How to (not) cancel varieties

Bob Lutz

February 22, 2015

Abstract
Since the early 1970s, various authors have made steady progress toward resolutions of the algebraic cancellation problem, an easily-stated yet compellingly-elusive question whose difficulty emphasizes the limitations of our understanding of the geometry of objects as fundamental to algebraists as complex affine n-space. We survey these partial resolutions and examine the first known counterexample [5] in light of developments since the last prominent expository note [16] written on the subject.

1 Introduction
An exceedingly natural question:

Given varieties $X$, $Y$ and $Z$ with $X \times Z \cong Y \times Z$, must $X \cong Y$?

This is a generic form of the “cancellation problem,” often misattributed to Zariski. It is stated frequently in the affine special case.

Given a variety $X$ with $X \times \mathbb{A}^m \cong \mathbb{A}^m \times \mathbb{A}^n$, must $X \cong \mathbb{A}^n$?

In what follows, we hope to provide a survey of results and counterexamples addressing special cases of cancellation. At times, we will follow loosely the wonderful exposition of [16, §3], adding detail at the whim of the author’s personal interest, or whenever the material is relatively accessible. In particular, we include a more scenic discussion of the surfaces of Danielewski and address cancellation results that have appeared since the publication of [16] in 1995.

Zariski’s 1949 question differed materially from the ones above.

If $F$ and $F'$ are finitely generated fields over some base field, does an isomorphism $F(x) \cong F'(x')$ imply an isomorphism $F \cong F'$?

The problem remains open as stated, although Belov–Yu solved it for two low-dimensional cases in [2]. Certain other special cases are well known and much easier to demonstrate, with some salient examples listed in [21]. The Zariski question continues to attract a fair amount of study and has inspired progress on a number of more general problems. Notably, Hochster showed in [12] that two commutative rings $R$ and $S$ need not be isomorphic despite an isomorphism $R[x] \cong S[x]$. Related affirmative results appear in [1, 3, 24].

While Hochster’s example is strikingly uncomplicated, a negative answer to the other question takes more work. We occupy ourselves with this formulation; when we write “cancellation,” we

---

1Including front and end matter, the article is just over one page in length.
will hereafter mean the former problem. For brevity, if $X \times Z \cong Y \times Z$ implies that $X \cong Y$, then we abbreviate the implication by saying that the triple $(X, Y, Z)$ cancels. Unless stated otherwise, we presume to work over the complex numbers $\mathbb{C}$, i.e., $\mathbb{A}^n = \mathbb{A}^n_{\mathbb{C}}$ and $\mathbb{G}_a = \mathbb{C}^+$.  

2 Early work

The first significant progress toward a resolution was affirmative. Early results of Fujita-Iitaka relied heavily on the notion of logarithmic Kodaira dimension $\kappa(Y)$ of a variety $Y$, an invariant taking values among $-\infty, 0, 1, 2, \ldots$. A definition would take us afield of our objectives in this note, but the motivated reader can pursue the details in [13]. It is worthwhile to note that at first, Fujita–Iitaka discussed only cases of induced cancellation, where the isomorphism $X \cong Y$ agrees with $X \times Z \cong Y \times Z$ in a sense prescribed below. This feature was omitted from subsequent results.

Theorem 2.1 ([14]). If $X$ and $Y$ are varieties with $\kappa(Y) \geq 0$, then $(X, Y, \mathbb{A}^n)$ cancels for all $n$. Moreover, if $X \times \mathbb{A}^n \cong Y \times \mathbb{A}^n$, then we have the following commutative diagram of isomorphisms and projections:

$$
\begin{array}{ccc}
X \times \mathbb{A}^n & \leftrightarrow & Y \times \mathbb{A}^n \\
\downarrow & & \downarrow \\
X & \leftrightarrow & Y
\end{array}
$$

One might naturally wonder how restrictive the nonnegativity condition is. It turns out to be easy to come up with varieties $Y$ such that $\kappa(Y) = -\infty$. For example, $\kappa(\mathbb{A}^n) = -\infty$ for all $n$. Less trivially, Gurjar-Miyanishi showed in [11] that any nonsingular surface $Y$ with $|\pi_1^\infty(Y)| < \infty$ (see Appendix A) also works. Surfaces with negative $\kappa$ are in fact fairly well understood, thanks to the “ruling theorem” of Miyanishi–Sugie–Russell (namely, [23, Theorem 1]). However, the obvious choices for such surfaces do not yield any counterexamples of cancellation; Danielewski would provide the first one, twelve years later.

