrm{d}\theta' \, \breve{\mathcal{R}}_{\rm iso}\Big(k\ell\sqrt{2(1-\cos(\theta-\theta'))}\Big)\big[\mathcal{W}(\theta') - \mathcal{W}(\theta)\big], \tag{3.12}$$

with

$$\breve{\mathcal{R}}_{\rm iso}(q) = \int_0^\infty \mathrm{d}s\, s\mathcal{R}_{\rm iso}(s)J_0(qs). \tag{3.13}$$

This is exactly (3.10) with the identification:

$$\mu_s p(\theta-\theta') = \frac{k^3\ell^2\alpha^2}{4}\breve{\mathcal{R}}_{\rm iso}\Big(k\ell\sqrt{2(1-\cos(\theta-\theta'))}\Big). \tag{3.14}$$

The scattering phase function (3.11) is a particular case of (3.14), corresponding to a Lorentzian for $\breve{\mathcal{R}}_{\rm iso}$, that is

$$\breve{\mathcal{R}}_{\rm iso}(q) = \frac{\breve{\mathcal{R}}_o}{1+q^2}. \tag{3.15}$$

This corresponds (through (3.13) and [14, formula 6.521.2]) to an autocovariance function of the form $\mathcal{R}_{\rm iso}(s) = \breve{\mathcal{R}}_o K_0(s)$, where K_0 is the Bessel function of the second kind of order zero. This is the zeroth von Kármán correlation function [17]. It has a logarithmic divergence at $s = 0$, which can be regularized by introducing an ultraviolet cutoff in (3.15). By substituting (3.11) and (3.15) into (3.14) we obtain the anisotropy parameter and total scattering cross section

$$g = 1 + \frac{1}{2(k\ell)^2} - \frac{1}{k\ell}\sqrt{1+\frac{1}{4(k\ell)^2}}, \qquad \mu_s = \Big(\frac{1-g}{1+g}\Big)\frac{\pi k^3\ell^2\alpha^2\breve{\mathcal{R}}_o}{2}.$$

The validity condition $g \sim 1$ in [13] is equivalent to $\lambda < \ell$. This completes the proof that (3.10) is a special case of our Eq. (3.7). It justifies the model (3.10), as our results in this paper show that it can be rigorously derived from the wave equation in random media, in the scaling regime (2.4).

3.2. Connection to the radiative transfer theory. To connect our transport equation (3.7) to the standard radiative transfer theory in random media [5, 25, 20], we let $d+1 = 3$ and adhere to the notation in [25]. Following [25, Eq. (3.42)], we define

$$f(\vec{\boldsymbol{\mathcal{K}}}, \vec{\boldsymbol{x}}) = \pi\Big[-\frac{i}{k}\frac{\vec{\boldsymbol{\mathcal{K}}}}{|\vec{\boldsymbol{\mathcal{K}}}|}\cdot\vec{\nabla}_{\vec{\boldsymbol{x}}}u(\vec{\boldsymbol{x}}) + u(\vec{\boldsymbol{x}})\Big],$$

where we use a different constant of proportionality than in [25], to simplify the relation in (3.20). The Wigner transform $W(\vec{\boldsymbol{\mathcal{K}}}, \vec{\boldsymbol{x}})$ introduced in [25, Eq. (3.41)] is

$$W(\vec{\boldsymbol{\mathcal{K}}}, \vec{\boldsymbol{x}}) = \int_{\mathbb{R}^3}\frac{\mathrm{d}\vec{\boldsymbol{y}}}{(2\pi)^3}\, f\Big(\vec{\boldsymbol{\mathcal{K}}}, \vec{\boldsymbol{x}} - \frac{\vec{\boldsymbol{y}}}{2}\Big)\overline{f\Big(\vec{\boldsymbol{\mathcal{K}}}, \vec{\boldsymbol{x}} + \frac{\vec{\boldsymbol{y}}}{2}\Big)}e^{i\vec{\boldsymbol{\mathcal{K}}}\cdot\vec{\boldsymbol{y}}}, \tag{3.16}$$

and satisfies the transport equation [25, Eq. (4.38)]

$$\vec{\nabla}_{\vec{\mathcal{K}}}\omega(\vec{\mathcal{K}})\cdot\vec{\nabla}_{\vec{\boldsymbol{x}}}W(\vec{\mathcal{K}},\vec{\boldsymbol{x}}) = \int_{\mathbb{R}^3} \mathrm{d}\vec{\mathcal{K}}'\,\sigma(\vec{\mathcal{K}},\vec{\mathcal{K}}')W(\vec{\mathcal{K}}',\vec{\boldsymbol{x}}) \\ -\Sigma(\vec{\mathcal{K}})W(\vec{\mathcal{K}},\vec{\boldsymbol{x}}), \tag{3.17}$$

with dispersion relation $\omega(\vec{\mathcal{K}}) = c_o|\vec{\mathcal{K}}|$, and integral kernel, the differential scattering cross-section,

$$\sigma(\vec{\mathcal{K}},\vec{\mathcal{K}}') = \frac{\pi c_o^2 k^2 \ell^3 \alpha^2}{2(2\pi)^3}\widetilde{\mathcal{R}}\big[\ell(\vec{\mathcal{K}}-\vec{\mathcal{K}}')\big]\delta[\omega(\vec{\mathcal{K}})-\omega(\vec{\mathcal{K}}')]. \tag{3.18}$$

The scalar $\Sigma(\vec{\mathcal{K}})$ is the total scattering cross section

$$\Sigma(\vec{\mathcal{K}}) = \int_{\mathbb{R}^3} \mathrm{d}\vec{\mathcal{K}}'\,\sigma(\vec{\mathcal{K}},\vec{\mathcal{K}}'). \tag{3.19}$$

Substituting (3.3) into (3.16), we obtain after some algebraic manipulations that

$$W(\vec{\mathcal{K}},\vec{\boldsymbol{x}}) = \frac{\delta\big[\mathcal{K}_z - k\beta(\boldsymbol{\mathcal{K}}/k)\big]}{\beta(\boldsymbol{\mathcal{K}}/k)}\mathcal{W}(\boldsymbol{\mathcal{K}}/k,\boldsymbol{x},z), \tag{3.20}$$

with $\mathcal{W}$ the Wigner transform (3.6). The Dirac factor in Eq. (3.20) expresses the fact that in our scaling regime, in which the wave field has the form (3.3), the forward scattering approximation is valid and the intensity resolved over directions of propagation is supported on the wave vectors $\boldsymbol{\mathcal{K}}$ with positive $\mathcal{K}_z$. Next we rewrite the three terms of (3.17) to show that the equation is equivalent to (3.7).

1) Since (3.20) gives that $W(\vec{\mathcal{K}},\vec{\boldsymbol{x}})$ is supported at vectors $\vec{\mathcal{K}}$ of the form $\vec{\mathcal{K}} = k\vec{\boldsymbol{\kappa}}$, with $\vec{\boldsymbol{\kappa}} = (\boldsymbol{\kappa},\beta(\boldsymbol{\kappa}))$, the operator on the left hand side of (3.17) is

$$\vec{\nabla}_{\vec{\mathcal{K}}}\omega(\vec{\mathcal{K}})\cdot\vec{\nabla}_{\vec{\boldsymbol{x}}} = c_o\beta(\boldsymbol{\kappa})\big[\partial_z - \nabla\beta(\boldsymbol{\kappa})\cdot\nabla_{\boldsymbol{x}}\big],$$

and we obtain that

$$\vec{\nabla}_{\vec{\mathcal{K}}}\omega(\vec{\mathcal{K}})\cdot\vec{\nabla}_{\vec{\boldsymbol{x}}}W(\vec{\mathcal{K}},\vec{\boldsymbol{x}}) = c_o\delta\big[\mathcal{K}_z - k\beta(\boldsymbol{\mathcal{K}}/k)\big] \\ \times\big[\partial_z - \nabla\beta(\boldsymbol{\mathcal{K}}/k)\cdot\nabla_{\boldsymbol{x}}\big]\mathcal{W}(\boldsymbol{\mathcal{K}}/k,\boldsymbol{x},z). \tag{3.21}$$

2) The integral kernel in (3.17) is supported at $\vec{\mathcal{K}}' = k\vec{\boldsymbol{\kappa}}'$, with $\vec{\boldsymbol{\kappa}}' = (\boldsymbol{\kappa}',\beta(\boldsymbol{\kappa}'))$, by (3.20), so the Dirac distribution in (3.18) is

$$\delta\big[\omega(\vec{\mathcal{K}}) - \omega(k\vec{\boldsymbol{\kappa}}')\big] = \frac{\delta\big[\mathcal{K}_z - k\beta(\boldsymbol{\mathcal{K}}/k)\big]}{c_o\beta(\boldsymbol{\mathcal{K}}/k)}.$$

Thus, we have

$$\int_{\mathbb{R}^3}\mathrm{d}\vec{\mathcal{K}}'\,\sigma(\vec{\mathcal{K}},\vec{\mathcal{K}}')W(\vec{\mathcal{K}}',\vec{\boldsymbol{x}}) = \frac{c_o k^2\ell^3\alpha^2}{4}\delta\big[\mathcal{K}_z - k\beta(\boldsymbol{\mathcal{K}}/k)\big] \\ \times\int_{|\boldsymbol{\kappa}'|<1}\frac{\mathrm{d}(k\boldsymbol{\kappa}')}{(2\pi)^2}\,\frac{\widetilde{\mathcal{R}}\big[\ell(\boldsymbol{\mathcal{K}}-k\boldsymbol{\kappa}'),k\ell(\beta(\boldsymbol{\mathcal{K}}/k)-\beta(\boldsymbol{\kappa}'))\big]}{\beta(\boldsymbol{\mathcal{K}}/k)\beta(\boldsymbol{\kappa}')}\mathcal{W}(\boldsymbol{\kappa}',\boldsymbol{x},z), \tag{3.22}$$

where $|\boldsymbol{\kappa}'| < 1$ because we have only propagating waves.

3) From (3.19) we find that

$$\Sigma(\vec{\mathcal{K}}) = \frac{c_o^2 k^2 \ell^3 \alpha^2}{4(2\pi)^2} \int_{\mathbb{R}^3} \mathrm{d}\vec{\mathcal{K}}' \delta\big[\omega(\vec{\mathcal{K}}') - \omega(\vec{\mathcal{K}})\big] \widetilde{\mathcal{R}}\big(\ell(\vec{\mathcal{K}} - \vec{\mathcal{K}}')\big),$$

so for $\vec{\mathcal{K}} = k(\boldsymbol{\kappa}, \beta(\boldsymbol{\kappa}))$,

$$\Sigma(\vec{\mathcal{K}}) W(\vec{\mathcal{K}}, \vec{\boldsymbol{x}}) = \frac{c_o k^2 \ell^3 \alpha^2}{4} \delta\big[\mathcal{K}_z - k\beta(\boldsymbol{\mathcal{K}}/k)\big] \int_{|\boldsymbol{\kappa}'|<1} \frac{\mathrm{d}(k\boldsymbol{\kappa}')}{(2\pi)^2}$$
$$\times \frac{\widetilde{\mathcal{R}}\big[\ell(\boldsymbol{\mathcal{K}} - k\boldsymbol{\kappa}'), \ell k(\beta(\boldsymbol{\mathcal{K}}/k) - \beta(\boldsymbol{\kappa}'))\big]}{\beta(\boldsymbol{\mathcal{K}}/k)\beta(\boldsymbol{\kappa}')} \mathcal{W}(\boldsymbol{\mathcal{K}}/k, \boldsymbol{x}, z). \tag{3.23}$$

Finally, substituting (3.21), (3.22), and (3.23) into the transport equation (3.17) satisfied by W, we obtain that the Wigner transform $\mathcal{W}$ satisfies the transport equation (3.7). This completes the proof that Eq. (3.7) can be obtained from the standard radiative transfer equation (3.17) by applying a forward scattering approximation. However, as stated before, there is no rigorous derivation of the standard radiative transfer equation from the wave equation in random media. In this paper we obtain a rigorous derivation of Eq. (3.7) from the wave equation in random media, in the scaling regime (2.4).

