1. Let \( M \) be a simply connected manifold, let \( \sim \) be an equivalence relation on \( M \) and let \( X = M/\sim \) be the quotient space. Suppose that

(i) Every equivalence class has exactly 2 points and

(ii) For every point \( x \in M \) there are open sets \( U(x), V(x) \) in \( M \) such that \( x \in U(x) \), \( U(x) \cap V(x) = \emptyset \), and for every \( u \in U(x) \) there is a \( v \) in \( V(x) \) with \( u \sim v \) and for every \( v \) in \( V(x) \) there is a \( u \in U(x) \) with \( v \sim u \).

Let \( x_0 \) be a point in \( X \) and compute \( \pi_1(X,x_0) \). Can you describe a generator of \( \pi_1(X,x_0) \)?

Define an action of \( \mathbb{Z}_2 \) on \( M \) by setting, for each \( x \in M \), \( 0 \cdot x = x \) and \( 1 \cdot x = y \), where \( y \neq x \) and \( x \sim y \). It follows that \( 0 \cdot U(x) = U(x) \) and \( 1 \cdot U(x) = V(x) \) for all \( x \in M \), so \((0 \cdot U(x)) \cap (1 \cdot U(x)) = \emptyset \). Hence \( \mathbb{Z}_2 \) is a covering space action in the language of Hatcher, i.e., \( X = M/\mathbb{Z}_2 \). Therefore \( \pi_1(X,x_0) = \mathbb{Z}_2 \). If \( \alpha \) is a simple path in \( M \) from \( x \) to \( y \), where \( x \neq y \) and \( [x] = [y] = x_0 \), then \( [\alpha] \) is a generator of \( \pi_1(X,x_0) \).

2. Let \( X \) be a compact Hausdorff space and let \( A_1, A_2, \ldots \) be a sequence of connected subspaces with \( \overline{A}_j - A_j \subset A_{j+1} \) for each \( j = 1, 2, \ldots \). Suppose also that for each \( x \) in \( X \) there is an open set \( U(x) \) containing \( x \) such that \( U(x) \cap A_j = \emptyset \) for all but a finite number of \( j \). Prove that \( \bigcup_{j=1}^{\infty} A_j \) is compact.

It suffices to show that \( X \setminus \bigcup A_j \) is open. Let \( x \in X \setminus \bigcup A_j \), and let \( J = \{ j : U(x) \cap A_j \neq \emptyset \} \). Further let \( V = U(x) \setminus \bigcup_{j \in J} \overline{A}_j \). We have

\[
V = U(x) \setminus \left( \bigcup_{j \in J} A_j \cup \left( \bigcup_{j \in J} \overline{A}_j \right) \right) \subset U(x) \setminus \left( \bigcup_{j \in J} A_j \cup A_{j+1} \right) \subset U(x) \setminus \bigcup_{j=1}^{\infty} A_j,
\]

so \( x \in V \). Since \( J \) is finite, \( V \) is open by construction, and we are done.

3. Let \( f : X \to Y \) be a differentiable map of smooth compact simply connected \( n \)-manifolds. Show that \( f \) is a submersion if and only if \( f \) is a diffeomorphism.

If \( f \) is a diffeomorphism, then it is a fortiori a submersion. If \( f \) is a submersion, then \( \text{rk } df_x = \dim T_{f(x)} Y = n = \dim T_x X \) for all \( x \in X \), so \( f \) is an immersion, hence a local diffeomorphism. Since \( X \) is compact and \( Y \) is Hausdorff, \( f \) is proper; since \( f \) is a local homeomorphism, it is a covering map. It follows that \( f \) is surjective, and moreover bijective, since \( Y \) is simply connected. A bijective local diffeomorphism is a diffeomorphism.
4. Let $F$ be a free group and $R$ be a normal subgroup. Assume that $F$ is finitely generated and $R$ is finitely normally generated, and put $\pi = F/R$.

(a) Construct a finite 2-dimensional complex $X$ such that $\pi_1(X, x_0) = \pi$.