In the meantime, Ramanujam gave in [22] a nice topological characterization of the affine plane; he proved that a surface $S$ is contractible and simply connected at infinity (see Appendix A) if and only if $S \cong \mathbb{A}^2$. This result led him to a promising candidate surface $R$ for non-cancellation; however, as Kraft points out in [16], Ramanujam’s example had no hope in light of Theorem 2.1 and the fact that $\kappa(R) = 2$.

Fujita was the first to publish affirmatively in the two-dimensional affine case with the following result, for which he credits Miyanishi with a substantial part of the argument. It relies critically on the nontrivial fact that if $X$ is any surface containing a “cylinderlike” open subset of the form $\mathbb{A}^1 \times C$, where $C$ is a curve, then $(X, \mathbb{A}^2, \mathbb{A}^1)$ cancels (cf. [19]). The final result does away, in essence, with Miyanishi’s hypothesis.

Theorem 2.2 ([8]). If $X$ is a variety, then $(X, \mathbb{A}^2, \mathbb{A}^1)$ cancels.

The result is often stated as its immediate corollary, i.e., that $(X, \mathbb{A}^2, \mathbb{A}^n)$ cancels for all $n$. Besides avoiding mention of the often-difficult-to-compute $\kappa$ in its statement (although not in its proof), one benefit of Theorem 2.2 is that it holds over any perfect field, as shown by Kambayashi in

\[2\text{This terminology does not appear in the literature, but it reduces clutter for our purposes.}\]
\[3\text{Fujita gives a lucid exposition in [10] but assumes some familiarity with the subject.}\]
\[4\text{As Sugie notes in [26, §3], the latter condition can be relaxed to }|\pi_1^\infty(X)| < \infty.\]
The interested reader can find a self-contained algebraic proof of the result over an arbitrary algebraically closed field by Crachiola–Makar-Limanov in [4].

Fujita presented another result in his paper, which Kambayashi later called the “strong cancellation theorem” (for obvious reasons, although by all counts this name did not persist) and also generalized to the case of a perfect field:

**Theorem 2.3** ([8]). \*If \( X \) is an affine surface and \( Z \) any variety, then \((X, \mathbb{A}^2, Z)\) cancels.\*

For mathematicians concerned only with affine cases of cancellation, this result dispensed with the problem in dimension two. Many authors refer to the “solution” of cancellation for affine surfaces; they are usually referring to either the preceding result or the next one, which followed several years later from a simple algebraic characterization of \( \mathbb{A}^2 \) provided by Miyanishi in [18]. His result can be retooled to say that if \( f \) is a smooth affine surface, then \( S \cong \mathbb{A}^2 \) if and only if \( S \) is factorial and there exists a dominant morphism \( \mathbb{A}^n \to S \) for some \( n \).

**Theorem 2.4.** \*If \( X \) is a surface and \( Z \) any variety with \( X \times Z \cong \mathbb{A}^n \) for some \( n \), then \( X \cong \mathbb{A}^2 \).\*

To date, this remains one of the strongest affirmative cancellation results. Fujita’s work on the problem extends beyond these affine cases, but the situation becomes more complicated and the results messier. One prominent example is the following theorem on cancellation of complete varieties, which appeared a year after the above result. It seems that Fujita invents the term “Picard independent” ad hoc; the earnest reader can find his definition in [9, Proposition 3], but the point is that the behavior of the Picard schemes of \( X \) and \( Z \) determines ultimately whether \((X, Y, Z)\) cancels.

