Singularities and K-stability

by

Harold Blum

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Doctoral Committee:

Professor Mircea Mustaţă, Chair
Professor Bhargav Bhatt
Professor Mattias Jonsson
Professor Venky Nagar
Professor Karen E. Smith

Harold Blum

blum@umich.edu

ORCID iD: 0000-0003-2865-392X

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Contents

Abstract

We study invariants of singularities that have arisen in connection with the K-stability of Fano varieties. The first invariant we consider is Li's normalized volume function on the space of valuations over a klt singularitiy. Proving a conjecture of Li, we show that there always exists a valuation over a klt singularity with smallest normalized volume. Next, we present joint work with Mattias Jonsson on the log canonical and stability thresholds of a line bundle. The latter notion generalizes an invariant recently introduced by Fujita and Odaka, and can be used to characterize when a Fano variety is K-semistable or uniformly K-stable. We express the two thresholds as infima of certain functionals on the space of valuations and systematically study these invariants.

Chapter 1

Introduction

1.1 Background

1.1.1 Kähler-Einstein Geometry

A fundamental problem in complex geometry is to find canonical metrics on (complex) projective varieties. The search for canonical metrics has lead to the study of Kähler-Einstein metrics, which are Kähler metrics whose Ricci curvarture is a constant multiple of the Kähler class. On smooth projective varieties with trivial or ample canonical bundle, such Kähler-Einstein metrics always exist by the work of Yau and Aubin [Yau78, Aub78].

The case of Fano varieties (i.e. smooth projective varieties with anti-ample canonical bundle) is more subtle, since not all Fano varieties have Kähler-Einstein metrics. For example, $\mathbf{P}^2$ and $\mathbf{P}^2$ blown up at a point are both Fano varieties, but only the first has a Kähler-Einstein metric.

Yau conjectured that the existence of a Kähler-Einstien metric on a Fano variety is related to some algebraically defined "stability" condition. By the recent work of Chen-Donaldson-Sun and Tian, we now know that a Fano variety has a Kähler-Einstein metric if and only if it is K-polystable [CDS15, Tia15]. However, the condition for a variety to be K-(poly)stable is difficult to verify.

It has long been understood that singularities from the Minimal Model Program play an important role in understanding the condition for a Fano variety to be K-stable. This dates back to Tian's α-invariant [Tia87], which Demailly interpreted as measuring singularities of pluri-anti-canonical divisors [CS08]. More recently, Odaka related the notion of K-stability to discrepancies, a measure of singularities used in the Minimal Model Program [Oda13].

1.1.2 Singularities of the MMP

A goal of birational geometry is to classify all varieties up to birational equivalence. The Minimal Model Program (MMP) approaches this goal by attaching to each birational equivalence class, a "simplest" variety. While one might be primarily interested in smooth projective varieties, in the MMP it is important to consider varieties with mild singularities.

One such class of mild singularities are *Kawamata log terminal* (klt) singularities. Examples of klt singularities include du Val singularities and normal $\mathbf{Q}$-Gorenstein toric singularities. An important example for our purposes stems from Fano varieties. If V is a Fano variety, then the cone over V with polarization $-K_V$ is a klt singularity.

Related to the previous definition is the *log canonical threshold*. Given an effective Cartier divisor D on a klt variety X, the log canonical threshold of D, denoted $\mathrm{lct}(X;D)$, measures the singularities of D. In the simplest case when $X = \mathbf{C}^n$ and $D = \{f = 0\}$, where $f \in \mathbf{C}[x_1, \ldots, x_n]$, we have

$$\mathrm{lct}(X;D) = \sup\left\{ c \,\middle|\, \frac{1}{|f|^{2c}} \text{ is locally integrable at every point } P \in \mathbf{C}^n \right\}.$$

A smaller log canonical threshold means D has worse singularities.