3.3. Connection to the paraxial theory. It is shown in [10] that if $\lambda \ll \ell \ll L$ so that the medium Fresnel number $\ell^2/(\lambda L) \sim 1$, and if the standard deviation α of the fluctuations is small so that $\alpha^2 \sim \lambda^2/(\ell L)$, then the inverse Fourier transform of the mode amplitudes, denoted by $a_{\rm pa}(\boldsymbol{\kappa}, z)$,

$$\check{a}_{\rm pa}(\boldsymbol{x}, z) = \int_{|\boldsymbol{\kappa}|<1} \frac{\mathrm{d}(k\boldsymbol{\kappa})}{(2\pi)^d}\, a_{\rm pa}(\boldsymbol{\kappa}, z) e^{ik\boldsymbol{\kappa}\cdot\boldsymbol{x}},$$

satisfies the random paraxial wave equation (or Itô-Schrödinger model) [10]

$$\mathrm{d}\check{a}_{\rm pa}(\boldsymbol{x}, z) = \frac{i}{2k} \Delta_{\boldsymbol{x}} \check{a}_{\rm pa}(\boldsymbol{x}, z)\mathrm{d}z + \frac{ik}{2} \check{a}_{\rm pa}(\boldsymbol{x}, z) \circ \mathrm{d}B(\boldsymbol{x}, z). \tag{3.24}$$

Here B is the Brownian field i.e., a Gaussian process with mean zero and covariance

$$\mathbb{E}[B(\boldsymbol{x}, z)B(\boldsymbol{x}', z')] = \alpha^2 \ell \min(z, z') \mathcal{C}\Big(\frac{\boldsymbol{x} - \boldsymbol{x}'}{\ell}\Big), \quad \mathcal{C}(\boldsymbol{r}) = \int_{-\infty}^{\infty} \mathrm{d}\zeta\, \mathcal{R}(\boldsymbol{r}, \zeta).$$

The symbol $\circ$ stands for the Stratonovich integral. This integral is the suitable form of stochastic integral for the Itô-Schrödinger model as shown in [10], and as could be predicted by the general Wong-Zakai theorem [27]. Alternatively, we can characterize $a_{\rm pa}(\boldsymbol{\kappa}, z)$ as the solution of

$$\mathrm{d}\check{a}_{\rm pa}(\boldsymbol{x}, z) = \frac{i}{2k} \Delta_{\boldsymbol{x}} \check{a}_{\rm pa}(\boldsymbol{x}, z)\mathrm{d}z + \frac{ik}{2} \check{a}_{\rm pa}(\boldsymbol{x}, z)\mathrm{d}B(\boldsymbol{x}, z) - \frac{k^2 \ell \alpha^2 \mathcal{C}(\boldsymbol{0})}{8} \check{a}_{\rm pa}(\boldsymbol{x}, z)\mathrm{d}z,$$

where the stochastic integral is now understood in the usual Itô's form.

The derivation of (3.24) from the wave equation in random media, given in [10], involves two main steps: first show that the forward scattering approximation is valid; second show that the effect of the fluctuations of the random medium on the wave field can be captured in distribution by a white noise (in z) model.

Using the Itô-Schrödinger model (3.24) we find by Itô's formula that the mean field $\check{A}_{\rm pa}(\boldsymbol{x},z) = \mathbb{E}[\check{a}_{\rm pa}(\boldsymbol{x},z)]$ satisfies

$$\partial_z \check{A}_{\rm pa}(\boldsymbol{x},z) = \frac{i}{2k}\Delta_{\boldsymbol{x}}\check{A}_{\rm pa}(\boldsymbol{x},z) - \frac{k^2\ell\alpha^2\mathcal{C}(\boldsymbol{0})}{8}\check{A}_{\rm pa}(\boldsymbol{x},z).$$

It decays with z on the scale

$$\mathcal{S}_{\rm pa} = \frac{8}{k^2\ell\alpha^2\mathcal{C}(\boldsymbol{0})} = \frac{8}{k^2\ell\alpha^2\int_{-\infty}^{\infty}{\rm d}\zeta\,\mathcal{R}(\boldsymbol{0},\zeta)},$$

which corresponds to the scattering mean free path $\mathcal{S}(\boldsymbol{\kappa})$ defined by (3.4-3.5), for $\lambda \ll \ell$ and $|\boldsymbol{\kappa}| = O(\lambda/\ell)$.

The Wigner transform is

$$\begin{aligned} W_{\rm pa}(\boldsymbol{\mathcal{K}},\boldsymbol{x},z) &= \int_{\mathbb{R}^d}{\rm d}\boldsymbol{y}\, e^{i\boldsymbol{\mathcal{K}}\cdot\boldsymbol{y}}\mathbb{E}\Big[\check{a}_{\rm pa}\big(\boldsymbol{x}-\frac{\boldsymbol{y}}{2},z\big)\overline{\check{a}_{\rm pa}\big(\boldsymbol{x}+\frac{\boldsymbol{y}}{2},z\big)}\Big] \\ &= \int_{\mathbb{R}^d}\frac{{\rm d}(k\boldsymbol{q})}{(2\pi)^d}\, e^{ik\boldsymbol{q}\cdot\boldsymbol{x}}\mathbb{E}\Big[a_{\rm pa}\big(\frac{\boldsymbol{\mathcal{K}}}{k}+\frac{\boldsymbol{q}}{2},z\big)\overline{a_{\rm pa}\big(\frac{\boldsymbol{\mathcal{K}}}{k}-\frac{\boldsymbol{q}}{2},z\big)}\Big], \end{aligned}$$

which corresponds to (3.6) for $\boldsymbol{\mathcal{K}} = k\boldsymbol{\kappa}$ and $|\boldsymbol{\kappa}| = O(\lambda/\ell)$. Using Itô's formula it is shown in [10] to satisfy the transport equation

$$\begin{aligned} &\partial_z W_{\rm pa} + \frac{1}{k}\boldsymbol{\mathcal{K}}\cdot\nabla_{\boldsymbol{x}}W_{\rm pa} = \\ &\frac{k^2\ell^{d+1}\alpha^2}{4}\int_{\mathbb{R}^d}\frac{{\rm d}\boldsymbol{\mathcal{K}}'}{(2\pi)^d}\,\widetilde{\mathcal{R}}\big(\ell(\boldsymbol{\mathcal{K}}-\boldsymbol{\mathcal{K}}'),0\big)\big[W_{\rm pa}(\boldsymbol{\mathcal{K}}') - W_{\rm pa}(\boldsymbol{\mathcal{K}})\big], \end{aligned} \tag{3.25}$$

with differential scattering cross section

$$\mathcal{Q}_{\rm pa}(\boldsymbol{\mathcal{K}},\boldsymbol{\mathcal{K}}') = \frac{k^2\ell^{d+1}\alpha^2}{4}\widetilde{\mathcal{R}}\big(\ell(\boldsymbol{\mathcal{K}}-\boldsymbol{\mathcal{K}}'),0\big)$$

corresponding to (3.8) for $\boldsymbol{\mathcal{K}} = k\boldsymbol{\kappa}$, $\boldsymbol{\mathcal{K}}' = k\boldsymbol{\kappa}'$, and $|\boldsymbol{\kappa}|, |\boldsymbol{\kappa}'| = O(\lambda/\ell)$.

This establishes the connection between Eq. (3.7) and the transport equation (3.25) derived in [10]. Together with the result in section 3.2 it completes the proof that Eq. (3.7) is a bridge between the radiative transfer and paraxial approximation regimes.

We end the section with the note that, as shown for instance in [16, Chapter 13], the radiative transfer equation in the white-noise paraxial regime (3.25) can also be derived heuristically from the standard radiative transfer equation in the "approximation of large particles", or equivalently in the "small angle approximation", which corresponds to a random medium with large correlation radius.

4. Derivation of results. To derive the transport equation (3.7) from the wave equation, we use multiscale analysis and probabilistic limit theorems. The asymptotic regime of separation of scales (2.4) is defined in terms of three small dimensionless parameters

$$\varepsilon = \frac{\lambda}{L}, \qquad \gamma = \frac{\lambda}{\ell}, \qquad \eta = \frac{\lambda}{X}, \tag{4.1}$$

ordered as

$$0 < \varepsilon \ll \gamma \sim \eta < 1,$$

and the standard deviation α of the fluctuations of the random medium is of order $\varepsilon^{\frac{1}{2}}$. We begin with the wave decomposition, and obtain a stochastic system of differential equations satisfied by the mode amplitudes. We consider both forward and backward going waves, but then show that we can neglect the backward waves in the limit $\varepsilon \to 0$ (subsection 4.3). The $\varepsilon \to 0$ limit of the mode amplitudes defines the Markov process whose expectation and Wigner transform are described in section 3.

4.1. Scaled equation. We let L be the reference length scale, which is similar to the distance of propagation, and introduce the scaled length variables $\boldsymbol{x}' = \boldsymbol{x}/(\varepsilon L)$, $z' = z/L$, $L' = L/L = 1$, $\ell' = \ell/L = \varepsilon/\gamma$ and $X' = X/L = \varepsilon/\eta$. The scaled standard deviation is $\alpha' = \alpha/\varepsilon^{1/2}$. The scaled wavenumber is $k' = kL\varepsilon = 2\pi$.

Let us denote the wave field by u^ε. Substituting in (2.1) and dropping all the primes, as all the variables are scaled henceforth, we obtain

$$\Big\{\partial_z^2 + \frac{1}{\varepsilon^2}\Delta_{\boldsymbol{x}} + \frac{k^2}{\varepsilon^2}\Big[1 + \varepsilon^{\frac{1}{2}}\alpha\nu\Big(\gamma\boldsymbol{x}, \frac{\gamma z}{\varepsilon}\Big)\Big]\Big\} u^\varepsilon(\boldsymbol{x}, z) = -\frac{1}{\varepsilon}F\big(\eta\boldsymbol{x}\big)\delta(z), \tag{4.2}$$

for $0 \le z \le L$. At ranges $z < 0$ and $z > L$ the equations are simpler, as the term involving the process ν vanishes. Since the wave field depends linearly on the source, we scaled F by $1/\varepsilon$ to obtain an order one result in the limit $\varepsilon \to 0$.

4.2. Wave decomposition. We decompose the field $u^\varepsilon(\boldsymbol{x}, z)$ in plane waves using the Fourier transform with respect to $\boldsymbol{x} \in \mathbb{R}^d$, as in (3.1):

$$\widehat{u}^\varepsilon(\boldsymbol{\kappa}, z) = \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{x}\, u^\varepsilon(\boldsymbol{x}, z) e^{-ik\boldsymbol{\kappa}\cdot\boldsymbol{x}}. \tag{4.3}$$

The transformed field $\widehat{u}^\varepsilon(\boldsymbol{\kappa}, z)$ is a superposition of forward and backward going waves (modes) along z, as explained next. To ease the explanation we begin with the reference case in the homogeneous medium, and then consider the random medium.

