1. A space $X$ is constructed from two disjoint copies of $\mathbb{RP}^3$ and a copy of the unit interval $I$ by gluing one end of $I$ to a point of one copy of $\mathbb{RP}^3$, and gluing the other end of $I$ to the other copy of $\mathbb{RP}^3$.

(a) Describe the universal cover $\tilde{X}$ of $X$.

(b) Compute the homology groups of $\tilde{X}$.

(a) Let $T$ denote translation by 1 along the first axis of $\mathbb{R}^4$, so that

$$\tilde{X} \cong \bigcup_{n \in \mathbb{Z}} T^{3n}(S^3) \cup \bigcup_{n \in \mathbb{Z}} T^{3n+1}(I \times \{0\}^3).$$

(b) It is obvious that $\tilde{X} \cong \bigvee_{n \in \mathbb{Z}} S^3$, so

$$H_k(\tilde{X}) = \begin{cases} 
\mathbb{Z} & \text{if } k = 0 \\
\mathbb{Z}^\omega & \text{if } k = 3 \\
0 & \text{else.}
\end{cases}$$

2. Let $D$ denote the closed unit disk in the plane with boundary the unit circle $S^1$. Let $f : D \to D$ be a continuous map whose restriction to $S^1$ is the identity map. Show that $f$ must be surjective.

Since $D$ is contractible, $f$ is nullhomotopic. Therefore if $\alpha$ is a simple loop around $S^1$, then the image of $f \circ \alpha$, namely $S^1$, is contractible in $f(D)$. The image of any homotopy between $S^1 = \partial D$ and a point in $D$ must contain $D$, so in particular $D \subset f(D)$. 

3. Let $X$ denote the space $S^2 \cup A$, where $A = \{(x,0,0) \in \mathbb{R}^3 : 1 \leq x \leq 2\}$. Show that if $p : X \to Y$ is a covering map, then $p$ must be a homeomorphism, i.e. $X$ cannot cover anything except itself.

Suppose that $p : X \to Y$ is a covering map. Let $U$ be an evenly covered neighborhood of $p(0,0,1)$ in $Y$, and let $V$ be the sheet over $U$ containing $(0,0,1)$. Notice that $V$ is a union of a 1-dimensional manifold and a 2-dimensional manifold, while any neighborhood in $X \setminus V$ is either a 1-manifold or a 2-manifold. Since $V$ is disjoint from and homeomorphic to each other sheet over $U$, it follows that $V$ is the unique sheet over $U$. In particular, this implies that $p$ is of degree 1, whence $Y \cong X$ necessarily.

4. Let $K$ denote the Klein bottle. You may assume without proof that $K$ is the union of two copies $M$ and $N$ of the Moebius band with boundaries glued by a homeomorphism. Compute $H_1(K,M)$ and $H_2(K,M)$.

If $U$ is a neighborhood of the central circle of $M$, then $(K \setminus U, M \setminus U) \simeq (N, \partial N)$, so $H_*(K,M) = H_*(N, \partial N)$. The sequence

$$
H_2(N) \to H_2(N, \partial N) \to H_1(\partial N) \to H_1(N) \to H_1(N, \partial N) \to H_0(\partial N) \to H_0(N, \partial N) \to 0
$$

is exact. We have $N \simeq \partial N \cong S^1$, so $H_2(N) = 0$ and $H_1(N) = H_1(\partial N) = H_0(\partial N) = \mathbb{Z}$, so the sequence becomes

$$
0 \to H_2(N, \partial N) \to \mathbb{Z} \xrightarrow{z \to 2z} \mathbb{Z} \to H_1(N, \partial N) \to \mathbb{Z} \xrightarrow{\text{id} z} \mathbb{Z} \to H_0(N, \partial N) \to 0.
$$

A diagram-chase gives $H_1(N, \partial N) = \mathbb{Z}_2$ and $H_2(N, \partial N) = 0$.

5. Let $X$ denote the quotient space $\mathbb{R}/\mathbb{Q}$ of the real line obtained by identifying all the rationals to a single point. (This is not the group theoretic quotient.)

(a) Is $X$ Hausdorff?

(b) Is $X$ compact?