The previous definitions may be interpreted in terms of valuations. Let $\pi : Y \to X$ be a proper birational morphism of normal varieties and E a prime divisor on Y (called a *prime divisor over* X). Associated to E is a valuation $\mathrm{ord}_E : K(X)^\times \to \mathbf{Z}$ that sends a rational function on X to its order of vanishing along E. We call ord_E a *divisorial* valuation.

To a valuation ord_E, as above, we associate a couple of invariants. The *center* of ord_E is the generic point of $\pi(E)$. The *log discrepancy* of ord_E is given by

$$A_X(\mathrm{ord}_E) = 1 + \text{ the coefficient of } E \text{ in } K_{Y/X}\ ,$$

where $K_{Y/X}$ is the *relative canonical divisor* of π. In the case when X and Y are smooth, $K_{Y/X}$ is effective and locally cut out by the determinant of the Jacobian matrix of π.

We can now define our terms. A variety X is klt if it is normal, $\mathbf{Q}$-Gorenstein, and $A_X(\mathrm{ord}_E) > 0$ for all prime divisors E over X. If X is a klt variety and D an effective Cartier divisor, then

$$\mathrm{lct}(X;D) = \inf_E \frac{A_X(\mathrm{ord}_E)}{\mathrm{ord}_E(D)}$$

where the infimum runs through all prime divisors E over X. Using a similar formula, we can also define the log canonical threshold of a nonzero ideal $\mathfrak{a} \subset \mathcal{O}_X$ on a klt variety X.

Recent work has shown relations between the log canonical threshold, valuations, and non-archimedean geometry [BFJ08, JM12, BdFFU15]. Given a variety X, we write Val_X for the set of real-valued valuations $v : K(X)^\times \to \mathbf{R}$ with center on X. Recall that v has *center* $\xi \in X$ if v is ≥ 0 on $\mathcal{O}_{X,\xi}$ and > 0 on the maximal ideal of $\mathcal{O}_{X,\xi}$. While log canonical thresholds are usually defined using only divisorial valuations, the larger valuation space provides a richer environment for studying singularities.

1.2 Main Results

This dissertation concerns the following invariants of singularities that have arisen in connection with Kähler-Einstein geometry.

- Normalized volume of a valuation: Li introduced the normalized volume function on the space of valuations over a klt singularity and proposed the problem of studying its minimizer [Li15a]. The minimizer is related to a conjecture of Donaldson and Sun concerning singularities appearing on Gromov-Hausdorff limits of Kähler-Einstein Fano varieties [DS17].

- The α- and δ-invariants. These two invariants measure the singularities of linear systems. The α-invariant dates back to the work of Tian [Tia87] and has received considerable attention. The δ-invariant was recently introduced by Fujita and Odaka in [FO16] and was designed to detect the K-(semi)stability of Fano varieties.

Remark 1.2.1. Our main results are stated for klt pairs, a generalization of klt varieties. A klt pair (X, B) is composed of a normal variety X and an effective $\mathbf{Q}$-divisor B on X such that $K_X + B$ is $\mathbf{Q}$-Cartier. (See Section 2.2.7 for the complete definition.) If X is a klt variety, then $(X, 0)$ is a klt pair. The definitions of the log canonical threshold and log discrepancy can be extended to this setting.

1.2.1 Normalized volume of a valuation

Fix a klt pair (X, B) and $x \in X$ a closed point. We write $\mathrm{Val}_{X,x}$ for the set of real valuations on X with center equal to x. In [Li15a], Li introduced the *normalized volume* function

$$\widehat{\mathrm{vol}} : \mathrm{Val}_{X,x} \longrightarrow \mathbf{R}_{>0} \cup \{+\infty\},$$

which sends a valuation v to its *normalized volume*, denoted $\widehat{\mathrm{vol}}(v)$. To define the normalized volume, we recall the following. Given a valuation $v \in \mathrm{Val}_{X,x}$, we have valuation ideals

$$\mathfrak{a}_m(v)_x = \{f \in \mathcal{O}_{X,x} \mid v(f) \geq m\} \subseteq \mathcal{O}_{X,x}$$