4.2.1. Homogeneous media. The transformed field in homogeneous media $\widehat{u}_o^\varepsilon(\boldsymbol{\kappa}, z)$ satisfies the ordinary differential equation

$$\partial_z^2 \widehat{u}_o^\varepsilon(\boldsymbol{\kappa}, z) + \frac{k^2}{\varepsilon^2}\beta(\boldsymbol{\kappa})^2 \widehat{u}_o^\varepsilon(\boldsymbol{\kappa}, z) = -\frac{1}{\varepsilon\eta^d}\widehat{F}\Big(\frac{k\boldsymbol{\kappa}}{\eta}\Big)\delta(z), \tag{4.4}$$

with $\beta(\boldsymbol{\kappa})$ defined in (3.2) and $\widehat{F}$ the Fourier transform of F,

$$\widehat{F}(\boldsymbol{q}) = \int_{\mathbb{R}^d} F(\boldsymbol{r}) e^{-i\boldsymbol{q}\cdot\boldsymbol{r}} \mathrm{d}\boldsymbol{r}. \tag{4.5}$$

The solution is outgoing and bounded away from the source, and it is given explicitly, for $z \ne 0$, by

$$\widehat{u}_o^\varepsilon(\boldsymbol{\kappa}, z) = \frac{a_o(\boldsymbol{\kappa})}{\beta^{\frac{1}{2}}(\boldsymbol{\kappa})} e^{\frac{ik}{\varepsilon}\beta(\boldsymbol{\kappa})z} 1_{(0,\infty)}(z) + \frac{b_o(\boldsymbol{\kappa})}{\beta^{\frac{1}{2}}(\boldsymbol{\kappa})} e^{-\frac{ik}{\varepsilon}\beta(\boldsymbol{\kappa})z} 1_{(-\infty,0)}(z). \tag{4.6}$$

Thus, the wave field

$$u_o^\varepsilon(\boldsymbol{x}, z) = \int_{|\boldsymbol{\kappa}|<1} \frac{\mathrm{d}(k\boldsymbol{\kappa})}{(2\pi)^d} \widehat{u}_o^\varepsilon(\boldsymbol{\kappa}, z) e^{ik\boldsymbol{\kappa}\cdot\boldsymbol{x}}$$

is a synthesis of plane waves with wave vectors $k\big(\boldsymbol{\kappa}, \pm\beta(\boldsymbol{\kappa})\big)$. The plus sign corresponds to forward going waves, and the negative sign to backward going waves. The amplitudes are determined by the jump conditions at the source

$$\widehat{u}_o^\varepsilon(\boldsymbol{\kappa}, 0+) - \widehat{u}_o^\varepsilon(\boldsymbol{\kappa}, 0-) = 0,$$
$$\partial_z \widehat{u}_o^\varepsilon(\boldsymbol{\kappa}, 0+) - \partial_z \widehat{u}_o^\varepsilon(\boldsymbol{\kappa}, 0-) = -\frac{1}{\varepsilon\eta^d}\widehat{F}\Big(\frac{k\boldsymbol{\kappa}}{\eta}\Big),$$

which gives

$$a_o(\boldsymbol{\kappa}) = b_o(\boldsymbol{\kappa}) = \frac{i}{2k\eta^d \beta^{\frac{1}{2}}(\boldsymbol{\kappa})}\widehat{F}\Big(\frac{k\boldsymbol{\kappa}}{\eta}\Big). \tag{4.7}$$

The radius of the support of $\widehat{F}(\boldsymbol{q})$ is one, so the scaling parameter η controls the support in $\boldsymbol{\kappa}$ of the wave modes generated by the source i.e., the opening angle of the initial wave beam. Consistent with (2.4) and (4.1), we assume henceforth that

$$\frac{\eta}{k} < 1. \tag{4.8}$$

so that in (4.7) we have $|\boldsymbol{\kappa}| \le \eta/k < 1$. Then $\beta(\boldsymbol{\kappa})$ defined by (3.2) is real valued, and there are no evanescent waves in the decomposition (4.6).

4.2.2. Random media. The field $\widehat{u}^\varepsilon(\boldsymbol{\kappa}, z)$ in the random medium satisfies the equation

$$\partial_z^2 \widehat{u}^\varepsilon + \frac{k^2}{\varepsilon^2}\beta(\boldsymbol{\kappa})^2 \widehat{u}^\varepsilon + 1_{(0,L)}(z)\,\mathcal{M}^\varepsilon \widehat{u}^\varepsilon = -\frac{1}{\varepsilon\eta^d}\widehat{F}\Big(\frac{k\boldsymbol{\kappa}}{\eta}\Big)\delta(z), \tag{4.9}$$

derived from (4.2), with radiation conditions at $z < 0$ and $z > L$, and source conditions at $z = 0$. The leading $O(1/\varepsilon^2)$ term in the right hand side is the same as in the homogeneous medium, so we can use a similar wave decomposition to that in section 4.2.1. The random perturbation is in the operator $\mathcal{M}^\varepsilon$ defined by

$$\mathcal{M}^\varepsilon \widehat{u}^\varepsilon(\boldsymbol{\kappa}, z) = \frac{ik\alpha}{\varepsilon^{\frac{1}{2}}\gamma^d}\int \frac{\mathrm{d}(k\boldsymbol{\kappa}')}{(2\pi)^d}\,\frac{\widehat{\nu}\Big(\frac{k(\boldsymbol{\kappa}-\boldsymbol{\kappa}')}{\gamma}, \frac{\gamma z}{\varepsilon}\Big)}{[\beta(\boldsymbol{\kappa})\beta(\boldsymbol{\kappa}')]^{\frac{1}{2}}}\widehat{u}^\varepsilon(\boldsymbol{\kappa}', z),$$

where $\widehat{\nu}$ is the Fourier transform of ν with respect to the first argument in $\mathbb{R}^d$ as in (4.5).

The wave decomposition is

$$a^\varepsilon(\boldsymbol{\kappa}, z) = \frac{1}{2}\Big(\beta(\boldsymbol{\kappa})^{\frac{1}{2}}\widehat{u}^\varepsilon(\boldsymbol{\kappa}, z) + \frac{\varepsilon}{ik\beta(\boldsymbol{\kappa})^{\frac{1}{2}}}\partial_z \widehat{u}^\varepsilon(\boldsymbol{\kappa}, z)\Big)e^{-\frac{ik}{\varepsilon}\beta(\boldsymbol{\kappa})z},$$
$$b^\varepsilon(\boldsymbol{\kappa}, z) = \frac{1}{2}\Big(\beta(\boldsymbol{\kappa})^{\frac{1}{2}}\widehat{u}^\varepsilon(\boldsymbol{\kappa}, z) - \frac{\varepsilon}{ik\beta(\boldsymbol{\kappa})^{\frac{1}{2}}}\partial_z \widehat{u}^\varepsilon(\boldsymbol{\kappa}, z)\Big)e^{\frac{ik}{\varepsilon}\beta(\boldsymbol{\kappa})z},$$

so that we can write as in the homogeneous medium

$$\widehat{u}^\varepsilon(\boldsymbol{\kappa}, z) = \frac{1}{\beta(\boldsymbol{\kappa})^{\frac{1}{2}}}\Big(a^\varepsilon(\boldsymbol{\kappa}, z)e^{\frac{ik}{\varepsilon}\beta(\boldsymbol{\kappa})z} + b^\varepsilon(\boldsymbol{\kappa}, z)e^{-\frac{ik}{\varepsilon}\beta(\boldsymbol{\kappa})z}\Big), \tag{4.10}$$

$$\partial_z \widehat{u}^\varepsilon(\boldsymbol{\kappa}, z) = \frac{ik\beta(\boldsymbol{\kappa})^{\frac{1}{2}}}{\varepsilon}\Big(a^\varepsilon(\boldsymbol{\kappa}, z)e^{\frac{ik}{\varepsilon}\beta(\boldsymbol{\kappa})z} - b^\varepsilon(\boldsymbol{\kappa}, z)e^{-\frac{ik}{\varepsilon}\beta(\boldsymbol{\kappa})z}\Big). \tag{4.11}$$

The forward and backward going wave amplitudes $a^\varepsilon(\boldsymbol{\kappa}, z)$ and $b^\varepsilon(\boldsymbol{\kappa}, z)$ are no longer constant, but random fields due to scattering in the range interval $z \in (0, L)$. The medium is homogeneous outside this interval and we have the radiation conditions

$$a^\varepsilon(\boldsymbol{\kappa}, z) = 0 \text{ if } z < 0 \quad \text{and} \quad b^\varepsilon(\boldsymbol{\kappa}, z) = 0 \text{ if } z \geq L. \tag{4.12}$$

Moreover, $a^\varepsilon(\boldsymbol{\kappa}, z) = a^\varepsilon(\boldsymbol{\kappa}, L)$ for $z > L$, and $b^\varepsilon(\boldsymbol{\kappa}, z) = b^\varepsilon(\boldsymbol{\kappa}, 0-)$ for $z < 0$.

The jump conditions at the source are as in section 4.2.1, and give

$$a^\varepsilon(\boldsymbol{\kappa}, 0+) = a_o(\boldsymbol{\kappa}) \quad \text{and} \quad b^\varepsilon(\boldsymbol{\kappa}, 0-) = b_o(\boldsymbol{\kappa}) + b^\varepsilon(\boldsymbol{\kappa}, 0+). \tag{4.13}$$

As expected, the forward going waves leaving the source are the same as in the homogeneous medium, because the scattering effects in the random medium manifest only at long distances of propagation. The waves at $z < 0$ are given by the superposition of those emitted by the source, modeled by $b_o(\boldsymbol{\kappa})$, and the waves backscattered by the random medium, modeled by $b^\varepsilon(\boldsymbol{\kappa}, 0+)$.

To determine the amplitudes in the random medium, we substitute equations (4.10)-(4.11) into (4.9). We obtain that

$$\begin{aligned}\partial_z \begin{pmatrix} a^\varepsilon(\boldsymbol{\kappa}, z) \\ b^\varepsilon(\boldsymbol{\kappa}, z) \end{pmatrix} = \frac{ik\alpha}{2\gamma^d \varepsilon^{\frac{1}{2}}} \int \frac{\mathrm{d}(k\boldsymbol{\kappa}')}{(2\pi)^d} \, \widehat{\nu}\Big(\frac{k(\boldsymbol{\kappa} - \boldsymbol{\kappa}')}{\gamma}, \frac{\gamma z}{\varepsilon}\Big) \\ \times \boldsymbol{\Gamma}\Big(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \frac{z}{\varepsilon}\Big) \begin{pmatrix} a^\varepsilon(\boldsymbol{\kappa}', z) \\ b^\varepsilon(\boldsymbol{\kappa}', z) \end{pmatrix},\end{aligned} \tag{4.14}$$

in $z \in (0, L)$, with boundary conditions (4.12)-(4.13). We are interested in the propagating waves, corresponding to $|\boldsymbol{\kappa}| < 1$ in (4.14), and we explain in section 4.3 that in our regime the evanescent waves may be neglected. The 2×2 complex matrices

$$\boldsymbol{\Gamma}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \zeta) = \begin{pmatrix} \Gamma^{aa}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \zeta) & \Gamma^{ab}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \zeta) \\ \Gamma^{ba}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \zeta) & \Gamma^{bb}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \zeta) \end{pmatrix}, \tag{4.15}$$

couple the mode amplitudes. The superscripts on their entries indicate which types of waves they couple. We have

$$\begin{aligned}\Gamma^{aa}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \zeta) =& \frac{e^{ik[\beta(\boldsymbol{\kappa}') - \beta(\boldsymbol{\kappa})]\zeta}}{\beta^{\frac{1}{2}}(\boldsymbol{\kappa})\beta^{\frac{1}{2}}(\boldsymbol{\kappa}')}, \quad \Gamma^{ab}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \zeta) = \frac{e^{-ik[\beta(\boldsymbol{\kappa}') + \beta(\boldsymbol{\kappa})]\zeta}}{\beta^{\frac{1}{2}}(\boldsymbol{\kappa})\beta^{\frac{1}{2}}(\boldsymbol{\kappa}')}, \\ \Gamma^{bb}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \zeta) =& -\overline{\Gamma^{aa}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \zeta)}, \quad \Gamma^{ba}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \zeta) = -\overline{\Gamma^{ab}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \zeta)},\end{aligned} \tag{4.16}$$

where the bar denotes complex conjugate, and substituting in (4.14) we obtain the energy conservation identity

$$\int_{|\boldsymbol{\kappa}| < 1} \frac{\mathrm{d}(k\boldsymbol{\kappa})}{(2\pi)^d} \Big[|a^\varepsilon(\boldsymbol{\kappa}, z)|^2 - |b^\varepsilon(\boldsymbol{\kappa}, z)|^2 \Big] = \text{ constant in } z.$$

4.3. The Markov limit. Here we describe the $\varepsilon \to 0$ limit of the solution of (4.14) with boundary conditions (4.12)-(4.13). We begin in section 4.3.1 by writing the solution in terms of propagator matrices, and show in section 4.3.2 that we can neglect the backward and evanescent waves. The limit of the forward going amplitudes is in section 4.3.3.