(b) If $Y$ is a space such that $\pi_1(Y, y_0) = \pi$, prove that there is a map $f : (X, x_0) \to (Y, y_0)$ such that $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

(a) Since $F$ is finitely generated, so is $\pi$. Since $R$ is finitely generated, $\pi$ is finitely presented, say $\pi = \langle g_1, \ldots, g_n \mid r_1, \ldots, r_m \rangle$. Let $X$ consist of one 0-cell $e^0 = x_0$; $n$ 1-cells $e^1_i$, attached via constant maps $S^0_i \to e^0$; and $m$ 2-cells $e^2_j$. By an abuse of notation, we treat each $e^1_i$ as both the cell itself and the path in $X$ that traverses its closure simply. With this in mind, if $r_j = g_{j_1} \cdots g_{j_k}$, we define the attaching maps from $S^1_{j_1}$ to the concatenation $e^1_{j_1} \ast \cdots \ast e^1_{j_k}$ via a suitable parametrization.

(b) We define $f : X \to Y$ on cells of increasing dimension. First, let $f(x_0) = y_0$. Since the fundamental groups of each pointed space are equal, they have identical presentations. There are obvious maps from the images of generators of $\pi_1(X, x_0)$ to the images of generators of $\pi_1(Y, y_0)$, given by $g_i(t) \mapsto g_i(t)$ for all $i = 1, \ldots, n$ and $t \in [0,1]$. Now, for each relation $r_j = g_{j_1} \cdots g_{j_k}$, there exists a homotopy $H_j$ between $g_{j_1} \ast \cdots \ast g_{j_k}$ and the constant loop at $y_0$. By identifying $e^2_j$ with $[0,1]^2$ via a suitable parametrization, we can set $f(e^2_j) = H([0,1]^2)$ in a manner consistent with our previous definition of $f$ on the boundary 1-cells.

5.

(a) Compute the singular homology groups $H_*(S^n/S^k)$ where $S^k = \mathbb{R}^{k+1} \cap S^n$ and $S^n$ is the unit sphere in $\mathbb{R}^{n+1}$.

(b) Prove carefully that $H_*(S^n - S^k) = H_*(S^n - T)$ where $T$ is a small tubular neighborhood of $S^k$ in $S^n$. Note that $T$ is diffeomorphic to $S^k \times D^{n-k}$.

(c) Compute the singular homology groups $H_*(S^n - S^k)$.

(a) If $k = n$, then $S^n/S^k$ is contractible, so $H_i(S^n/S^k) = \mathbb{Z}$ if $i = 0$ and is trivial otherwise. Suppose that $k < n$, and notice that $S^n/S^k = S^n/\partial D^{k+1} \cong S^n \cup D^{k+1}$, since $D^{k+1}$ is contractible. Contracting this space about a $(k+1)$-cell $e_{k+1} \subset S^n$ whose boundary is $\partial D^{k+1}$ reveals a homotopy $S^n \cup D^{k+1} \cong S^n \vee S^{k+1}$. Hence

$$H_i(S^n/S^{n-1}) = \begin{cases} 
\mathbb{Z} & \text{if } i = 0 \\
\mathbb{Z}^2 & \text{if } i = n \\
0 & \text{else,}
\end{cases}$$

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and for $k < n - 1$,

$$H_i(S^n / S^k) = \begin{cases} \mathbb{Z} & \text{if } i = 0, k + 1, \text{ or } n \\ 0 & \text{else.} \end{cases}$$

(b) We have $T \cong S^k \times D^{n-k} \cong S^k \times * \cong S^k$. The result follows now from homotopy invariance of singular homology.

(c) If $k = n$, then $S^n \setminus S^k = \emptyset$. If not, then $S^n \setminus S^k \cong (\mathbb{R}^{k+1})^* \cap S^n \cong \mathbb{R}^{n-k} \cap S^n = S^{n-k-1}$. Hence

$$H_i(S^n \setminus S^{n-1}) = \begin{cases} \mathbb{Z}^2 & \text{if } i = 0 \\ 0 & \text{else,} \end{cases}$$

and for $k < n - 1$,

$$H_i(S^n \setminus S^k) = \begin{cases} \mathbb{Z} & \text{if } i = n - k - 1 \\ 0 & \text{else.} \end{cases}$$

Afternoon Session, 2:00–5:00

1. Let $X$ be $\mathbb{R}^2$ with the standard topology $\tau_1$ and let $Y$ be $\mathbb{R}^2$ with the topology $\tau_2$ given by: $C$ is closed in $Y$ if $C \cap L$ is closed in $L$ (standard topology) for every straight line $L$ in $\mathbb{R}^2$.

