1. If $M$ is a manifold with boundary, then the double of $M$ is defined by identifying two copies of $M$ along their boundaries by the identity map. Let $M = D^2 - \bigcup_i D_\varepsilon(x_i)$ where $\{D_\varepsilon(x_i)\}$ are $n$ mutually disjoint open disc [sic] of radius $\varepsilon$ in the interior of $D^2$ centered at $\{x_i\}$. Let $W$ be the double of $M$. Determine the fundamental group and Euler characteristic of $W$.

There is an obvious homotopy between $W$ and the compact orientable surface of genus $n$, which, by Seifert-van Kampen, has fundamental group

$$\langle a_1, b_1, a_2, b_2, \ldots, a_n, b_n \mid \prod_{i=1}^n [a_i, b_i] \rangle$$

and, by construction as a CW complex with 1 0-cell, $n$ 1-cells and 1 2-cell, has Euler characteristic $2 - 2n$.

2. Let $U_1, U_2, \ldots$ be a countable open covering of a metric space $X$. A refinement is an open covering $V_1, V_2, \ldots$ of $X$ such that for each $i$, $V_i \subset U_i$ for some $j$. Show that there exists a refinement $V_1, V_2, \ldots$ which is star-finite, i.e., for each $i$, $V_i \cap V_j \neq \emptyset$ for at most finitely many values of $j$.

For each $x$ in $X$, there is some $i_x$ in $\mathbb{N}$ and open set $O_x$ such that $x \in O_x \subset \overline{O_x} \subset U_{i_x}$. Since $X$ is paracompact, $\{O_x\}$ has a locally finite refinement $\{V_\alpha\}_{\alpha \in \mathcal{A}}$. For every $\alpha$ in $\mathcal{A}$, there exists $x_\alpha$ such that $V_\alpha \subset O_{x_\alpha}$. For each $x$, let $W_x$ be the union of all $V_\alpha$ such that $x_\alpha = x$. By construction, $\{W_x\}$ is a locally finite open cover of $X$ and $W_x \subset U_i$ for all $x$ in $X$.

For each $i$, let $B_i$ be the union of all $W_x$ such that $i_x = i$, and notice that if $i_x = i$, then $W_x \subset O_x$, so $B_i \subset U_i$ for all $i$. Hence $\{B_i\}$ is a locally finite open refinement of $\{U_i\}$. Since closures and unions of locally finite collections commute, we have $\overline{B_i} \subset U_i$ for all $i$.

For each $i$ in $\mathbb{N}$, let $f_i$ be a Urysohn function with $f(\overline{B_i}) = \{0\}$ and $f(X \setminus U_i) = \{1\}$. For each $i$ and $n = i, i+1, \ldots$, let $G_i^n = f_i^{-1}(0, 1 - \frac{1}{n}))$, so that $B_i \subset G_i^n \subset \overline{G_i^n} \subset G_i^{n+1}$. Further let $X_n = \bigcup_{i=1}^n G_i^n$ for each $n$ in $\mathbb{N}$, so that $\{X_n\}$ is an increasing cover of $X$. Now let $H_n = X_n \setminus X_{n-1}$ and $K_n = X_n \setminus X_{n-1}$ for each $n$, where we set $X_0 = \emptyset$ for $n \leq 0$. Then $K_n \subset H_{n+1}$ for each $n$, and $\{K_n\}$ covers $X$. Since $K_n = \bigcup_{i=1}^n (K_n \cap \overline{G_i^n})$, the collection $\{K_n \cap \overline{U_i^m} : i = 1, \ldots, n \text{ and } n = 1, 2, \ldots\}$ is a closed covering of $X$, so $\{H_{n+1} \cap U_i^{n+1} : i = 1, \ldots, n \text{ and } n = 1, 2, \ldots\}$ is an open covering of $X$. The latter covering refines $\{U_i\}$ obviously, and is star-finite, since $H_m \cap H_n = \emptyset$ for $|m-n| \geq 3$.\footnote{This argument, essentially due to Morita [2, Thm. 2.16], establishes the (stronger) result for paracompact Hausdorff spaces. It seems too involved to be expected as a QR response, but I am unaware of a slicker proof in the case of metric spaces. In the literature, the property in question is sometimes called “countable hypocompactness” [1].}
3. Let $M$ be a compact smooth manifold of dimension $n$, and let $f : M \to \mathbb{R}^n$ be a smooth map. Show that $f$ has a singular [sic] point.

Suppose that $f$ has no critical points. Since $\dim M = \dim \mathbb{R}^n = n$, it follows that $f$ is a local diffeomorphism, hence an open map. Therefore $f(M)$ is compact and open in $\mathbb{R}^n$. It follows that $f(M) = \emptyset$, a contradiction. ■

4. Let $\mathbb{R}P^2$ and $T$ denote, in this order, the real projective plane and the torus $S^1 \times S^1$. Prove that any map $f : \mathbb{R}P^2 \to T$ is homotopic to a covering map.