4.3.1. Propagator matrices. The 2×2 propagator matrices $\mathbf{P}^\varepsilon(\boldsymbol{\kappa}, z; \boldsymbol{\kappa}_o)$ are solutions of

$$\partial_z \mathbf{P}^\varepsilon(\boldsymbol{\kappa}, z; \boldsymbol{\kappa}_o) = \frac{ik\alpha}{2\gamma^d \varepsilon^{\frac{1}{2}}} \int_{|\boldsymbol{\kappa}'|<1} \frac{\mathrm{d}(k\boldsymbol{\kappa}')}{(2\pi)^d} \widehat{\nu}\Big(\frac{k(\boldsymbol{\kappa}-\boldsymbol{\kappa}')}{\gamma}, \frac{\gamma z}{\varepsilon}\Big) \times \mathbf{\Gamma}\Big(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \frac{z}{\varepsilon}\Big) \mathbf{P}^\varepsilon(\boldsymbol{\kappa}', z; \boldsymbol{\kappa}_o), \tag{4.17}$$

for $z > 0$, with initial condition $\mathbf{P}^\varepsilon(\boldsymbol{\kappa}, z = 0; \boldsymbol{\kappa}_o) = \delta(\boldsymbol{\kappa} - \boldsymbol{\kappa}_o)\mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix. They allow us to write the solution of (4.14) as

$$\begin{pmatrix} a^\varepsilon(\boldsymbol{\kappa}, z) \\ b^\varepsilon(\boldsymbol{\kappa}, z) \end{pmatrix} = \int_{|\boldsymbol{\kappa}_o|<1} \mathrm{d}\boldsymbol{\kappa}_o \, \mathbf{P}^\varepsilon(\boldsymbol{\kappa}, z; \boldsymbol{\kappa}_o) \begin{pmatrix} a_o(\boldsymbol{\kappa}_o) \\ b^\varepsilon(\boldsymbol{\kappa}_o, 0) \end{pmatrix}, \tag{4.18}$$

for all $z > 0$. In particular, when $z = L$, the backward going amplitude $b^\varepsilon(\boldsymbol{\kappa}, L)$ in the left hand side vanishes by (4.12).

4.3.2. The forward scattering approximation. Equation (4.18) shows that the interaction of the forward and backward going wave amplitudes a^ε and b^ε depends on the coupling of the entries of the propagator. The $\varepsilon \to 0$ limit of the propagator

$$\mathbf{P}^\varepsilon(\boldsymbol{\kappa}, z; \boldsymbol{\kappa}_o) = \begin{pmatrix} P^{aa,\varepsilon}(\boldsymbol{\kappa}, z; \boldsymbol{\kappa}_o) & P^{ab,\varepsilon}(\boldsymbol{\kappa}, z; \boldsymbol{\kappa}_o) \\ P^{ba,\varepsilon}(\boldsymbol{\kappa}, z; \boldsymbol{\kappa}_o) & P^{bb,\varepsilon}(\boldsymbol{\kappa}, z; \boldsymbol{\kappa}_o) \end{pmatrix}$$

can be obtained and identified as a Markov process that satisfies a system of stochastic differential equations. We refer to [21, 23] and appendix A for details. Here we state the results.

The stochastic differential equations for the limit entries of $P^{ab,\varepsilon}(\boldsymbol{\kappa}, z; \boldsymbol{\kappa}_o)$ and $P^{ba,\varepsilon}(\boldsymbol{\kappa}, z; \boldsymbol{\kappa}_o)$ are coupled to the limit entries of $P^{aa,\varepsilon}(\boldsymbol{\kappa}', z; \boldsymbol{\kappa}_o)$ and $P^{bb,\varepsilon}(\boldsymbol{\kappa}', z; \boldsymbol{\kappa}_o)$ through the coefficients

$$\widetilde{\mathcal{R}}\left(\frac{k(\vec{\boldsymbol{\kappa}} - \vec{\boldsymbol{\kappa}}'^{-})}{\gamma}\right) = \widetilde{\mathcal{R}}\left(\frac{k(\boldsymbol{\kappa} - \boldsymbol{\kappa}')}{\gamma}, \frac{k(\beta(\boldsymbol{\kappa}) + \beta(\boldsymbol{\kappa}'))}{\gamma}\right),$$

where $\widetilde{\mathcal{R}}$ is the power spectral density (2.3) and $\vec{\boldsymbol{\kappa}} = (\boldsymbol{\kappa}, \beta(\boldsymbol{\kappa}))$ and $\vec{\boldsymbol{\kappa}}^- = (\boldsymbol{\kappa}, -\beta(\boldsymbol{\kappa}))$ are the wave vectors of the forward and backward going waves. The second argument in these coefficients comes from the phase factors $\pm k(\beta(\boldsymbol{\kappa}) + \beta(\boldsymbol{\kappa}'))\zeta$ in the matrices Γ^{ab} and Γ^{ba}. The coupling between $P^{aa,\varepsilon}(\boldsymbol{\kappa}, z; \boldsymbol{\kappa}_o)$ and $P^{aa,\varepsilon}(\boldsymbol{\kappa}', z; \boldsymbol{\kappa}_o)$ is through the coefficients

$$\widetilde{\mathcal{R}}\left(\frac{k(\vec{\boldsymbol{\kappa}} - \vec{\boldsymbol{\kappa}}')}{\gamma}\right) = \widetilde{\mathcal{R}}\left(\frac{k(\boldsymbol{\kappa} - \boldsymbol{\kappa}')}{\gamma}, \frac{k(\beta(\boldsymbol{\kappa}) - \beta(\boldsymbol{\kappa}'))}{\gamma}\right),$$

because the phase factors in matrices Γ^{aa} are $k(\beta(\boldsymbol{\kappa}) - \beta(\boldsymbol{\kappa}'))\zeta$. The matrices Γ^{bb} have the same factors so the same coefficients couple the entries $P^{bb,\varepsilon}$.

We conclude that the coupling of the entries of the propagator and therefore the interaction of the waves depends on the decay of the power spectral density $\widetilde{\mathcal{R}}$. We now explain that when the mode amplitudes are supported initially at $|\boldsymbol{\kappa}| \le \eta/k < 1$, and γ is as in (4.1), we can neglect the backward going waves over distances of propagation of order L.

The power spectral density $\widetilde{\mathcal{R}}(\vec{\boldsymbol{q}})$ is negligible when $|\vec{\boldsymbol{q}}| > 1$, so $\widetilde{\mathcal{R}}(k\vec{\boldsymbol{\kappa}}/\gamma)$ is negligible when $|\vec{\boldsymbol{\kappa}}| > \gamma/k$. From (2.4) and (4.1), it is possible to choose some $\kappa_{_M} \in (\eta/k, 1)$ such that γ satisfies

$$\frac{k\beta(\kappa_M)}{\gamma} > 1. \tag{4.19}$$

Then, for all $\boldsymbol{\kappa}'$ satisfying $|\boldsymbol{\kappa}'| < \kappa_{_M}$, the coupling coefficients between $P^{aa,\varepsilon}$ and $P^{ab,\varepsilon}$ vanish because

$$\frac{k|\vec{\boldsymbol{\kappa}} - \vec{\boldsymbol{\kappa}}^{'-}|}{\gamma} \geq \frac{k(\beta(\boldsymbol{\kappa}) + \beta(\boldsymbol{\kappa}'))}{\gamma} \geq \frac{k\beta(\kappa_M)}{\gamma} > 1,$$

and $\widetilde{\mathcal{R}}\big(k(\vec{\boldsymbol{\kappa}} - \vec{\boldsymbol{\kappa}}^{'-})/\gamma\big)$ is negligible. This implies the asymptotic decoupling of a^ε and b^ε, and due to the homogeneous boundary condition $b^\varepsilon(\boldsymbol{\kappa}, L) = 0$, we conclude that we can neglect the backward going waves in the limit $\varepsilon \to 0$.

The forward going amplitudes interact with each other, because the coupling coefficients of the entries $P^{aa,\varepsilon}$ of the propagator are large for at least a subset of transverse wave vectors satisfying $|\boldsymbol{\kappa}|, |\boldsymbol{\kappa}'| \leq \kappa_{_M}$ and

$$|\boldsymbol{\kappa} - \boldsymbol{\kappa}'|,\ |\beta(\boldsymbol{\kappa}) - \beta(\boldsymbol{\kappa}')| < \frac{\gamma}{k}.$$

Due to this coupling there is diffusion of energy from the waves emitted by the source with $|\boldsymbol{\kappa}| < \eta/k$, to waves at larger values of $|\boldsymbol{\kappa}|$. This is why we take $\kappa_{_M} > \eta/k$ in (4.19). By assuming that $a^\varepsilon(\boldsymbol{\kappa}, z)$ are supported at $|\boldsymbol{\kappa}| \leq \kappa_{_M} < 1$ we essentially restrict z by Z_M, so that the energy does not diffuse to waves with $|\boldsymbol{\kappa}| > \kappa_{_M}$ for $z \leq Z_{_M}$. Physically, the wave vectors $(\boldsymbol{\kappa}, \beta(\boldsymbol{\kappa}))$ of the forward going waves remain within a cone with opening angle smaller than 180 degrees.

We will see that the evolution of the $\boldsymbol{\kappa}$-distribution of the wave energy is described by a radiative transfer equation, which means that the wave energy undergoes a random walk (or diffusion). We can estimate from Eq. (4.30) that the diffusion coefficient is of the order $\alpha^2\gamma$, so the $\boldsymbol{\kappa}$-distribution of the wave energy reaches $\kappa_{_M}$ after a propagation distance of the order of $Z_{_M}$, such that $\alpha^2\gamma Z_{_M} = \kappa_{_M}^2$. In dimensional units, this means $\alpha^2 Z_{_M}/\ell = \kappa_{_M}^2$. Since $\alpha^2 L/\ell = (\alpha^2 L/\lambda)(\lambda/\ell) < 1$ by (2.4), it is possible to choose $Z_{_M} \sim L$ and a suitable $\kappa_{_M} < 1$.

The evanescent waves can only couple with the propagating waves with wave vectors of magnitude close to 1. Thus, as long as the energy of the wave is supported at $|\boldsymbol{\kappa}| < \kappa_{_M}$, assumption (4.19) implies that the evanescent waves do not get excited.