**Theorem 2.5** ([9]). \*If \( X \) and \( Z \) are Picard independent varieties and \( Z \) is projective, then \((X, Y, Z)\) cancels for any variety \( Y \).\*

Unfortunately, little is known about the affine problem in dimensions three and higher. The following result, due to Sugie, represents the state of the art in three-dimensional affine cancellation, despite being several decades old. We provide its statement primarily for anecdotal value; the reader is invited to skim the details.

**Theorem 2.6** ([26]). \*Let \( A \) be a 3-dimensional affine domain, \( X = \text{Spec} A \), and \( Y = \text{Spec}(A^G) \). If \( X \) is affine 1-ruled and a certain morphism\(^5\) \( f : X \to Y \) is geometrically irreducible in codimension 1 over \( Y \), then \((X, \mathbb{A}^3, \mathbb{A}^n)\) cancels for any \( n \).

This result is markedly unwieldier than its counterparts in dimension two; such is the difficulty of cancellation in higher dimensions. The main impediment is a lack of any nice algebraic or topological characterizations of \( \mathbb{A}^n \) for \( n > 2 \), as the results concerning the plane do not generalize in any obvious way. Excepting Theorem 2.1, even less is known about the non-affine situation.

### 3 The surfaces of Danielewski

It turns out that affine cancellation fails in dimension one. Danielewski provided the first example in [5], which has been generalized several times over, as we will see momentarily. The original example consists of smooth surfaces arising as principal \( \mathbb{G}_a \)-bundles over a suitable space. Their

\(^5\)Specifically, \( f \) is induced by the inclusion \( A^G \hookrightarrow A \).
construction is relatively elementary, but Danielewski’s argument is ingenious. We are particularly interested in this example, since it forms the basis for most negative cancellation results to date.

Danielewski’s intuition for finding non-cancellative varieties relies on the observation that if $X$ and $Y$ are affine varieties and $X \to V$ and $Y \to V$ are principal $G_a$-bundles, then the pullback bundle $X \times_V Y$ is trivial. We restate this result below. At the heart is the simple fact that the first Čech cohomology group $H^1(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ denotes the structure sheaf of $X$, is trivial whenever $X$ is affine.

**Proposition 3.1.** If $E$ and $E'$ are affine total spaces of principal $G_a$-bundles over the same base space, then $E \times \mathbb{A}^1 \cong E' \times \mathbb{A}^1$.

For a terse proof, cf. [7, Remark 1.5]. Serre conducts a more detailed discussion in [25, §2.3]. In light of this proposition, Danielewski’s search narrows considerably; in order to find non-cancellative varieties, he now need look for fiber bundles over a space that is easy to work with but still somewhat-poorly behaved. It turns out that the familiar “line with two origins” $V$ fits the bill. Concretely, $V$ is the pushout (of topological spaces) of $\mathbb{A}^1 \leftarrow \mathbb{A}^1 \setminus \{0\} \rightarrow \mathbb{A}^1$. The key faulty property of $V$ is non-separatedness.

Notice that $V$ has an open affine covering by two copies of $\mathbb{A}^1$ whose intersection is the algebraic torus $\text{Spec}(\mathbb{C}[x, x^{-1}])$. The relevant portion of the Čech complex corresponding to this cover is

$$0 \to \mathbb{C}[x] \oplus \mathbb{C}[x] \xrightarrow{\partial^0} \mathbb{C}[x, x^{-1}] \to 0,$$

where the coboundary map $\partial^0$ is $(f, g) \mapsto f - g$. We see now that

$$H^1(V, \mathcal{O}_V) = \mathbb{C}[x, x^{-1}]/\mathbb{C}[x].$$

Since principal $G_a$-bundles over $V$ are in correspondence with the elements of $H^1(V, \mathcal{O}_V)$ (cf. [5, Proposition 3.1]), it follows that there are many nontrivial such bundles. Danielewski studied those of the form