for all positive integers m. The volume of v is given by

$$\mathrm{vol}(v) = \limsup_{m\to\infty} \frac{\ell(\mathcal{O}_{X,x}/\mathfrak{a}_m(v)_x)}{m^n/n!},$$

where n is the dimension of X. The normalized volume of v is

$$\widehat{\mathrm{vol}}(v) = A_{X,B}(v)^n \,\mathrm{vol}(v),$$

where $A_{X,B}(v)$ is the log discrepancy of v. Since (X, B) is klt, $A_{X,B} > 0$ on $\mathrm{Val}_{X,x}$, and, thus, $\widehat{\mathrm{vol}} > 0$ as well.

The K-semistablity of a Fano variety can be phrased in terms of the minimizer of $\widehat{\mathrm{vol}}$. Let V be a smooth Fano variety. Consider the cone $C(V, -K_V) := \mathrm{Spec}(\oplus_{m\geq 0} H^0(V, -mK_V))$ and write $p \in C(V, -K_V)$ for the cone point. The blowup of $C(V, -K_V)$ at p has a unique exceptional divisor, which we denote by $\widetilde{V}$ and gives a valuation $\mathrm{ord}_{\widetilde{V}} \in \mathrm{Val}_{C,p}$.

Theorem 1.2.2 ([Li15b] [LiLiu16] [LiX16]). *Let V be a smooth Fano variety. The following are equivalent:*

(i) The Fano variety V is K-semistable.

(ii) The function $\widehat{\mathrm{vol}} : \mathrm{Val}_{C,p} \to \mathbf{R} \cup \{+\infty\}$ is minimized at $\mathrm{ord}_{\widetilde{V}}$.

Thus, if V is K-semistable, there exists a valuation centered at $p \in C(V, -K_V)$ that minimizes the normalized volume function. Li conjectured the following statement.

Conjecture 1.2.3 ([Li15a]). If (X, B) is a klt pair and $x \in X$ a closed point, then there exists a valuation $v_* \in \mathrm{Val}_{X,x}$ that minimizes $\widehat{\mathrm{vol}} : \mathrm{Val}_{X,x} \to \mathbf{R} \cup \{+\infty\}$. Furthermore, such a minimizer v_* is unique (up to scaling) and quasimonomial.

The above statement is a major component of the *Stable Degeneration Conjecture* described in [LiX17]. The conjecture further states that the minimizer of the normalized volume function should gives a unique degeneration to K-semistable Fano cone singularity.

Conjecture 1.2.3 holds when X is a smooth variety and $B = 0$. As observed in [Li15a], if x is a smooth point, then $\widehat{\mathrm{vol}}$ is minimized at ord_x, the valuation that sends a function $f \in \mathcal{O}_{X,x}$ to the maximum m such that $f \in \mathfrak{m}_x^m$. Thus,

$$n^n = \widehat{\mathrm{vol}}(\mathrm{ord}_x) \leq \widehat{\mathrm{vol}}(v)$$

for all $v \in \mathrm{Val}_{X,x}$. The above observation relies on the following result of de Fernex-Ein-Mustaţă.

Theorem 1.2.4 ([dFEM04]). *Let X be smooth variety of dimension n and $x \in X$ a closed point. If $\mathfrak{a}$ is an $\mathfrak{m}_x$-primary ideal, then*

$$n^n = \mathrm{lct}(X; \mathfrak{m}_x)^n \,\mathrm{e}(\mathfrak{m}_x) \leq \mathrm{lct}(X; \mathfrak{a})^n \,\mathrm{e}(\mathfrak{a}),$$

where $\mathrm{lct}(X; \mathfrak{a})$ and $\mathrm{e}(\mathfrak{a})$ denote the log canonical threshold and Hilbert-Samuel multiplicity of $\mathfrak{a}$.

Our main result on the normalized volume function is the following statement.

