1. Let $M$ be a smooth $n$-manifold with boundary $\partial M$. A retraction from $M$ to $\partial M$ is a map $f : M \to \partial M$ which fixes each point of $\partial M$.

(a) Show that if $M$ is compact, there cannot be a smooth retraction from $M$ to $\partial M$.

(b) Give an example of a smooth manifold which does admit a retraction to its boundary.

(a) Suppose that such a retraction $f$ exists. Sard gives a regular value $p \in \partial M$, so that $f^{-1}(p)$ is a smooth submanifold of $M$ of codimension $\dim \partial M = n - 1$. Since $M$ is compact, $f$ is proper; since $\partial M$ is Hausdorff, $\{p\}$, hence also $f^{-1}(p)$, is compact. Any compact manifold of dimension 1 is a finite union of circles and compact intervals, so in particular, $|\partial f^{-1}(p)|$ is even. But $\partial f^{-1}(p) = \partial M \cap f^{-1}(p) = f(f^{-1}(p)) = \{p\}$, a contradiction.

(b) Consider the constant retraction $f : [0,1) \to \{0\}$, which is clearly smooth.

2. Let $X$ be the union of the three coordinate axes in $\mathbb{R}^3$, and let $X_0$ be obtained from $X$ by deleting the origin. Compute the relative homology of the pair $(X, X_0)$.

Notice that $X \simeq *$ and $X_0 \simeq \bigsqcup_{i=1}^6 *$. The relative homology sequence

$$0 \to H_1(X, X_0) \to \mathbb{Z}^6 \to \mathbb{Z} \to H_0(X, X_0) \to 0$$

is exact. Since the homomorphism from $\mathbb{Z}^6$ to $\mathbb{Z}$ is surjective, a diagram-chase yields $H_1(X, X_0) = \mathbb{Z}^5$ and $H_0(X, X_0) = 0$. Clearly, if $n \geq 2$, then $H_n(X, X_0) = 0$.

3. Let $X$ be a compact Hausdorff space, and let $X_1 \supset X_2 \supset \ldots \supset X_{n-1} \supset X_n \supset \ldots$ be a nested sequence of closed, non-empty connected subsets of $X$. Show that $\bigcap_{i=1}^\infty X_i$ is nonempty and connected.

For each $n$, let $x_n \in X_n$. Since $X$ is compact and $X_1$ is closed, the sequence $\{x_n\} \subset X_1$ has a convergent subsequence, say $x_{n_k} \to x$. Since each $X_n$ contains all its limit points, it follows that $x \in \bigcap X_i$.

Now suppose that $\{A, B\}$ is a disconnection of $\bigcap X_i$. Since $X$ is compact and Hausdorff, it is normal, so there exist disjoint open sets $U$ and $V$ with $A \subset U$ and $B \subset V$. This gives a nested sequence $\{X_i \setminus (U \cup V)\}$ of compact sets with empty intersection, so by the above, $X_k \setminus (U \cup V) = \emptyset$ for some $k$. Hence $X_k \subset U \cup V$; but $X_k \cap A$ and $X_k \cap B$ are nonempty by construction, so $\{U, V\}$ disconnects $X_k$, a contradiction.
4. Let $x$ and $y$ be two distinct points of $S^2 \times S^2$, and let a space $Z$ be obtained from $S^2 \times S^2$ by identifying $x$ and $y$. Describe the universal cover of $Z$, and compute $\pi_1(Z)$.

The universal cover is the space obtained from $\coprod_{i=-\infty}^{\infty} S^2 \times S^2$ by identifying $x$ in the $i$-th copy of $S^2 \times S^2$ with $y$ in the $(i+1)$-th copy for every $i$. The group of deck transformations is generated by the isomorphism that maps the $i$-th copy of $S^2 \times S^2$ identically onto the $(i+1)$-th copy for every $i$. Hence $\pi_1(Z) = \mathbb{Z}$. ■

5. Let $S^1 = \{x \in \mathbb{C} \mid ||z|| = 1\}$. Let $X$ be the quotient of $S^1 \times [0, 1]$ by the identification $(u, z) \sim (u, e^{2\pi i/3}z)$, $u \in \{0, 1\}$. Compute the homology of $X$.