4.3.3. Markov limit of the forward going mode amplitudes. We just explained that in the limit $\varepsilon \to 0$ we can can neglect all the backward going waves and the evanescent ones. It remains to describe the limit of the forward going wave amplitudes $a^\varepsilon(\boldsymbol{\kappa}, z)$ which satisfy the initial value problem

$$\partial_z a^\varepsilon(\boldsymbol{\kappa}, z) = \frac{ik\alpha}{2\gamma^d \varepsilon^{\frac{1}{2}}} \int_{|\boldsymbol{\kappa}'|<1} \frac{\mathrm{d}(k\boldsymbol{\kappa}')}{(2\pi)^d}\, \widehat{\nu}\Big(\frac{k(\boldsymbol{\kappa} - \boldsymbol{\kappa}')}{\gamma}, \frac{\gamma z}{\varepsilon}\Big) \times \Gamma^{aa}\Big(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \frac{z}{\varepsilon}\Big) a^\varepsilon(\boldsymbol{\kappa}', z), \tag{4.20}$$

for $z > 0$, and the initial condition $a^\varepsilon(\boldsymbol{\kappa}, 0) = a_o(\boldsymbol{\kappa})$. These equations conserve energy, meaning that for all $\varepsilon > 0$ and all $z \geq 0$,

$$\int_{|\boldsymbol{\kappa}|<1} \frac{\mathrm{d}(k\boldsymbol{\kappa})}{(2\pi)^d} |a^\varepsilon(\boldsymbol{\kappa}, z)|^2 = \int_{|\boldsymbol{\kappa}|<1} \frac{\mathrm{d}(k\boldsymbol{\kappa})}{(2\pi)^d} |a_o(\boldsymbol{\kappa})|^2 . \tag{4.21}$$

The details of the $\varepsilon \to 0$ limit of $a^\varepsilon(\boldsymbol{\kappa}, z)$ are in appendix A. In particular, we explain there that the process

$$\boldsymbol{X}^\varepsilon(z) = \begin{pmatrix} \mathrm{Re}\big(a^\varepsilon(\boldsymbol{\kappa}, z)\big) \\ \mathrm{Im}\big(a^\varepsilon(\boldsymbol{\kappa}, z)\big) \end{pmatrix}_{\boldsymbol{\kappa}\in\mathcal{O}} \quad \text{for } \mathcal{O} = \{\boldsymbol{\kappa} \in \mathbb{R}^d,\, |\boldsymbol{\kappa}| < 1\}, \tag{4.22}$$

converges weakly in $\mathcal{C}([0,L], \mathcal{D}')$ to a Markov process $\boldsymbol{X}(z)$, where $\mathcal{D}'$ is the space of distributions, dual to the space $\mathcal{D}(\mathcal{O}, \mathbb{R}^2)$ of infinitely differentiable vector valued functions in $\mathbb{R}^2$, with compact support. The generator of $\boldsymbol{X}(z)$ is given in appendix A, and we denote henceforth the limit amplitudes by $(a(\boldsymbol{\kappa}, z))_{\boldsymbol{\kappa}\in\mathcal{O}} = X_1(z) + iX_2(z)$. Their first and second moments are described in the next two sections.

4.4. The coherent field. The coherent wave field is

$$\mathbb{E}\big[u^\varepsilon\big(\frac{\boldsymbol{x}}{\varepsilon}, z\big)\big] \approx \int_{|\boldsymbol{\kappa}|<1} \frac{\mathrm{d}(k\boldsymbol{\kappa})}{(2\pi)^d} \frac{\mathbb{E}[a(\boldsymbol{\kappa}, z)]}{\beta^{\frac{1}{2}}(\boldsymbol{\kappa})} e^{i\frac{k}{\varepsilon}\vec{\boldsymbol{\kappa}}\cdot\vec{\boldsymbol{x}}},$$

where we replaced $\mathbb{E}[a^\varepsilon(\boldsymbol{\kappa}, z)]$ by its $\varepsilon \to 0$ limit $\mathbb{E}[a(\boldsymbol{\kappa}, z)]$. As explained in appendix A, the mean field $\mathcal{A}(\boldsymbol{\kappa}, z) = \mathbb{E}[a(\boldsymbol{\kappa}, z)]$ satisfies the initial value problem

$$\partial_z \mathcal{A}(\boldsymbol{\kappa}, z) = \mathrm{Q}(\boldsymbol{\kappa})\mathcal{A}(\boldsymbol{\kappa}, z), \quad z > 0, \tag{4.23}$$

with initial condition $\mathcal{A}(\boldsymbol{\kappa}, 0) = a_o(\boldsymbol{\kappa})$, and $\mathrm{Q}(\boldsymbol{\kappa})$ given by

$$\begin{aligned} \mathrm{Q}(\boldsymbol{\kappa}) = &-\frac{k^2\alpha^2}{4\gamma^{d+1}} \int_{|\boldsymbol{\kappa}'|<1} \frac{\mathrm{d}(k\boldsymbol{\kappa}')}{(2\pi)^d} \frac{1}{\beta(\boldsymbol{\kappa})\beta(\boldsymbol{\kappa}')} \\ &\times \int_0^\infty \mathrm{d}\zeta \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{r}\, \mathcal{R}(\boldsymbol{r}, \zeta) e^{-i\frac{k}{\gamma}\big(\boldsymbol{\kappa}-\boldsymbol{\kappa}', \beta(\boldsymbol{\kappa})-\beta(\boldsymbol{\kappa}')\big)\cdot(\boldsymbol{r},\zeta)}. \end{aligned} \tag{4.24}$$

This is the same as (3.4) in our scaling.

The solution of (4.23) is

$$\mathcal{A}(\boldsymbol{\kappa}, z) = \exp\big[\mathrm{Q}(\boldsymbol{\kappa})z\big] a_o(\boldsymbol{\kappa}), \tag{4.25}$$

so as stated in section 3, the random medium effects do not average out. The mean amplitudes are not the same as the amplitudes in the homogeneous medium at $z > 0$, and they decay with z on the $\boldsymbol{\kappa}$ dependent scales $\mathcal{S}(\boldsymbol{\kappa}) = -1/\mathrm{Re}[\mathrm{Q}(\boldsymbol{\kappa})]$, the scattering mean free paths. The real part of $\mathrm{Q}(\boldsymbol{\kappa})$, which is non-positive, is an effective diffusion term in (4.23), which removes energy from the mean field and gives it to the incoherent fluctuations. This is due to the randomization or loss of coherence of the waves. The imaginary part of $Q(\boldsymbol{\kappa})$ is an effective dispersion term, which does not remove energy from the mean field and ensures causality[3].

4.5. The one-way radiative transfer equations. The mean intensity in the direction of $\boldsymbol{\kappa}$ is

$$\mathcal{I}(\boldsymbol{\kappa}, z) = \lim_{\varepsilon\to 0} \mathbb{E}\left[|a^\varepsilon(\boldsymbol{\kappa}, z)|^2\right], \tag{4.26}$$

and it evolves in $z > 0$ as modeled by equation

$$\partial_z \mathcal{I}(\boldsymbol{\kappa}, z) = \int_{|\boldsymbol{\kappa}'|<1} \frac{\mathrm{d}(k\boldsymbol{\kappa}')}{(2\pi)^d}\, \mathcal{Q}(\boldsymbol{\kappa}, \boldsymbol{\kappa}')\big[\mathcal{I}(\boldsymbol{\kappa}', z) - \mathcal{I}(\boldsymbol{\kappa}, z)\big], \tag{4.27}$$

[3]If we write the coherent wave fields in the time domain, using the inverse Fourier transform with respect to the frequency ω, we obtain a causal result.

with initial condition $\mathcal{I}(\boldsymbol{\kappa},0)=|a_o(\boldsymbol{\kappa})|^2$ (see Appendix A). The differential scattering cross section

$$\mathcal{Q}(\boldsymbol{\kappa},\boldsymbol{\kappa}')=\frac{k^2\alpha^2}{4\gamma^{d+1}\beta(\boldsymbol{\kappa})\beta(\boldsymbol{\kappa}')}\widetilde{\mathfrak{R}}\Big(\frac{k}{\gamma}\big(\boldsymbol{\kappa}-\boldsymbol{\kappa}',\beta(\boldsymbol{\kappa})-\beta(\boldsymbol{\kappa}')\big)\Big)$$

is the same as (3.8) in our scaling, and from (4.24) we see that $-2\mathrm{Re}[\mathrm{Q}(\boldsymbol{\kappa})]$ equals the total scattering cross section

$$-2\mathrm{Re}[\mathrm{Q}(\boldsymbol{\kappa})]=\int_{|\boldsymbol{\kappa}'|<1}\frac{\mathrm{d}(k\boldsymbol{\kappa}')}{(2\pi)^d}\,\mathcal{Q}(\boldsymbol{\kappa},\boldsymbol{\kappa}'). \tag{4.28}$$

We also note that the intensities satisfy the conservation identity

$$\int_{|\boldsymbol{\kappa}|<1}\frac{\mathrm{d}(k\boldsymbol{\kappa})}{(2\pi)^d}\mathcal{I}(\boldsymbol{\kappa},z)=\int_{|\boldsymbol{\kappa}|<1}\frac{\mathrm{d}(k\boldsymbol{\kappa})}{(2\pi)^d}|a_o(\boldsymbol{\kappa})|^2,\quad \text{for all } z>0,$$

which is consistent with (4.21).

Using the generator of the Markov limit process $\mathbf{X}(z)$ given in appendix A, we can also calculate the $\varepsilon\to 0$ limit of the second moments $\mathbb{E}\left[a^\varepsilon(\boldsymbol{\kappa},z)\overline{a^\varepsilon(\boldsymbol{\kappa}',z)}\right]$ of the mode amplitudes. We obtain that when $\boldsymbol{\kappa}\neq\boldsymbol{\kappa}'$,

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[a^\varepsilon(\boldsymbol{\kappa},z)\overline{a^\varepsilon(\boldsymbol{\kappa}',z)}\right]=\lim_{\varepsilon\to 0}\mathbb{E}[a^\varepsilon(\boldsymbol{\kappa},z)]\overline{\mathbb{E}[a^\varepsilon(\boldsymbol{\kappa}',z)]},$$

meaning that the waves traveling in different directions are asymptotically decorrelated [4]. This is because these waves see different regions of the random medium. It is only when the waves propagate in similar directions i.e., $|\boldsymbol{\kappa}'-\boldsymbol{\kappa}|=O(\varepsilon)$, that the mode amplitudes are correlated, so we define the energy density (Wigner transform) as

$$\begin{aligned}\mathcal{W}(\boldsymbol{\kappa},\boldsymbol{x},z)=\lim_{\varepsilon\to 0}\int\frac{\mathrm{d}(k\boldsymbol{q})}{(2\pi)^d}\,\exp\Big[ik\boldsymbol{q}\cdot\big(\nabla\beta(\boldsymbol{\kappa})z+\boldsymbol{x}\big)\Big]\\ \times\mathbb{E}\left[a^\varepsilon\big(\boldsymbol{\kappa}+\frac{\varepsilon\boldsymbol{q}}{2},z\big)\overline{a^\varepsilon\big(\boldsymbol{\kappa}-\frac{\varepsilon\boldsymbol{q}}{2},z\big)}\right].\end{aligned} \tag{4.29}$$

It satisfies the transport equation

$$\begin{aligned}&\partial_z\mathcal{W}(\boldsymbol{\kappa},\boldsymbol{x},z)-\nabla\beta(\boldsymbol{\kappa})\cdot\nabla_{\boldsymbol{x}}\mathcal{W}(\boldsymbol{\kappa},\boldsymbol{x},z)\\ &=\int_{|\boldsymbol{\kappa}'|<1}\frac{\mathrm{d}(k\boldsymbol{\kappa}')}{(2\pi)^d}\,\mathcal{Q}(\boldsymbol{\kappa},\boldsymbol{\kappa}')\big[\mathcal{W}(\boldsymbol{\kappa}',\boldsymbol{x},z)-\mathcal{W}(\boldsymbol{\kappa},\boldsymbol{x},z)\big],\end{aligned} \tag{4.30}$$

for $z>0$, as stated in section 3. When the initial condition $a_o(\boldsymbol{\kappa})$ is smooth in $\boldsymbol{\kappa}$, we have from (4.29) that

$$\mathcal{W}(\boldsymbol{\kappa},\boldsymbol{x},0)=\delta(\boldsymbol{x})|a_o(\boldsymbol{\kappa})|^2,$$

and therefore at $z>0$

$$\mathcal{W}(\boldsymbol{\kappa},\boldsymbol{x},z)=\delta\big(\boldsymbol{x}+\nabla\beta(\boldsymbol{\kappa})z\big)\mathcal{I}(\boldsymbol{\kappa},z).$$

This shows that the energy is transported on the characteristic

$$\boldsymbol{x}=-\nabla\beta(\boldsymbol{\kappa})z=\frac{\boldsymbol{\kappa}}{\beta(\boldsymbol{\kappa})}z.$$

[4] It can also be shown that the waves decorrelate over frequency offsets larger than ε. Thus, one can study the energy density resolved over both time and space i.e., the space-time Wigner transform.