$$W_n^h = \mathbb{C}[x, y, z]/(x^n y - z^2 - h z),$$

where $n \geq 1$ and $h \in \mathbb{C}[x]$. Fittingly, these have come to be known as Danielewski surfaces. If $h = 1$, then we write $W_n = W_n^1$ for brevity. If $h$ and $x$ are coprime, then the $W_n^h$ have a convenient description in terms of Čech cocycles; specifically, $W_n^h$ is the principal bundle obtained from the cocycle $hx^{-n}$, in the sense that the surface can be formed by gluing two copies of $\mathbb{A}^1 \times G_a$ along $\{0\} \times G_a$ using the map $(s, t) \mapsto (s, t + hx^{-n})$.

In addition to providing Danielewski with his counterexample, the following easy corollary of Proposition 3.1 and the comments above served as a launching pad for later generalizations of his work by others.

**Corollary 3.2.** If $g, h \in \mathbb{C}[x]$ are each coprime to $x$, then

$$W_m^g \times \mathbb{A}^1 \cong W_n^h \times \mathbb{A}^1,$$

for all $m$ and $n$.

In order to distinguish between the isomorphism classes of the surfaces themselves, Danielewski needed an invariant. He invoked the so-called first homology at infinity, denoted $H^\infty_1(-)$, to show

For a definition and brief discussion, we refer the reader to Appendix A.
ultimately that $W_1$ and $W_2$ are not even homeomorphic, let alone isomorphic as varieties. Armed with this, Danielewski could set about establishing his counterexample.

To begin, he exhibits an isomorphism $W_1 \cong (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$, where $\Delta$ is the diagonal. Since $\Delta$ is a divisor of bidegree $(1, 1)$, it has self-intersection number 2, whence it follows from an observation of Mumford in [20, p. 253] that the torsion subgroup of $H^\infty_1(W_1)$ has order 2. Tom Dieck provides in [27] a geometric argument of a different flavor\footnote{Kraft gives a nice gloss in [16].} showing that $\pi^\infty_1(W_1) = \mathbb{Z}/2\mathbb{Z}$ (cf. Theorem 4.1).

To address the other surface, Danielewski capitalizes again on the work of Mumford, but the argument requires more work and is ultimately less descriptive. He first writes $W_2$ as the zero locus of the homogenization $x^2y - z^2w - zw^2$ in $\mathbb{P}^3$ and decomposes the boundary $\partial W_2$ as the union of the lines $x = w = 0$ and $y = w = 0$. Then, after decomposing further by taking blowups, he is able to transform the intersection matrix of the resulting curves into an integral matrix. Elementary linear algebra now allows him to deduce that the determinant of the intersection matrix, hence the order of the torsion subgroup of $H^\infty_1(W_2)$, is divisible by 4. The discussion in Appendix A implies easily that $W_1$ and $W_2$ are not homeomorphic. Hence, a fortiori:

**Example 3.3.** The triple $(X, Y, \mathbb{A}^1)$ does not cancel in general.

## 4 Generalizations and extensions

The result of Danielewski has since inspired many further examples of non-cancellation in the one-dimensional affine case. Many are direct generalizations or developments of the original surfaces. In his paper, Danielewski acknowledges the following result of Fieseler.

**Theorem 4.1 ([7]).** We have $H^\infty_1(W_n) = \mathbb{Z}/2n\mathbb{Z}$ for all $n$.

Fieseler therefore obtains, in combination with Corollary 3.2, infinitely many counterexamples of cancellation. His motivation lies elsewhere, however, and he goes as far as classifying $G_a$-actions on normal affine surfaces. Tom Dieck demonstrates a related, but broader, family of isomorphism classes of $\mathbb{Q}$-homology planes whose cylinders are isomorphic, comparing them via the fundamental group at infinity.

In [17], Makar-Limanov introduced the AK invariant to prove that the Russell cubic is not isomorphic to $\mathbb{A}^3$. The invariant has proven useful in issues of cancellation as well; notably, Crachiola used it to compute the automorphism groups of the Danielewski surfaces $W_n^m$ and show that they are in fact non-cancellative over any field. The AK invariant, true to its original purpose, has been effective in distinguishing other exotic (hyper)surfaces from affine space.