Let $U = (S^1 \times [0, \frac{2}{3}))/ \sim$ and $V = (S^1 \times (\frac{1}{3}, 1))/ \sim$, so that $U \simeq V \simeq U \cap V \simeq S^1$. The associated Mayer-Vietoris sequence is

$$0 \to H_2(X) \to \mathbb{Z} \xrightarrow{1 \to (3, 3)} \mathbb{Z}^2 \to H_1(X) \to 0.$$ 

A diagram-chase gives $H_2(X) = \mathbb{Z}$ and

$$H_1(X) = \mathbb{Z}^2 / \langle (3, 3) \rangle = (\langle (1, 0) \rangle \oplus \langle (1, 1) \rangle) / \langle (3, 3) \rangle = \mathbb{Z} \oplus \mathbb{Z}_3.$$

It is clear that $H_n(X) = 0$ if $n \geq 3$. ■

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### Afternoon Session, 2:00–5:00

1. Let $X$ be the topological space obtained from the real line $\mathbb{R}$ by identifying the rationals $\mathbb{Q}$ to a point.

(a) Is $X$ Hausdorff?

(b) Is $X$ compact?

Justify your answers.

(a) No; let $q : \mathbb{R} \to X$ be the given quotient map, and let $U$ be a neighborhood of $[0]$ in $X$. If $V$ is any other neighborhood in $X$, then $q^{-1}(V)$ is open and nonempty. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, the intersection $q^{-1}(V) \cap \mathbb{Q}$ is nonempty; but $\mathbb{Q} \subset q^{-1}(U)$, so $U \cap V = q(q^{-1}(U)) \cap q(q^{-1}(V))$ is nonempty.

(b) No; for each $n \in \mathbb{Z}$, let $U_n = \mathbb{R} \setminus \{k\pi \}^{\infty}_{k=n}$. Each $q(U_n)$ is open in $X$, since $U_n = q^{-1}(q(U_n))$, and the collection $\{q(U_n)\}_{n \in \mathbb{Z}}$ covers $X$. However, for any finite subset $A \subset \mathbb{Z}$, we have $[\pi \max A] \notin \bigcup_{n \in A} q(U_n)$. ■

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2. For exactly which values of $c$ is the intersection of the unit sphere with the hyperboloid $x^2 + y^2 - z^2 = c$ a smooth submanifold of $\mathbb{R}^3$? Prove that your answer is correct.

Write the intersection as

$$\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = \frac{c+1}{2} \text{ and } z = \pm \sqrt{1-c^2}\}.$$ 

If $|c| > 1$, then the intersection is empty, hence a smooth manifold. If $c = -1$, then the intersection is $\{(0,0,1),(0,0,-1)\}$, a smooth 0-manifold. If $c = 1$, then the intersection is the unit circle in the $xy$-plane, a smooth 1-manifold. If $|c| < 1$, then the intersection is the disjoint union of two circles given by the equation above, again a smooth 1-manifold. ■

3. Let $W$ be a finite simplicial complex of Euler characteristic 1. Are the following statements true for all such $W$? Give a proof or a counterexample:

(a) Every continuous map $f : W \to W$ has a fixed point.

(b) Every continuous map $f : W \to W$ which is homotopic to the identity has a fixed point.

(a) Consider $W = S^1 \vee S^2$. Let $g : W \to W$ be the identity map on $S^2$ and the constant map from $S^1$ to the wedge point, and let $h : S^2 \to S^2$ be the reflection map. Clearly $f := h \circ g$ is a composition of continuous maps that has no fixed points.

(b) We have $\Lambda(f) = \chi(W) = 1 \neq 0$, so $f$ has a fixed point by Lefschetz. ■

4. Let $f : S^3 \to S^3$ be a smooth immersion. Prove that $f$ is a diffeomorphism.

Since $\dim T_pS^3 = \dim T_{f(p)}S^3$ and $df_p$ is injective for all $p \in S^3$, it is surjective as well. Hence $f$ is a submersion, and is therefore a local diffeomorphism. Since $S^3$ is compact, $f$ is proper; since $f$ is a local homeomorphism, it follows that $f$ is a covering map. But $S^3$ is simply connected, so $f$ is a homeomorphism. A bijective local diffeomorphism is a diffeomorphism. ■
5. A space $X$ is obtained from the convex hull $Y$ of $e^{2\pi i k/6}$, $k = 0, \ldots, 5$ by making the following identifications of the edges of $Y$, for each value of $k$,

$$te^{2\pi ik/6} + (1 - t)e^{2\pi i(k+1)/6} \sim te^{2\pi i(k+4)/6} + (1 - t)e^{2\pi i(k+3)/6}, t \in [0, 1].$$

Compute $\pi_1(X)$. Show that $X$ is a surface, and decide which compact surface is homeomorphic to $X$. [You may quote the classification of compact surfaces.]

Consider the following (crude) gluing diagram and surgery of $X$:

From this it is clear that $X \cong S^1 \times S^1$ is indeed a surface, and that $\pi_1(X) = \mathbb{Z}^2$.\(^1\) 

\(^1\)This nice solution is due to Trevor Hyde.