5. The high-frequency limit. In the high-frequency limit $\gamma \to 0$ the transport equations simplify. We quantify the scattering mean free paths in this limit, and show how to derive the diffusion approximation and paraxial model from the transport equations (4.30).

5.1. Quantification of scattering mean free paths. If we expand in powers of γ the right hand side of (4.28), we obtain the following expression of the scattering mean free paths

$$\mathcal{S}(\boldsymbol{\kappa}) = -\frac{1}{\mathrm{Re}\big[\mathrm{Q}(\boldsymbol{\kappa})\big]} = \frac{8\gamma\beta^2(\boldsymbol{\kappa})}{k^2\alpha^2\int_{-\infty}^{\infty}\mathrm{d}\zeta\,\mathcal{R}\Big(\frac{\boldsymbol{\kappa}\zeta}{\beta(\boldsymbol{\kappa})},\zeta\Big)} + O(\gamma^2).$$

They are of order γ and decrease as the negative power of 2 with the frequency $\omega = kc_o$, meaning that higher frequency waves lose coherence faster. We also expect that $\mathcal{S}(\boldsymbol{\kappa})$ decrease monotonically with $|\boldsymbol{\kappa}|$, because a plane wave mode with wavevector $k(\boldsymbol{\kappa},\beta(\boldsymbol{\kappa}))$ travels the distance $z/\beta(\boldsymbol{\kappa})$ in the random medium when it propagates up to z. The closer $|\boldsymbol{\kappa}|$ is to one, the longer the distance and thus, the faster the loss of coherence quantified by the scale $\mathcal{S}(\boldsymbol{\kappa})$. The monotone dependence of $\mathcal{S}(\boldsymbol{\kappa})$ on $|\boldsymbol{\kappa}|$ can be seen explicitly in statistically isotropic media, where $\mathcal{R}(\vec{\boldsymbol{x}}) = \mathcal{R}_{\rm iso}(|\vec{\boldsymbol{x}}|)$, and

$$\mathcal{R}\Big(\frac{\boldsymbol{\kappa}\zeta}{\beta(\boldsymbol{\kappa})},\zeta\Big) = \mathcal{R}_{\rm iso}\left(\sqrt{\frac{|\boldsymbol{\kappa}|^2\zeta^2}{\beta^2(\boldsymbol{\kappa})} + \zeta^2}\right) = \mathcal{R}_{\rm iso}\Big(\frac{|\zeta|}{\beta(\boldsymbol{\kappa})}\Big).$$

Then

$$\mathcal{S}(\boldsymbol{\kappa}) = \frac{4\gamma\beta(\boldsymbol{\kappa})}{k^2\alpha^2\int_0^{\infty}\mathrm{d}\zeta\,\mathcal{R}_{\rm iso}(\zeta)} + O(\gamma^2),$$

and the decay with $|\boldsymbol{\kappa}|$ is captured by $\beta(\boldsymbol{\kappa}) = \sqrt{1-|\boldsymbol{\kappa}|^2}$.

5.2. The diffusion approximation. The mean mode intensities $\mathcal{I}(\boldsymbol{\kappa},z)$ defined in (4.26) satisfy (4.27), with initial condition at $z=0$ derived from (4.7):

$$\mathcal{I}(\boldsymbol{\kappa},0) = \frac{1}{4k^2\beta(\boldsymbol{\kappa})\eta^{2d}}\Big|\widehat{F}\Big(\frac{k\boldsymbol{\kappa}}{\eta}\Big)\Big|^2.$$

This is independent of γ and for fixed η.

The diffusion model is obtained by expanding Eq. (4.27) in powers of γ. We obtain that

$$\partial_z\mathcal{I}(\boldsymbol{\kappa},z) \approx \gamma\Big[\sum_{j,l=1}^{d}A_{jl}(\boldsymbol{\kappa})\partial^2_{\kappa_j\kappa_l} + \gamma\sum_{j=1}^{d}B_j(\boldsymbol{\kappa})\partial_{\kappa_j}\Big]\mathcal{I}(\boldsymbol{\kappa},z), \tag{5.1}$$

where the approximation means that we neglect higher powers in γ, and the diffusion and drift coefficients are independent of k and γ:

$$A_{jl}(\boldsymbol{\kappa}) = -\frac{\alpha^2}{8\beta(\boldsymbol{\kappa})^2}\int_{-\infty}^{\infty}\mathrm{d}\zeta\,\partial^2_{r_jr_l}\mathcal{R}\Big(\frac{\boldsymbol{\kappa}\zeta}{\beta(\boldsymbol{\kappa})},\zeta\Big), \qquad j,l=1,\ldots,d,$$

and

$$B_j(\boldsymbol{\kappa}) = \sum_{l,m=1}^{d}\frac{\alpha^2\partial^2_{\kappa_l\kappa_M}\beta(\boldsymbol{\kappa})}{8\beta(\boldsymbol{\kappa})^2}\int_{-\infty}^{\infty}\mathrm{d}\zeta\,\zeta\partial^3_{r_jr_lr_m}\mathcal{R}\Big(\frac{\boldsymbol{\kappa}\zeta}{\beta(\boldsymbol{\kappa})},\zeta\Big)$$
$$-\sum_{l=1}^{d}\frac{\alpha^2\kappa_l}{4\beta(\boldsymbol{\kappa})^4}\int_{-\infty}^{\infty}\mathrm{d}\zeta\,\partial^2_{r_jr_l}\mathcal{R}\Big(\frac{\boldsymbol{\kappa}\zeta}{\beta(\boldsymbol{\kappa})},\zeta\Big), \qquad j=1,\ldots,d.$$

Note that the diffusion is the dominant term in (5.1).

5.3. The paraxial approximation. The paraxial (beam-like) propagation model is for a large diameter X of the support of the source with respect to the wavelength, so that $\eta \to 0$. The result depends on the order in which we take the limits $\eta \to 0$ and $\gamma \to 0$, as we now explain.

In regimes with $\lambda \ll \ell = X$, where $\eta = \gamma$, the rescaled intensity

$$\mathcal{I}_{\rm res}(\boldsymbol{\kappa}, z) = \gamma^{2d} \mathcal{I}\big(\gamma \boldsymbol{\kappa}, \gamma z\big)$$

satisfies in the limit $\gamma \to 0$ the equation

$$\partial_z \mathcal{I}_{\rm res} = \frac{k^2 \alpha^2}{4} \int_{\mathbb{R}^d} \frac{{\rm d}(k\boldsymbol{\kappa}')}{(2\pi)^d} \widetilde{\mathcal{R}}\big(k(\boldsymbol{\kappa} - \boldsymbol{\kappa}'), 0\big) \big[\mathcal{I}_{\rm res}(\boldsymbol{\kappa}') - \mathcal{I}_{\rm res}(\boldsymbol{\kappa})\big], \tag{5.2}$$

with initial condition $\mathcal{I}_{\rm res}(\boldsymbol{\kappa}, 0) = \big|\widehat{F}(k\boldsymbol{\kappa})\big|^2 / [4k^2 \beta(\boldsymbol{\kappa})]$. This is the transport equation for the random paraxial wave equation, as explained in subsection 3.3.

In regimes with $\lambda \ll \ell \ll X$, analyzed with the sequence of limits $\gamma \to 0$, followed by $\eta \to 0$, the rescaled intensity

$$\mathcal{I}_{\rm res}(\boldsymbol{\kappa}, z) = \eta^{2d} \mathcal{I}\Big(\eta \boldsymbol{\kappa}, \frac{\eta^2}{\gamma} z\Big)$$

satisfies the diffusion equation

$$\partial_z \mathcal{I}_{\rm res} = \sum_{j,l=1}^{d} D_{{\rm res},jl} \partial^2_{\kappa_j \kappa_l} \mathcal{I}_{\rm res}, \tag{5.3}$$

with initial condition $\mathcal{I}_{\rm res}(\boldsymbol{\kappa}, 0) = \big|\widehat{F}(k\boldsymbol{\kappa})\big|^2 / [4k^2 \beta(\boldsymbol{\kappa})]$ and diffusion tensor $D_{{\rm res},jl}$ given by

$$D_{{\rm res},jl} = -\frac{\alpha^2}{8} \int_{-\infty}^{\infty} {\rm d}\zeta \, \partial^2_{r_j r_l} \mathcal{R}(\mathbf{0}, \zeta) = \lim_{|\boldsymbol{\kappa}| \to 0} A_{jl}(\boldsymbol{\kappa}),$$

for $j, l = 1, \dots, d$. This result was derived in [7, 22, 9, 10] starting from the paraxial wave equation. We recovered it here because in the regime with $\lambda \ll \ell \ll X$ we have a narrow cone beam propagating through a random medium.

Note that equation (5.3) can also be derived formally from the radiative transfer equation (3.17). First, one considers that scattering is sharply peaked in the forward scattering direction, so that it is possible to take the Fokker-Planck approximation, that is to say, the right-hand side of (3.17) can be approximated by a diffusion operator in $\vec{\mathfrak{K}}$ [24, 18]. Second, one considers that the source emission is sharply peaked and that the propagation distance is short enough so that the wave remains in the form of a narrow cone beam.

6. Summary. The one-way radiative transfer equation describes the evolution of the intensity of the waves resolved over directions, the Wigner transform, in forward-peaked scattering regimes. We derived it using multiscale analysis and probabilistic limits, starting from the wave equation in random media. The scattering regime with small random fluctuations of the wave speed and long distances of propagation over which cumulative scattering becomes significant leads to waves propagating forward in a wide angular cone. It bridges between two known regimes: The first is the radiative transfer regime where waves propagate in all directions and the Wigner transform satisfies a boundary value problem. The second is the paraxial regime, where waves propagate in a narrow angle cone. We established this bridge by connecting the one-way radiative transfer equation to the equations for the Wigner transform in these two regimes.