(a) No; let $q : \mathbb{R} \to X$ be the given quotient map, and let $U$ be a neighborhood of $[0]$ in $X$. If $V$ is any other neighborhood in $X$, then $q^{-1}(V)$ is open and nonempty. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, the intersection $q^{-1}(V) \cap \mathbb{Q}$ is nonempty; but $\mathbb{Q} \subset q^{-1}(U)$, so $U \cap V = q(q^{-1}(U)) \cap q(q^{-1}(V))$ is nonempty.

(b) No; for each $n \in \mathbb{Z}$, let $U_n = \mathbb{R} \setminus \{k\pi\}_{k=-n}^{\infty}$. Each $q(U_n)$ is open in $X$, since $U_n = q^{-1}(q(U_n))$, and the collection $\{q(U_n)\}_{n \in \mathbb{Z}}$ covers $X$. However, for any finite subset $A \subset \mathbb{Z}$, we have $[\pi \max A] \notin \bigcup_{n \in A} q(U_n)$. 


1. Identify the space of all $2 \times 2$ real matrices with $\mathbb{R}^4$ so that the matrix \[
\begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix}
\] corresponds to $(a, b, c, d)$. Show that the subspace $\Sigma$ of all matrices with determinant 1 is a smooth 3-dimensional manifold. Let $\Pi$ denote the hyperplane in $\mathbb{R}^4$ with equation $x_1 + x_2 + x_3 - x_4 = 0$. Does $\Pi$ intersect $\Sigma$ transversely [sic] at $I$?

Notice that $d \det(a, b, c, d) = (d, -c, -b, a)$ is surjective unless $a = b = c = d = 0$. Since $\det(0, 0, 0, 0) = 0$, it follows that 1 is a regular value of $\det$. Hence $\det^{-1}(1)$ is a smooth manifold of codimension 1 in $\mathbb{R}^4$.

Since $\dim \Pi + \dim \Sigma = 6 > 4$, the manifolds intersect transversally at $I$ unless their tangent spaces coincide. Since $\Pi$ has normal vector $(1, 1, 1, -1)$ and $T_I \Sigma$ has normal vector $(1, 0, 0, 1)$, this is not the case; the intersection is transversal. ■

2. The suspension of a space $Y$ is the quotient space $Y \times I$ obtained by identifying $Y \times \{0\}$ to a point and separately identifying $Y \times \{1\}$ to a point. Let $X$ denote the suspension of $\mathbb{R}P^2$.

(a) Compute $\pi_1(X)$.

(b) Compute all the homology groups of $X$.

(a) Let $U$ be $X$ minus one identified point, and let $V$ be $X$ minus the other identified point, so that $U \simeq V \simeq \ast$. Since $\pi_1(X)$ is the pushout of the diagram $0 \leftarrow \pi_1(U \cap V) \rightarrow 0$ by Seifert-van Kampen, we have $\pi_1(X) = 0$.

(b) Since $X$ is a quotient of a connected space, it is connected, so $H_0(X) = \mathbb{Z}$. Part (a) gives $H_1(X) = \text{Ab} \pi_1(X) = 0$. Take $U$ and $V$ as above, so that Mayer-Vietoris gives the exact sequence

$$H_2(U) \oplus H_2(V) \rightarrow H_2(X) \rightarrow H_1(U \cap V) \rightarrow H_1(U) \oplus H_1(V).$$

We have $H_2(U) = H_2(V) = H_1(U) = H_1(V) = 0$. Furthermore, $U \cap V \simeq \mathbb{R}P^2$, so $H_1(U \cap V) = \mathbb{Z}_2$. The sequence becomes

$$0 \rightarrow H_2(X) \rightarrow \mathbb{Z}_2 \rightarrow 0,$$

giving $H_2(X) = \mathbb{Z}_2$. After another application of Mayer-Vietoris, we conclude that

$$H_k(X) = \begin{cases} 
\mathbb{Z} & \text{if } k = 0 \\
\mathbb{Z}_2 & \text{if } k = 1 \\
0 & \text{else.}
\end{cases}$$

■
3. Let \( T = \{(z, w) \in \mathbb{C}^2 : |z| = |w| = 1\} \). Define a map \( f : T \to T \) by \( f(z, w) = (zw^3, w) \). Prove that \( f \) is a diffeomorphism. Choose a basis of \( H_1(T) \) and compute the matrix of \( f_* : H_1(T) \to H_1(T) \) with respect to this basis.