Theorem A. *Let (X, B) be a klt pair and $x \in X$ a closed point. There exists a valuation $v_* \in \mathrm{Val}_{X,x}$ that minimizes $\widehat{\mathrm{vol}} : \mathrm{Val}_{X,x} \to \mathbf{R} \cup \{+\infty\}$.*

In order to prove the above theorem, we first take a sequence of valuations $(v_i)_{i\in\mathbf{N}}$ such that $\lim_{i\to\infty} \widehat{\mathrm{vol}}(v_i) = \inf_{v\in\mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}(v)$. Ideally, we would would like to find a valuation v_* that is a limit point of the sequence $(v_i)_{i\in\mathbf{N}}$ and then argue that v_* is a minimizer of $\widehat{\mathrm{vol}}$. To proceed with such an argument, one would need to prove a conjecture of Li stating that $\widehat{\mathrm{vol}}$ is lower semicontinuous on $\mathrm{Val}_{X,x}$. Recall that $\widehat{\mathrm{vol}}(v) := A_{X,B}(v)^n \,\mathrm{vol}(v)$. While $v \mapsto A_{X,B}(v)$ is lower semicontinuous, $v \mapsto \mathrm{vol}(v)$ fails to be lower semicontinuous in general. To avoid this complication, we shift our focus.

Instead of studying valuations $v \in \mathrm{Val}_{X,x}$, we may consider ideals $\mathfrak{a} \subseteq \mathcal{O}_X$ that are $\mathfrak{m}_x$-primary. For an $\mathfrak{m}_x$-primary ideal, the *normalized multiplicity* of $\mathfrak{a}$ is given by $\mathrm{lct}(X, B; \mathfrak{a})^n \,\mathrm{e}(\mathfrak{a})$, as in Theorem 1.2.4.

We can also define a similar invariant for graded sequences of $\mathfrak{m}_x$-primary ideals. Recall that a graded sequence of ideals on X is a sequence of ideals $\mathfrak{a}_\bullet = (\mathfrak{a}_m)_{m\in\mathbf{Z}_{>0}}$ such that $\mathfrak{a}_p \cdot \mathfrak{a}_q \subseteq \mathfrak{a}_{p+q}$ for all p, q. The following statement relates the infimum of the normalized volume function to these invariants.

Proposition 1.2.5 ([Liu16])**.** *If (X,B) is a klt pair and $x \in X$ a closed point, then*

$$\inf_{v \in \mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}(v) = \inf_{\mathfrak{a}_\bullet\ \mathfrak{m}_x\text{-primary}} \mathrm{lct}(X,B;\mathfrak{a}_\bullet)^n \, \mathrm{e}(\mathfrak{a}_\bullet) = \inf_{\mathfrak{a}\ \mathfrak{m}_x\text{-primary}} \mathrm{lct}(X,B;\mathfrak{a})^n \, \mathrm{e}(\mathfrak{a}). \tag{1.1}$$

While our goal is to find $v_* \in \mathrm{Val}_{X,x}$ that achieves the first infimum of (1.1), we will instead find a graded sequence of $\mathfrak{m}_x$-primary ideals $\tilde{\mathfrak{a}}_\bullet$ that achieves the second infimum of the above equation. To this end, we use techniques from [dFM09, Kol08, dFEM10, dFEM11] on *generic limits* to construct a graded sequence of ideals $\tilde{\mathfrak{a}}_\bullet$ that can be thought of as a "limit point" of $(\mathfrak{a}_\bullet(v_i))_{i\in\mathbf{N}}$. After having constructed such a graded sequence $\tilde{\mathfrak{a}}_\bullet$, we can use [JM12, Theorem A] to find a valuation v_* that computes the log canonical threshold $\tilde{\mathfrak{a}}_\bullet$. This valuation will be the minimizer of the normalized volume function.

In proving Theorem A, we use some additional tools. Specifically, properness estimates from [Li15a] play an important role in the proof. In addition, we use ideas from [ELS03] to prove the following result concerning the volume of a valuation.