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Appendix A. The Markov limit. Let $\mathcal{O}$ be an open set in $\mathbb{R}^d$ and $\mathcal{D}(\mathcal{O},\mathbb{R}^2)$ the space of infinitely differentiable functions with compact support. We consider the process $\boldsymbol{X}^\varepsilon$ in the space $\mathcal{C}([0,L],\mathcal{D}'(\mathcal{O},\mathbb{R}^2))$ of continuous functions of z. It is the solution of

$$\frac{\mathrm{d}\boldsymbol{X}^\varepsilon}{\mathrm{d}z} = \frac{1}{\sqrt{\varepsilon}}\mathcal{F}\Big(\frac{z}{\varepsilon},\frac{z}{\varepsilon}\Big)\boldsymbol{X}^\varepsilon, \tag{A.1}$$

where $\mathcal{F}(\zeta,\zeta')$ is a random linear operator from $\mathcal{D}'$ to $\mathcal{D}'$. Here $\mathcal{D}'$ denotes the space of distributions, dual to $\mathcal{D}(\mathcal{O},\mathbb{R}^2)$. We assume that the mapping $\zeta \to \mathcal{F}(\zeta,\zeta')$ is stationary and possesses strong ergodic properties, and that $\mathcal{F}(\zeta,\zeta')$ has mean zero. Moreover, the mapping $\zeta' \to \mathcal{F}(\zeta,\zeta')$ is periodic.

We are interested in particular in equation (4.20), that can be put into the form (A.1) if we define the process $\boldsymbol{X}^\varepsilon$ as (4.22) and the operator $\mathcal{F}(\zeta,\zeta')$ as

$$\begin{aligned}\langle \mathcal{F}(\zeta,\zeta')\boldsymbol{X},\boldsymbol{\phi}\rangle &= \sum_{j=1}^2 \int_{\mathcal{O}} \frac{\mathrm{d}(k\boldsymbol{\kappa})}{(2\pi)^d}[\mathcal{F}(\zeta,\zeta')\boldsymbol{X}]_j(\boldsymbol{\kappa})\phi_j(\boldsymbol{\kappa}) \\ &= \int_{\mathcal{O}} \frac{\mathrm{d}(k\boldsymbol{\kappa})}{(2\pi)^d}\boldsymbol{\phi}(\boldsymbol{\kappa})\cdot\int_{\mathcal{O}} \frac{\mathrm{d}(k\boldsymbol{\kappa}')}{(2\pi)^d}\boldsymbol{\mathcal{F}}(\boldsymbol{\kappa},\boldsymbol{\kappa}',\zeta,\zeta')\boldsymbol{X}(\boldsymbol{\kappa}'),\end{aligned} \tag{A.2}$$

for $\boldsymbol{\phi} \in \mathcal{D}(\mathcal{O},\mathbb{R}^2)$ with components ϕ_j and $\boldsymbol{X} \in \mathcal{D}'(\mathcal{O},\mathbb{R}^2)$ with components X_j. The kernel matrix $\boldsymbol{\mathcal{F}}(\boldsymbol{\kappa},\boldsymbol{\kappa}',\zeta,\zeta')$ is given by

$$\boldsymbol{\mathcal{F}} = \begin{pmatrix} \mathcal{F}^r & -\mathcal{F}^i \\ \mathcal{F}^i & \mathcal{F}^r \end{pmatrix}, \tag{A.3}$$

in terms of

$$\mathcal{F}^{\mathrm{r}}(\boldsymbol{\kappa},\boldsymbol{\kappa}',\zeta,\zeta') = \mathrm{Re}\Big[\frac{ik\alpha}{2\gamma^d}\widehat{\nu}\Big(\frac{k(\boldsymbol{\kappa}-\boldsymbol{\kappa}')}{\gamma},\gamma\zeta\Big)\Gamma^{aa}\big(\boldsymbol{\kappa},\boldsymbol{\kappa}',\zeta'\big)\Big], \tag{A.4}$$

$$\mathcal{F}^{\mathrm{i}}(\boldsymbol{\kappa},\boldsymbol{\kappa}',\zeta,\zeta') = \mathrm{Im}\Big[\frac{ik\alpha}{2\gamma^d}\widehat{\nu}\Big(\frac{k(\boldsymbol{\kappa}-\boldsymbol{\kappa}')}{\gamma},\gamma\zeta\Big)\Gamma^{aa}\big(\boldsymbol{\kappa},\boldsymbol{\kappa}',\zeta'\big)\Big], \tag{A.5}$$

where we recall from (4.16) the expression of $\Gamma^{aa}(\boldsymbol{\kappa},\boldsymbol{\kappa}',\zeta')$. The adjoint operator $\mathcal{F}^*(\zeta,\zeta')$ is defined by

$$\langle \mathcal{F}(\zeta,\zeta')\boldsymbol{X},\boldsymbol{\phi}\rangle = \langle \boldsymbol{X},\mathcal{F}^*(\zeta,\zeta')\boldsymbol{\phi}\rangle$$

for $\boldsymbol{\phi} \in \mathcal{D}(\mathcal{O},\mathbb{R}^2)$ and $\boldsymbol{X} \in \mathcal{D}'(\mathcal{O},\mathbb{R}^2)$, and has matrix kernel $\boldsymbol{\mathcal{F}}^*(\boldsymbol{\kappa},\boldsymbol{\kappa}',\zeta,\zeta') = \boldsymbol{\mathcal{F}}(\boldsymbol{\kappa}',\boldsymbol{\kappa},\zeta,\zeta')^T$, where the superscript T stands for transpose.

To obtain the Markov limit we use the results in [23] (the interested reader may first read [8, Chap. 6] for a self-contained introduction to such limit theorems). They give that $\boldsymbol{X}^\varepsilon(z)$ converges weakly in $\mathcal{C}([0,L],\mathcal{D}')$ to $\boldsymbol{X}(z)$, the solution of a martingale problem with generator $\mathcal{L}$ defined by

$$\begin{aligned}
&\mathcal{L}f(\langle \boldsymbol{X},\boldsymbol{\phi}\rangle) = \\
&\int_0^\infty \mathrm{d}\zeta \lim_{Z\to\infty}\frac{1}{Z}\int_0^Z \mathrm{d}h\,\mathbb{E}\big[\,\langle \boldsymbol{X},\mathcal{F}^*(0,h)\boldsymbol{\phi}\rangle\,\langle \boldsymbol{X},\mathcal{F}^*(\zeta,\zeta+h)\boldsymbol{\phi}\rangle\,\big] \\
&\quad\times f''(\langle \boldsymbol{X},\boldsymbol{\phi}\rangle) \\
&+\int_0^\infty \mathrm{d}\zeta \lim_{Z\to\infty}\frac{1}{Z}\int_0^Z \mathrm{d}h\,\mathbb{E}\big[\,\langle \boldsymbol{X},\mathcal{F}^*(0,h)\mathcal{F}^*(\zeta,\zeta+h)\boldsymbol{\phi}\rangle\,\big] \\
&\quad\times f'(\langle \boldsymbol{X},\boldsymbol{\phi}\rangle),
\end{aligned}\tag{A.6}$$

for any $\boldsymbol{X}\in\mathcal{D}'(\mathcal{O},\mathbb{R}^2)$, $\boldsymbol{\phi}\in\mathcal{D}(\mathcal{O},\mathbb{R}^2)$, and smooth $f:\mathbb{R}\to\mathbb{R}$. This means that, for any $\boldsymbol{\phi}\in\mathcal{D}(\mathcal{O},\mathbb{R}^2)$ and smooth function $f:\mathbb{R}\to\mathbb{R}$, the real-valued process

$$f\big(\,\langle \boldsymbol{X}(z),\boldsymbol{\phi}\rangle\,\big) - \int_0^z \mathrm{d}z'\,\mathcal{L}f\big(\,\langle \boldsymbol{X}(z'),\boldsymbol{\phi}\rangle\,\big)$$

is a martingale. More generally, if $n\in\mathbb{N}$, $\boldsymbol{\phi}^{(1)},\dots,\boldsymbol{\phi}^{(n)}\in\mathcal{D}(\mathcal{O},\mathbb{R}^2)$, and $f:\mathbb{R}^n\to\mathbb{R}$ is a smooth function, then

$$\begin{aligned}
&f\big(\,\Big\langle \boldsymbol{X}(z),\boldsymbol{\phi}^{(1)}\Big\rangle,\dots,\Big\langle \boldsymbol{X}(z),\boldsymbol{\phi}^{(n)}\Big\rangle\,\big) \\
&-\int_0^z \mathrm{d}z'\,\mathcal{L}^{(n)}f\big(\,\Big\langle \boldsymbol{X}(z'),\boldsymbol{\phi}^{(1)}\Big\rangle,\dots,\Big\langle \boldsymbol{X}(z'),\boldsymbol{\phi}^{(n)}\Big\rangle\,\big)
\end{aligned}\tag{A.7}$$

is a martingale, where

$$\begin{aligned}
&\mathcal{L}^{(n)}f\big(\,\Big\langle \boldsymbol{X},\boldsymbol{\phi}^{(1)}\Big\rangle,\dots,\Big\langle \boldsymbol{X},\boldsymbol{\phi}^{(n)}\Big\rangle\,\big) = \\
&\sum_{j,l=1}^n\int_0^\infty \mathrm{d}\zeta \lim_{Z\to\infty}\frac{1}{Z}\int_0^Z \mathrm{d}h\,\mathbb{E}\big[\,\Big\langle \boldsymbol{X},\mathcal{F}^*(0,h)\boldsymbol{\phi}^{(j)}\Big\rangle \\
&\quad\times\Big\langle \boldsymbol{X},\mathcal{F}^*(\zeta,\zeta+h)\boldsymbol{\phi}^{(l)}\Big\rangle\,\big]\,\partial^2_{jl}f(\Big\langle \boldsymbol{X},\boldsymbol{\phi}^{(1)}\Big\rangle,\dots,\Big\langle \boldsymbol{X},\boldsymbol{\phi}^{(n)}\Big\rangle\,) \\
&+\sum_{j=1}^n\int_0^\infty \mathrm{d}\zeta \lim_{Z\to\infty}\frac{1}{Z}\int_0^Z \mathrm{d}h\,\mathbb{E}\big[\,\Big\langle \boldsymbol{X},\mathcal{F}^*(0,h)\mathcal{F}^*(\zeta,\zeta+h)\boldsymbol{\phi}^{(j)}\Big\rangle\,\big] \\
&\quad\times\partial_j f(\Big\langle \boldsymbol{X},\boldsymbol{\phi}^{(1)}\Big\rangle,\dots,\Big\langle \boldsymbol{X},\boldsymbol{\phi}^{(n)}\Big\rangle\,).
\end{aligned}\tag{A.8}$$

To calculate the first moment of the limit process $\boldsymbol{X}(z)$, let $n=1$ and $f(y)=y$ in (A.7)-(A.8). We find that

$$\frac{\mathrm{d}\,\mathbb{E}\big[\,\langle \boldsymbol{X}(z),\boldsymbol{\phi}\rangle\,\big]}{\mathrm{d}z} = \mathbb{E}\big[\,\langle \boldsymbol{X}(z),\mathcal{H}^*\boldsymbol{\phi}\rangle\,\big],$$

where

$$\mathcal{H}^* = \int_0^\infty \mathrm{d}\zeta \lim_{Z\to\infty}\frac{1}{Z}\int_0^Z \mathrm{d}h\,\mathbb{E}\big[\mathcal{F}^*(0,h)\mathcal{F}^*(\zeta,\zeta+h)\big].$$