Notice that \( f \) is smooth, and that

\[
df_{(z, w)} = \begin{pmatrix} w^3 & 3zw^2 \\ 0 & 1 \end{pmatrix}
\]

is invertible for all \((z, w) \in T\), since \(|\det df_{(z, w)}| = |w|^3 = 1\), proving that \( f \) is a local diffeomorphism. Moreover, \( f \) has smooth inverse \((x, y) \mapsto (xy^{-3}, y)\), so \( f \) is bijective, hence a diffeomorphism.

Consider \( T \) as \( S^1 \times S^1 \), and let \( \alpha = S^1 \times \{1\} \) and \( \beta = \{1\} \times S^1 \). With respect to the ordered basis \( \{\alpha, \beta\} \) of \( H_1(T) \), the induced map \( f_* \) has matrix representation

\[
f_* = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.
\]

\[
\blacksquare
\]

4. Let \( L \) denote a triangulated 3-dimensional lens space, so that \( L \) is the quotient of \( S^3 \) by a free action of a finite cyclic group \( \mathbb{Z}_n \), and the projection map \( S^3 \to L \) is a covering map.

Suppose that \( L \) is the union of two connected subcomplexes \( H \) and \( K \). Show that \( H \cap K \) must be connected.

Mayer-Vietoris gives the exact sequence

\[
H_1(L) \to H_0(H \cap K) \to H_0(H) \oplus H_0(K) \to H_0(L) \to 0.
\]

Since \( S^3 \) is simply connected and the action of \( \mathbb{Z}_n \) on \( S^3 \) is free, we have \( \pi_1(L) = \mathbb{Z}_n \).

Since each of \( H \) and \( K \) is connected, we have \( H_0(H) = H_0(K) = \mathbb{Z} \). Since \( L \) is a quotient of a connected space, it is connected, so \( H_0(L) = \mathbb{Z} \). Letting \( k \) denote the number of connected components of \( H \cap K \), the sequence now becomes

\[
\mathbb{Z}_n \to \mathbb{Z}^k \to \mathbb{Z}^2 \to \mathbb{Z} \to 0.
\]

The image of \( \mathbb{Z}_n \) is a finite subgroup of \( \mathbb{Z}^k \), and is therefore trivial. A straightforward diagram-chase now gives \( k = 1 \), as desired.

\[
\blacksquare
\]
5. A topological space is countably compact if every countable open cover has a finite sub cover. Prove that a metric space is countably compact if and only if every infinite sequence in $X$ has a convergent subsequence.

Suppose that every infinite sequence in $X$ has a countable subsequence. Let $\{ U_n \}$ be a countably infinite open cover of $X$, and suppose that it has no finite subcover. For each $n$, let $x_n \in X \setminus \bigcup_{i=1}^n U_i$, and let $x$ be the limit of a subsequence of $\{ x_n \}$. Since $\{ U_n \}$ covers $X$, there exists some $k$ such that $x \in U_k$. However, this implies that $U_k$ contains infinitely many terms of the subsequence, hence infinitely many terms of $\{ x_n \}$, a contradiction.

Conversely\(^1\), suppose that $X$ is countably compact, and let $\{ x_n \}$ be an infinite sequence in $X$. If $\{ x_n \}$ contains no limit points, then each $x_n$ has a neighborhood containing no other points in the sequence. This constitutes an open cover of $\{ x_n \}$ with no finite subcover, showing that $\{ x_n \}$ is not countably compact and therefore not closed in $X$. Hence the sequence has a limit point that it does not contain. Let $x$ be a limit point of $\{ x_n \}$. For each $k$ in $\mathbb{N}$, let $n_k$ be such that $d(x, x_{n_k}) \leq \frac{1}{k}$. Since the set $\{ x_n : d(x, x_n) \leq \frac{1}{k} \}$ is infinite for each $k$, we may choose the sequence $n_k$ to be strictly increasing. It is clear that $\{ x_{n_k} \}$ is a convergent subsequence of $\{ x_n \}$.

\(^1\)This argument is rather ungainly. If you know a more elegant one, let me know!