Note that the homomorphism $f_* : \mathbb{Z}_2 \to \mathbb{Z}_2$ is trivial, so there exists a lift $\tilde{f} : \mathbb{R}P^2 \to \mathbb{R}^2$ with respect to the universal cover $p : \mathbb{R}^2 \to T$. But $\mathbb{R}^2$ is contractible, so $\tilde{f}$ is nullhomotopic. Therefore $p \circ \tilde{f} = f$ is nullhomotopic as well. ■

5. Consider the covering map $f : S^2 \to \mathbb{R}P^2$. Let $X$ be the homotopy pushout of the diagram

\[
\mathbb{R}P^2 \leftarrow S^2 \xrightarrow{f} \mathbb{R}P^2.
\]

Calculate the homology groups of $X$. (Recall that a homotopy pushout of a diagram

\[
Z \xleftarrow{g} X \xrightarrow{f} Y
\]

is $(X \times [0,1]) \cup Y \cup Z/\sim$ with the quotient topology, where $\sim$ is the smallest equivalence relation satisfying $(x,0) \sim f(x), (x,1) \sim g(x)$ for every $x \in X$.

Let $U = (S^2 \times [0,\frac{2}{3}) \cup \mathbb{R}P^2 \cup \mathbb{R}P^2/\sim$ and $V = (S^2 \times (\frac{1}{3},1]) \cup \mathbb{R}P^2 \cup \mathbb{R}P^2/\sim$, so that $X = U \cup V$. Clearly $X$ is connected, so $H_0(X) = \mathbb{Z}$. The Mayer-Vietoris sequence

\[
H_3(U) \oplus H_3(V) \to H_3(X) \to H_2(U \cap V) \to H_2(U) \oplus H_2(V) \to H_2(X) \to H_1(U \cap V)
\]

is exact. Noting that $U \simeq V \simeq \mathbb{R}P^2$ and $U \cap V \simeq S^2$ by the obvious deformation retractions, so the sequence is

\[
0 \to H_3(X) \to \mathbb{Z} \to 0 \to H_2(X) \to 0,
\]
giving $H_3(X) = \mathbb{Z}$ and $H_2(X) = 0$. Seifert-van Kampen gives $\pi_1(X)$ as the pushout of $\mathbb{Z}_2 \leftarrow 0 \to \mathbb{Z}_2$, so $H_1(X) = \text{Ab}(\pi_1(X)) = \text{Ab}(\mathbb{Z}_2 * \mathbb{Z}_2) = \mathbb{Z}_2^2$. Since $H_n(U), H_n(V)$ and $H_n(U \cap V)$ are trivial for $n \geq 3$, it follows that $H_n(X) = 0$ for $n \geq 4$. ■
Let $X$ be obtained by gluing two solid tori $D^2 \times S^1$ along their boundary via the map $f : \partial D^2 \times S^1 \to \partial D^2 \times S^1$ given by $f(x,y) = (y^p x, y)$ where $p$ is a fixed positive integer.

(a) For which values of $p$ can $X$ be given the structure of a topological manifold?

(b) Compute $\pi_1(X)$.

(a) For all $p$, notice that $X = (D^2 \times S^1) \cup_f (D^2 \times S^1)$, where $f$ is a diffeomorphism of boundaries whose inverse is given by $(x,y) \mapsto (y^{-p}x,y)$. Since each $D^2 \times S^1$ is a smooth manifold, it follows that $X$ can be given a smooth manifold structure for all values of $p$.

(b) Let each of $U$ and $V$ be a fattening of each solid torus, so that $U \simeq V \simeq S^1$ and $U \cap V \simeq S^1 \times S^1$. Seifert-van Kampen now gives $\pi_1(X)$ as the pushout of $\mathbb{Z} \leftarrow \mathbb{Z}^2 \to \mathbb{Z}$, where the induced homomorphisms are both given on generators by $(1,0) \mapsto 1$ and $(0,1) \mapsto 1$. That is, $\pi_1(X) = \langle a, b \mid ab^{-1} \rangle = \mathbb{Z}$.

2. Consider the space

$$O_{n+1,2} = \{(x_1, x_2) \mid \langle x_1, x_2 \rangle = 0\} \subset S^n \times S^n$$

where $S^n$ is the unit sphere in the Euclidean space $\mathbb{R}^{n+1}$ with standard inner product. Denote by $p : O_{n+1,2} \to S^n$ the projection on the first factor. Prove that there is a section $s : S^n \to O_{n+1,2}$ (i.e. a continuous maps $s$ such that $ps = \text{Id}$) if and only if $n$ is odd.