This shows that

$$\boldsymbol{\mathcal{X}}(z) = \mathbb{E}\big[\boldsymbol{X}(z)\big]$$

satisfies a closed system of ordinary differential equations

$$\frac{\mathrm{d}\left\langle \boldsymbol{\mathcal{X}}(z),\boldsymbol{\phi}\right\rangle}{\mathrm{d}z}=\left\langle \boldsymbol{\mathcal{X}}(z),\mathcal{H}^*\boldsymbol{\phi}\right\rangle,$$

or, equivalently in $\mathcal{D}'$,

$$\frac{\mathrm{d}\boldsymbol{\mathcal{X}}(z)}{\mathrm{d}z}=\mathcal{H}\boldsymbol{\mathcal{X}}(z), \tag{A.9}$$

where $\mathcal{H}$ is the adjoint of $\mathcal{H}^*$. The kernel matrix of $\mathcal{H}$ is $\boldsymbol{\mathcal{H}}(\boldsymbol{\kappa}',\boldsymbol{\kappa})=\boldsymbol{\mathcal{H}}^*(\boldsymbol{\kappa},\boldsymbol{\kappa}')^T$. Recalling from (A.2)-(A.5) the expression of the kernel $\boldsymbol{\mathcal{F}}(\boldsymbol{\kappa}',\boldsymbol{\kappa},\zeta,\zeta')^T$ of $\mathcal{F}^*(\zeta,\zeta')$, we obtain that the matrix kernel $\boldsymbol{\mathcal{H}}^*(\boldsymbol{\kappa}',\boldsymbol{\kappa})$ of $\mathcal{H}^*$ is

$$\begin{aligned}\mathcal{H}^*_{jl}(\boldsymbol{\kappa},\boldsymbol{\kappa}')=\sum_{q=1}^{2}\int_{\mathcal{O}}\frac{\mathrm{d}(k\boldsymbol{\kappa}'')}{(2\pi)^d}\int_0^\infty \mathrm{d}\zeta\lim_{Z\to\infty}\frac{1}{Z}\int_0^Z\mathrm{d}h\\ \times\mathbb{E}\big[\mathcal{F}_{lq}(\boldsymbol{\kappa}',\boldsymbol{\kappa}'',\zeta,\zeta+h)\mathcal{F}_{qj}(\boldsymbol{\kappa}'',\boldsymbol{\kappa},0,h)\big],\end{aligned}$$

for $j,l=1,2$. For instance,

$$\begin{aligned}\mathcal{H}^*_{11}(\boldsymbol{\kappa},\boldsymbol{\kappa}')=&\int_{\mathcal{O}}\frac{\mathrm{d}(k\boldsymbol{\kappa}'')}{(2\pi)^d}\int_0^\infty \mathrm{d}\zeta\lim_{Z\to\infty}\frac{1}{Z}\int_0^Z\mathrm{d}h\\ &\times\mathbb{E}\big[\mathcal{F}^{\mathrm{r}}(\boldsymbol{\kappa}',\boldsymbol{\kappa}'',\zeta,\zeta+h)\mathcal{F}^{\mathrm{r}}(\boldsymbol{\kappa}'',\boldsymbol{\kappa},0,h)\big]\\ &-\int_{\mathcal{O}}\frac{\mathrm{d}(k\boldsymbol{\kappa}'')}{(2\pi)^d}\int_0^\infty \mathrm{d}\zeta\lim_{Z\to\infty}\frac{1}{Z}\int_0^Z\mathrm{d}h\\ &\times\mathbb{E}\big[\mathcal{F}^{\mathrm{i}}(\boldsymbol{\kappa}',\boldsymbol{\kappa}'',\zeta,\zeta+h)\mathcal{F}^{\mathrm{i}}(\boldsymbol{\kappa}'',\boldsymbol{\kappa},0,h)\big],\end{aligned}$$

and using (A.4)-(A.5), we get

$$\begin{aligned}\mathcal{H}^*_{11}(\boldsymbol{\kappa},\boldsymbol{\kappa}')=&\mathrm{Re}\Big\{\Big(\frac{ik\alpha}{2\gamma^d}\Big)^2\int_{\mathcal{O}}\frac{\mathrm{d}(k\boldsymbol{\kappa}'')}{(2\pi)^d}\int_0^\infty \mathrm{d}\zeta\lim_{Z\to\infty}\frac{1}{Z}\int_0^Z\mathrm{d}h\\ &\times\mathbb{E}\Big[\widehat{\nu}\Big(\frac{k(\boldsymbol{\kappa}'-\boldsymbol{\kappa}'')}{\gamma},\gamma\zeta\Big)\widehat{\nu}\Big(\frac{k(\boldsymbol{\kappa}''-\boldsymbol{\kappa})}{\gamma},0\Big)\Big]\\ &\times\big[\Gamma^{aa}(\boldsymbol{\kappa}',\boldsymbol{\kappa}'',\zeta+h)\Gamma^{aa}(\boldsymbol{\kappa}'',\boldsymbol{\kappa},h)\big]\Big\}.\end{aligned}$$

Moreover, using the identity

$$\begin{aligned}&\mathbb{E}\Big[\widehat{\nu}\Big(\frac{k(\boldsymbol{\kappa}'-\boldsymbol{\kappa}'')}{\gamma},\gamma\zeta\Big)\widehat{\nu}\Big(\frac{k(\boldsymbol{\kappa}''-\boldsymbol{\kappa})}{\gamma},0\Big)\Big]=\\ &\big(2\pi\gamma\big)^d\delta\big(k(\boldsymbol{\kappa}-\boldsymbol{\kappa}')\big)\widehat{\mathcal{R}}\Big(\frac{k(\boldsymbol{\kappa}-\boldsymbol{\kappa}'')}{\gamma},\gamma\zeta\Big),\end{aligned}$$

with

$$\widehat{\mathcal{R}}(\boldsymbol{q},\zeta)=\int_{\mathbb{R}^d}\mathcal{R}(\boldsymbol{r},\zeta)e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{r},$$

derived from the definition of the autocovariance function with straightforward algebraic manipulations, and obtaining from (4.16) that

$$\Gamma^{aa}(\boldsymbol{\kappa},\boldsymbol{\kappa}'',\zeta+h)\Gamma^{aa}(\boldsymbol{\kappa}'',\boldsymbol{\kappa},h)=\frac{1}{\beta(\boldsymbol{\kappa})\beta(\boldsymbol{\kappa}'')}e^{ik(\beta(\boldsymbol{\kappa}'')-\beta(\boldsymbol{\kappa}))\zeta},$$

we get

$$\mathcal{H}_{11}^*(\boldsymbol{\kappa},\boldsymbol{\kappa}') = -\frac{k^2\alpha^2}{4\gamma^d}\mathrm{Re}\Big\{ \int_{\mathcal{O}} \frac{\mathrm{d}(k\boldsymbol{\kappa}'')}{(2\pi)^d} \int_0^\infty \mathrm{d}\zeta\, \widehat{\mathcal{R}}\Big(\frac{k(\boldsymbol{\kappa}-\boldsymbol{\kappa}'')}{\gamma},\gamma\zeta\Big) \times e^{ik(\beta(\boldsymbol{\kappa}'')-\beta(\boldsymbol{\kappa}))\zeta} \frac{(2\pi)^d}{\beta(\boldsymbol{\kappa})\beta(\boldsymbol{\kappa}'')}\,\delta\big(k(\boldsymbol{\kappa}-\boldsymbol{\kappa}')\big)\Big\}.$$

The expressions of the other components of $\mathcal{H}_{jl}^*(\boldsymbol{\kappa},\boldsymbol{\kappa}')$ are of the same type. Substituting into (A.9) we obtain the explicit expression of the differential equations satisfied by the mean wave amplitudes. This is equation (4.23), written in complex form.

The calculation of the second moments is similar, by letting $n = 1$ and $f(y) = y^2$ in (A.8), and carrying the lengthy calculations.

REFERENCES

[1] Simon R. Arridge and John C. Schotland, *Optical tomography: forward and inverse problems*, Inverse Problems, 25 (2009), p. 123010 (59pp).

[2] Guillaume Bal and Kui Ren, *Transport-based imaging in random media*, SIAM Journal on Applied Mathematics, 68 (2008), pp. 1738–1762.

[3] Yu N Barabanenkov and VM FinkelBerg, *Radiation transport equation for correlated scatterers*, Sov. Phys. JETP, 26 (1968), pp. 587–591.

[4] A. Cazé and John Schotland, *Origins of radiative transport theory: diagrammatic and multiscale approaches*, Journal of the Optical Society of America A, 32 (2015), pp. 1475–1484.

[5] Subrahmanyan Chandrasekhar, *Radiative Transfer*, Dover, New York, 1960.

[6] H. Dehghani, D. T. Delpy, and S. R. Arridge, *Photon migration in non-scattering tissue and the effects on image reconstruction*, Phys. Med. Biol., 44 (1999), pp. 2897–2906.

[7] Albert C. Fannjiang, *Self-averaging radiative transfer for parabolic waves*, Comptes Rendus Mathematiques, 342 (2006), pp. 109–114.

[8] Jean-Pierre Fouque, Josselin Garnier, George Papanicolaou, and Knut Sølna, *Wave Propagation and Time Reversal in Randomly Layered Media*, Springer, New York, 2007.

[9] Josselin Garnier and Knut Sølna, *Coupled paraxial wave equations in random media in the white-noise regime*, Annals of Applied Probability, 19 (2009), pp. 318–346.

[10] ———, *Paraxial coupling of electromagnetic waves in random media*, SIAM Multiscale Modeling and Simulation, 7 (2009), pp. 1928–1955.

[11] Iosif Gikhman and Anatoliy Skorokhod, *The Theory of Stochastic Processes*, Springer-Verlag, New York, 1974.

[12] P González-Rodríguez, AD Kim, and M Moscoso, *Robust depth selectivity in mesoscopic scattering regimes using angle-resolved measurements*, Optics letters, 38 (2013), pp. 787–789.

[13] Pedro Gonzalez-Rodriguez and Arnold D. Kim, *Diffuse optical tomography using the one-way radiative transfer equation*, Biomedical Optics Express, 6 (2015), pp. 2006–2021.

[14] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Sums, Series, and Products*, Academic Press, New York, 1980.

[15] Jenni Heino, Simon Arridge, Jan Sikora, and Erkki Somersalo, *Anisotropic effects in highly scattering media*, Phys. Rev. E, 68 (2003), p. 031908.

[16] Akira Ishimaru, *Wave Propagation and Scattering in Random Media*, IEEE Press, Pitscataway, 1997.

[17] Ludek Klimes, *Correlation functions of random media*, Pure Appl. Geophys., 159 (2002), pp. 1811–1831.

[18] E. W. Larsen, *The linear boltzmann equation in optically thick systems with forward-peaked scattering*, Prog. Nucl. Energy, 107 (1999), pp. 413–423.

[19] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics*, Cambridge University Press, Cambridge, 1995.

[20] George Papanicolaou, Guillaume Bal, and Leonid Ryzhik, *Probabilistic theory of transport processes with polarization*, SIAM Journal on Applied Mathematics, 60 (2000), pp. 1639–1666.

[21] G Papanicolaou and W Kohler, *Asymptotic theory of mixing stochastic ordinary differential equations*, Communications on Pure and Applied Mathematics, 27 (1974), pp. 641–668.

[22] George Papanicolaou, Leonid Ryzhik, and Knut Sølna, *Self-averaging from lateral diversity in the ito-schroedinger equation*, SIAM Multiscale Modeling and Simulation, 6 (2007), pp. 468–492.

[23] George Papanicolaou and Sophie Weinryb, *A functional limit theorem for waves reflected by a random medium*, Applied Mathematics and Optimization, 30 (1994), pp. 307–334.

[24] G. C. Pomraning, *The fokker-planck operator as an asymptotic limit*, Math. Models Methods Appl. Sci., 2 (1992), pp. 21–36.

[25] Leonid Ryzhik, George Papanicolaou, and Joseph B Keller, *Transport equations for elastic and other waves in random media*, Wave motion, 24 (1996), pp. 327–370.

[26] Dieter Vollhardt and Peter Wölfle, *Diagrammatic, self-consistent treatment of the anderson localization problem in d 2 dimensions*, Physical Review B, 22 (1980), p. 4666.

[27] Eugne Wong and Moshe Zakai, *On the convergence of ordinary integrals to stochastic integrals*, Ann. Math. Statist., 36 (1965), pp. 1560–1564.