Dubouloz provided in [6] an elegant analog of Danielewski surfaces in any finite dimension. He proved that if $(m_1, \ldots, m_n), (m'_1, \ldots, m'_n) \in \mathbb{Z}_{>1}^n$ are distinct and no two entries of $(\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r$ are equal, where $r > 1$, then while the subvarieties of $\mathbb{A}^{n+2}$ given by the zero loci of

\[
\begin{align*}
    x_{n+1} \prod_{i=1}^n x_i^{m_i} - \prod_{i=1}^r (x_{n+2} - \lambda_i) & \quad \text{and} \quad & x_{n+1} \prod_{i=1}^n x_i^{m'_i} - \prod_{i=1}^r (x_{n+2} - \lambda_i),
\end{align*}
\]

are not isomorphic, their cylinders are isomorphic. He goes on to give a convenient description of such varieties in terms of Čech cocycles.
To close, it bears mentioning that not all negative progress in cancellation has followed from the work of Danielewski. Kraft claims in [16] to have achieved an infinite pairwise non-isomorphic family $Z_n$ of 5-dimensional affine varieties whose cylinders are each isomorphic to $\mathbb{A}^6$. He conjectures that $Z_n \not\cong \mathbb{A}^5$ for all $n$, but no proof exists yet. These examples remain intriguing in contrast to the examples of Douboloz, due to the surprising fact that each $Z_n$ is diffeomorphic to $\mathbb{A}^3$.

Generalization notwithstanding, these examples all tell us essentially the same thing, each in stronger terms than the last: cancellation fails badly in one dimension. We are, however, left with the obvious questions. What of triples $(X, \mathbb{A}^n, Z)$, where $n > 2$? What more can be said about $(X, Y, Z)$ when $Y$ is not affine? We leave the reader with these.

A Loops at infinity

Two of the invariants mentioned in the main article feature prominently enough to warrant a minor comment. We take this opportunity to clarify the notions of fundamental group and first homology at infinity.

**Definition.** For a topological space $X$, we set

$$H_1^\infty(X) = \lim_{\leftarrow} H_1(X \setminus K, \mathbb{Z}),$$

where the inverse limits are taken over all compact $K \subset X$ and the maps are induced by inclusion. If $X$ is path-connected, we also set

$$\pi_1^\infty(X) = \lim_{\leftarrow} \pi_1(X \setminus K).$$

It is perhaps instructive to think first about the situation in Euclidean space (or its one-point compactification), where we can take the $K$ literally to be compact neighborhoods of infinity. In the standard topology, we have $\pi_1^\infty(\mathbb{C}) = H_1^\infty(\mathbb{C}) = \mathbb{Z}$. If $X$ is compact, it is easy to see that $\pi_1^\infty(X) = H_1^\infty(X) = 0$. In case $\pi_1^\infty(X) = 0$, we say that $X$ is *simply connected at infinity*.

One important note is that while the fundamental group and first homology at infinity are topological invariants, neither is homotopy invariant. As an example, let $D$ be the the usual (contractible) open unit disk in $\mathbb{C}$. Clearly $H_1^\infty(D)$ and $\pi_1^\infty(D)$ are nontrivial, but a singleton is compact, so $\pi_1^\infty(*) = H_1^\infty(*) = 0$. These invariants allow us therefore to distinguish between spaces that are homeomorphic but not homotopy equivalent.

B Acknowledgments

The author extends his gratitude to David Speyer for helpful comments at the outset of writing, as well as to Pierre-Marie Poloni for providing the evanescent paper of Danielewski.

References


[17] L. Makar-Limanov. On the hypersurface $x + x^2y + z^2 + t^3 = 0$ in $\mathbb{C}^4$ or a $\mathbb{C}^3$-like threefold which is not $\mathbb{C}^3$. *Israel Journal of Mathematics*, 96:419–429, 1996.