Theorem B. *Let (X,B) be a klt pair of dimension n and $x \in X$ a closed point. For $\varepsilon > 0$ and constants $B, s \in \mathbf{N}^*$, there exists $N = N(\varepsilon, B, s)$ such that the following holds: If $v \in \mathrm{Val}_{X,x}$ satisfies $A_{X,B}(v) \leq B$ and $v(\mathfrak{m}_x) \geq \frac{1}{s}$, then*

$$\mathrm{vol}(v) \leq \frac{\mathrm{e}(\mathfrak{a}_m(v))}{m^n} < \mathrm{vol}(v) + \varepsilon$$

for all $m \geq N$.

1.2.2 Thresholds, Valuations, and K-stability

The results in this section are from [BlJ17] and are joint with Mattias Jonsson. The author thanks him for allowing the joint work to be reproduced in this setting.

Let (X, B) be a klt pair and L a big line bundle on X. We consider two natural "thresholds" of L, both involving the asymptotics of the singularities of the linear system $|mL|$ as $m \to \infty$.

The first measures the *worst* singularities appearing in the linear systems. For $m \geq 1$, define $\alpha_m(L) = \inf\{m \cdot \operatorname{lct}(X, B; D) \mid D \in |mL|\}$. The *log canonical threshold* of L is

$$\alpha(L) = \inf_{m \geq 1} \alpha_m(L).$$

This is an algebraic version of the α-invariant defined analytically by Tian [Tia97] when X is Fano and $L = -K_X$.

The second invariant measures the "average" singularities and was introduced by Fujita and Odaka in the Fano case, where it is relevant for K-stability, see [FO16, PW16]. Following [FO16] we say that an effective $\mathbf{Q}$-divisor $D \sim_{\mathbf{Q}} L$ on X is of *m-basis type*, where $m \geq 1$, if there exists a basis $s_1, \ldots, s_{N_m}$ of $H^0(X, mL)$ such that

$$D = \frac{\{s_1 = 0\} + \{s_2 = 0\} + \cdots + \{s_{N_m} = 0\}}{mN_m},$$

where $N_m = h^0(X, mL)$. Define

$$\delta_m(L) = \inf\{\operatorname{lct}(D) \mid D \sim_{\mathbf{Q}} L \text{ of } m\text{-basis type}\}.$$

We prove the following result.

Theorem C. *For any big line bundle L, the limit $\delta(L) = \lim_{m \to \infty} \delta_m(L)$ exists, and*

$$\alpha(L) \leq \delta(L) \leq (n+1)\alpha(L).$$

Further, the numbers $\alpha(L)$ and $\delta(L)$ are strictly positive and only depend on the numerical equivalence class of L. When L is ample, the stronger inequality $\delta(L) \geq \frac{n+1}{n}\alpha(L)$ holds.

We call $\delta(L)$ the (adjoint) *stability threshold*[1] of L. It can also be defined for $\mathbf{Q}$-line bundles L by $\delta(L) := r\delta(rL)$ for any $r \geq 1$ such that rL is a line bundle; see Remark 5.3.5.

[1]The idea of the stability threshold $\delta(L)$, with a slightly different definition, was suggested to the second author of [BlJ17] by R. Berman.

The following result, which verifies Conjecture 0.4 and strengthens Theorem 0.3 of [FO16], relates the stability threshold to the K-stability of a log Fano pair.

Theorem D. *Let (X,B) be a log Fano pair.*

(i) (X,B) is K-semistable if and only if $\delta(-(K_X+B)) \geq 1$;

(ii) (X,B) is uniformly K-stable if and only if $\delta(-(K_X+B)) > 1$.

The notion of uniform K-stability was introduced in [BHJ15, Der16]. As a special case of the Yau-Tian-Donaldson conjecture, it was proved in [BBJ15] that a smooth Fano manifold X without nontrivial vector fields is uniformly K-stable iff X admits a Kähler-Einstein metric.