(a) Show that, in fact, $\tau_2$ is a topology on $\mathbb{R}^2$.

(b) Are the topologies $\tau_1$ and $\tau_2$ comparable? Are they equal?

(a) Let $\{F_a\}_{a \in A}$ be a closed collection in $Y$. For any straight line $L \subset \mathbb{R}^2$, it is clear that $(F_a \cup F_{a'}) \cap L = (F_a \cap L) \cup (F_{a'} \cap L)$ is a finite union of closed sets in $L$, hence closed. Also, $L \cap \bigcap_{a \in A} F_a = \bigcap_{a \in A} (F_a \cap L)$ is an intersection of closed sets in $L$, hence closed. Clearly $\emptyset \cap L = \emptyset$ and $Y \cap L = L$ are closed in $L$, so $\tau_2$ is indeed a topology.

(b) We claim that $\tau_1 \subset \subset \tau_2$. If $A$ is not closed in $Y$, then there exists a line $L$ and a sequence $\{a_n\} \subset L \cap A$ such that $a_n \to a$ in $Y \setminus A$. But $a_n \to a$ in $X \setminus A$ as well, so $A$ is not closed in $X$. However, $S^1 \setminus \{(1,0)\}$ is not closed in $X$, but is closed in $Y$, since $|L \cap (S^1 \setminus \{(1,0)\})| \leq 2$ for any straight line $L$. 

■
2. Consider two embedded circles in the solid torus $M = S^1 \times D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \times \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}; \alpha = S^1 \times \{(0, 0)\} \text{ and } \beta = \{(0, 1)\} \times \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1/2\}$.

Let $T_\alpha, T_\beta$ be small open tubular neighborhood [sic] of $\alpha, \beta$ such that they are disjoint from each other and the boundary of $M$. Prove or disprove that $M \setminus T_\alpha, M \setminus T_\beta$ are homeomorphic.

The spaces are not homeomorphic. There are obvious homotopies $M \setminus T_\alpha \simeq S^1 \times S^1$ and $M \setminus T_\beta \simeq S^2 \vee S^1 \vee S^1$. Hence $\pi_1(M \setminus T_\alpha) = \mathbb{Z}^2 \neq \mathbb{Z} \ast \mathbb{Z} = \pi_1(M \setminus T_\beta)$.

3. Let $X$ denote the union of two circles meeting at one point $x_0$. Find a suitable connected 3-fold covering space of $X$ and use the cover to prove that $\pi_1(X, x_0)$ is not abelian.

Consider the covering $p : Y \rightarrow X$, where $Y$ is illustrated by

![Diagram of covering space]

and $p$ takes each vertex of $X$ to the wedge point of $Y$. No deck transformation takes an outer vertex to the center vertex, since each outer vertex is self-adjacent, while the center vertex is not. Hence $p$ is not regular, i.e., $p_*(\pi_1(Y))$ is not normal in $\pi_1(X)$. If $\pi_1(X)$ were Abelian, every subgroup would be normal.

4. Prove that there is no submersion of a smooth nonempty compact manifold into Euclidean space.

Suppose that $f : M \rightarrow \mathbb{R}^n$ is such a submersion. Since $f$ is a submersion, it is an open map, so $f(M)$ is open; since $f$ is continuous, $f(M)$ is compact. Hence $f(M) = \emptyset$, a contradiction.

5. Let $M$ be the smooth manifold given by $M = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 = 1, z^2 + w^2 = 1\}$ and let $f : M \rightarrow \mathbb{R}$ be given by $f(x, y, z, w) = x + z$. Find all the critical points of $f$ and the associated critical values.

We have $df_{(x,y,z,w)} = (1 \ 0 \ 1 \ 0)$ identically on $M$, and $TM = S^1 \times S^1$. The critical points of $f$ are precisely the set of points in $M$ whose tangent planes are orthogonal to the span of $(1, 0, 1, 0)$, i.e., all vectors in $S^1 \times S^1$ parallel to $(1, 0, 1, 0)$. This is

$$\{ (x, y, x, w) \in \mathbb{R}^4 \mid y^2 = w^2 = 1 - x^2 \},$$

whose associated critical values are $\{2x \in \mathbb{R} \mid |x| \leq 1\} = [-2, 2]$. ■