If $n$ is odd, then by hairy ball there exists a nonvanishing vector field on $S^n$, i.e., a nonvanishing section $X : S^n \to TS^n \setminus (S^n \times \{0\})$ given by $p \mapsto (p, X_p)$. The map $s : S^{n+1} \to O_{n+1,2}$ be given by $p \mapsto (p, X_p / ||X_p||)$ is a section. If, on the other hand, $n$ is even and $X' : S^n \to TS^n$ is a section given by $p \mapsto (p, X'_p)$, then $X'_p = 0$ for some $p \in S^n$, again by hairy ball. Since $O_{n+1,2} \subset TS^n \setminus (S^n \times \{0\})$, it follows that no section of $O_{n+1,2}$ exists.

3. Let $X, Y$ be topological spaces with $Y$ compact. Let $p : X \times Y \to X$ be the projection onto the first factor. Show that $p$ maps each closed subset of $X \times Y$ onto a closed subset of $X$.

Let $F \subset X \times Y$ be closed. Note that $X$ is closed in $X$, and suppose that $p(F) \neq X$. Let $x \in X \setminus p(F)$. For each $y$ in $Y$, let $U_y \times V_y$ be a basic neighborhood of $(x,y)$ contained in $(X \times Y) \setminus F$. Since $\{V_y\}$ covers $Y$, there exists a finite subcover $\{V_{y_i}\}$. Let $U = \bigcap U_{y_i}$ and note that $U$ is a neighborhood of $x$ contained in $X \setminus p(F)$.
4. Let $X$ be the union of the three coordinate axes in $\mathbb{R}^3$. Calculate the homology of $\mathbb{R}^3 - X$.

It is clear that $\mathbb{R}^3 - X$ deformation-retracts to $S^2$ minus six points, which is homeomorphic to $\mathbb{R}^2$ minus five points. This in turn is homotopic to $\bigvee_{i=1}^3 S^1$, so $H_n(\mathbb{R}^3 - X) = 0$ for $n \geq 2$, $H_1(\mathbb{R}^3 - X) = \mathbb{Z}^5$, and $H_0(\mathbb{R}^3 - X) = \mathbb{Z}$.  

5. Let $S^2 \subset \mathbb{R}^3$ be the standard unit sphere and

$$X = \{(x, y, z) \in S^2 : y^2 z = x^3 - xz^2\}.$$

Is $X$ a smooth submanifold of $\mathbb{R}^3$?

Yes. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(x, y, z) = (x^2 + y^2 + z^2, x^3 - xz^2 - y^2 z)$. Notice that $f$ is smooth and $X = f^{-1}(1, 0)$, so it suffices to verify that $(1, 0)$ is a regular value of $f$. Consider the differential

$$Df_{(x,y,z)} = \begin{pmatrix}
2x & 2y & 2z \\
3x^2 - z^2 & -2yz & -2xz - y^2
\end{pmatrix},$$

which is of rank 2 unless

$$\begin{pmatrix}
2x(-2yz) - 2y(3x^2 - z^2) \\
2x(-2xz - y^2) - 2z(3x^2 - z^2) \\
2y(-2xz - y^2) - 2z(-2yz)
\end{pmatrix} = \begin{pmatrix}0
\end{pmatrix}.$$

Notice that this occurs if $(x, y, z) = (0, 0, 0)$. Suppose that $(x, y, z) \neq (0, 0, 0)$ and that $\text{rk}(Df_{(x,y,z)}) < 2$. If $x = 0$, then $y = z = 0$, a contradiction, so suppose further that $x \neq 0$. If $y = 0$ and $z = 0$, then $x$ is free. If $y = 0$ and $z \neq 0$, then $z = \pm \sqrt{3}x$, where $x$ is free. If $y \neq 0$, then $z \in \{-x, 3x\}$. If $z = -x$, then $y = \pm 2x$, where $x$ is free. If $z = 3x$, then $y = \pm 2\sqrt{3}x$, where $x$ is free. To summarize, the critical points of $f$ are the lines through the origin spanned by each of the seven vectors

$$\begin{pmatrix}1 \\
0
\end{pmatrix}, \begin{pmatrix}1 \\
0
\end{pmatrix}, \begin{pmatrix}1 \\
\pm 2
\end{pmatrix}, \text{ and } \begin{pmatrix}1 \\
\pm 2\sqrt{3}
\end{pmatrix}.$$

We now verify that these lines do not intersect $X$ nontrivially. Notice that $(0, 0, 0) \notin S^2$. Let $g$ be the projection of $f$ onto the second coordinate. Since $g$ is homogeneous, it suffices to verify that none of the seven vectors above solves $g$. Indeed, we have $g(1, 0, 0) = 1$, $g(1, 0, \pm \sqrt{5}) = -g(1, \pm 2, -1) = -4$, and $g(1, \pm 2\sqrt{3}, 3) = -44$.  

References