Our approach to the two thresholds $\alpha(L)$ and $\delta(L)$ is through *valuations*. To any $v \in \mathrm{Val}_X$ we associate several invariants. First is the *log discrepancy*, denoted $A_{X,B}(v)$. Next, we have invariants that depend on a big line bundle L. These are the *maximal* and *average order of vanishing* of L along v, and are denoted by $T(v)$ and $S(v)$, respectively.

For a divisorial valuation ord_E, as above, $T(\mathrm{ord}_E)$ can be viewed as a pseudoeffective threshold:

$$T(\mathrm{ord}_E) = \sup\{t > 0 \mid \pi^*L - tE \text{ is pseudoeffective}\}$$

whereas $S(\mathrm{ord}_E)$ is an "integrated volume".

$$S(\mathrm{ord}_E) = \mathrm{vol}(L)^{-1} \int_0^\infty \mathrm{vol}(\pi^*L - tE)\, dt.$$

The invariants $S(\mathrm{ord}_E)$ and $T(\mathrm{ord}_E)$ play an important role in the work of K. Fujita [Fuj16b], C. Li [Li15b], and Y. Liu [Liu16], see Remark 5.2.10.

The next result shows that log canonical and stability thresholds can be computed using the above invariants of valuations :

Theorem E. *For any big line bundle L on X, we have*

$$\alpha(L) = \inf_v \frac{A_{X,B}(v)}{T(v)} = \inf_E \frac{A_{X,B}(\mathrm{ord}_E)}{T(\mathrm{ord}_E)} \quad \textit{and} \quad \delta(L) = \inf_v \frac{A_{X,B}(v)}{S(v)} = \inf_E \frac{A_{X,B}(\mathrm{ord}_E)}{S(\mathrm{ord}_E)},$$

where v ranges over nontrivial valuations with $A_{X,B}(v) < \infty$, and E over prime divisors over X.

While the formulas for $\alpha(L)$ follow quite easily from the definitions (see also [Amb16, §3.2]), the ones for $\delta(L)$ (as well as the fact that the limit $\delta(L) = \lim_m \delta_m(L)$ exists) are more subtle and use the concavity of the function on the Okounkov body of L defined by the filtration associated to the valuation v as in [BC11, BKMS16]. Theorem D follows from the second formula for $\delta(L)$ above and results in [Fuj16b] and [Li15b].

Next we investigate whether the infima in Theorem E are attained. We say that a valuation $v \in \mathrm{Val}_X$ *computes* the log canonical threshold if $\frac{A_{X,B}(v)}{T(v)} = \alpha(L)$. Similarly, v computes the stability threshold if $\frac{A_{X,B}(v)}{S(v)} = \delta(L)$.

Theorem F. *If L is ample, then there exist valuations with finite log discrepancy computing the log-canonical threshold and the stability threshold, respectively.*

This theorem can be viewed as a global analogue of Theorem A. The result is also related in spirit to recent results by Birkar [Bir16] on the existence of $\mathbf{Q}$-divisors achieving the infimum in the definition of $\mathrm{lct}(L)$ in the $\mathbf{Q}$-Fano case, and to the existence of optimal destabilizing test configurations [Don02, Szé08, Oda15, DS16].

Finally we treat the case when X is a toric variety, associated to a complete fan Δ, and B is a torus invariant $\mathbf{Q}$-divisor on X. The primitive lattice points of the 1-dimensional cones of Δ then correspond to torus invariant prime divisors D_i, for $1 \leq i \leq d$. Let $L = \mathcal{O}_X(D)$, where $D = \sum c_i D_i$ is an ample divisor on X. Associated to D is a polytope $P \subset M_{\mathbf{R}}$. To each $u \in P \cap M_{\mathbf{Q}}$, there is an associated effective torus invariant $\mathbf{Q}$-divisor $D_u \sim_{\mathbf{Q}} L$ on X.

Theorem G. *The log-canonical and stability thresholds of L are given by*

$$\alpha(L) = \min_{u \in \mathrm{Vert}(P)} \mathrm{lct}(D_u) \quad \textit{and} \quad \delta(L) = \mathrm{lct}(D_{\bar{u}}),$$

where $\bar{u} \in M_{\mathbf{Q}}$ denotes the barycenter of P, and $\mathrm{Vert}(P) \subset M_{\mathbf{Q}}$ the set of vertices of P. Furthermore, $\alpha(L)$ (resp. $\delta(L)$) is computed by one of the valuations $\mathrm{ord}_{D_1}, \dots, \mathrm{ord}_{D_d}$.

For $\alpha(L)$, the above statement is well known; see [CS08, Del15, Amb16]. For the case of $\delta(L)$, our proof uses global analogs of methods utilized in [Mus02, Blu16b].

When X is a toric $\mathbf{Q}$-Fano variety and $L = -K_X$, Theorem G implies that X is K-semistable iff the barycenter of P is the origin. For X smooth, this result was proven by analytic methods in [BB13, Berm16]. In general, it follows from [LiX16, Theorem 1.4], which was proven algebraically.

Chapter 2

Preliminaries

2.1 Conventions

Throughout, we work over an algebraically closed, uncountable, characteristic zero field k. By a *variety*, we will mean an integral, separated scheme of finite type over k.

An *ideal* on a variety X is a coherent ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$. If X is a variety and $Z \subset X$ a subscheme, we write $\mathcal{I}_Z \subset \mathcal{O}_X$ for the corresponding ideal. Similarly, if $x \in X$ is a closed point, we write $\mathfrak{m}_x \subset \mathcal{O}_X$ for the ideal of functions vanishing at x.

We use the convention $\mathbf{N} = \{0, 1, 2, \dots\}$, $\mathbf{N}^* = \mathbf{N} \setminus \{0\}$, $\mathbf{R}_+ = [0, +\infty)$, $\mathbf{R}_+^* = \mathbf{R}_+ \setminus \{0\}$. In an inclusion $A \subset B$ between sets, the case of equality is allowed.

2.2 Background material

2.2.1 Valuations

Let X be a variety. A *valuation on* X will mean a valuation $v\colon K(X)^\times \to \mathbf{R}$ that is trivial on k and has center on X. Recall, v has *center* on X if there exists a point $\xi \in X$ such that $v \geq 0$ on $\mathcal{O}_{X,\xi}$ and $v > 0$ on the maximal ideal of $\mathcal{O}_{X,\xi}$. Since X is assumed to be separated, such a point ξ is unique, and we say v has center $c_X(v) = \xi$. We use the convention that $v(0) = +\infty$.

Following [JM12, BdFFU15] we define Val_X as the set of valuations on X and equip it with the topology of pointwise convergence. We define a partial ordering on Val_X by $v \leq w$ if and only if $c_X(w) \in \overline{c_X(v)}$ and $v(f) \leq w(g)$ for $f, g \in \mathcal{O}_{X,c_X(w)}$. The unique minimal element is the *trivial* valuation on X. We write Val_X^* for the set of nontrivial valuations on X.

To any valuation $v \in \mathrm{Val}_X$ and $\lambda \in \mathbf{R}$ there is an associated *valuation ideal* defined locally by $\mathfrak{a}_\lambda(v)$ defined as follows. For an affine open subset $U \subset X$, $\mathfrak{a}_\lambda(v)(U) = \{f \in \mathcal{O}_X(U) \mid v(f) \geq \lambda\}$ if $c_X(v) \in U$ and $\mathfrak{a}_\lambda(v)(U) = \mathcal{O}_X(U)$ otherwise. Note that $\mathfrak{a}_\lambda(v) = \mathcal{O}_X$ for $\lambda \leq 0$. If v is divisorial, then Izumi's inequality (see [HS01]) shows that there exists $\varepsilon > 0$ such that $\mathfrak{a}_\lambda(v) \subset \mathfrak{m}_\xi^{\lceil \varepsilon\lambda \rceil}$ for any $\lambda \in \mathbf{R}_+$, where $\xi = c_X(v)$.

For an ideal $\mathfrak{a} \subset \mathcal{O}_X$ and $v \in \mathrm{Val}_X$, we set

$$v(\mathfrak{a}) := \min\{v(f) \mid f \in \mathfrak{a} \cdot \mathcal{O}_{X,c_X(v)}\} \in [0, +\infty].$$

The function $\mathrm{Val}_X \ni v \mapsto v(\mathfrak{a})$ is continuous on Val_X [JM12, Lemma 4.1].

We can also make sense of $v(s)$ when L is a line bundle and $s \in H^0(X, L)$. After trivializing L at $c_X(v)$, we write $v(s)$ for the value of the local function corresponding to s under this trivialization; this is independent of the choice of trivialization.

Similarly, we can define $v(D)$ when D is an effective $\mathbf{Q}$-Cartier divisor on X. Pick $m \geq 1$ such that mD is Cartier and set $v(D) = m^{-1}v(f)$, where f is a local equation of mD at the center of v on X. Note that $v(D) = m^{-1}v(\mathcal{O}_X(-mD))$.

2.2.2 Divisorial valuations

If $Y \to X$ is a proper birational morphism, with Y normal, and $E \subset Y$ is a prime divisor (called a *prime divisor over* X), then E defines a valuation $\mathrm{ord}_E \colon K(X)^\times \to \mathbf{Z}$ in Val_X given by the order of vanishing at the generic point of E. Any valuation of the form $v = c \cdot \mathrm{ord}_E$ with $c \in \mathbf{R}_{>0}$ will be called *divisorial*. We write $\mathrm{DivVal}_X \subset \mathrm{Val}_X$ for the set of divisorial valuations.

2.2.3 Quasimonomial valuations

A valuation in Val_X is quasimonomial if it becomes monomial on some proper birational model over X. Specifically, let $f : Y \to X$ be a proper birational morphism with Y smooth, and fix a point $\eta \in Y$. Given a regular system of parameters $y_1, \ldots, y_r \in \mathcal{O}_{Y,\eta}$ and $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbf{R}_+^r \setminus \{\mathbf{0}\}$, we define a valuation v_α as follows. For $f \in \mathcal{O}_{Y,\eta}$ we can write f in $\widehat{\mathcal{O}_{Y,\eta}}$ as $f = \sum_{\beta \in \mathbf{N}^r} c_\beta y^\beta$, with $c_\beta \in \widehat{\mathcal{O}_{Y,\eta}}$ either zero or unit. We set

$$v_\alpha(f) = \min\{\langle \alpha, \beta \rangle \mid c_\beta \neq 0\}.$$

A quasimonomial valuation is a valuation that can be written in the above form. Note that a valuation v_α, as above, is divisorial if there exists $\lambda \in \mathbf{R}_+^*$ such that $\lambda \cdot \alpha \in \mathbf{N}^r$. See

[JM12, Section 3.1] for further details.

2.2.4 Canonical divisors

Let X be a normal variety and X_{sing} denote its singular locus. Hence, $U = X \setminus X_{\text{sing}}$ is open in X and $\operatorname{codim}(X, X_{\text{sing}}) \geq 2$. A canonical divisor of X is a divisor K_X on X such that $\mathcal{O}_U(K_X|_U) = \omega_U$. Note that K_X is uniquely defined up to linear equivalence. If $\pi : Y \to X$ is a proper birational morphism of normal varieties, then it is straightforward to show that if K_Y is a canonical divisor on Y, then $\pi_* K_Y$ is a canonical divisor on X.

2.2.5 Pairs and triples

A *triple* $(X, B, \mathfrak{a}^c)$ is made up of a normal variety X, a $\mathbf{Q}$-divisor B on X such that $K_X + B$ is $\mathbf{Q}$-Cartier, a nonzero ideal $\mathfrak{a} \subset \mathcal{O}_X$, and $c \in \mathbf{Q}_{>0}$. When $c = 1$, we simply write $(X, B, \mathfrak{a})$.

A triple of the form $(X, B, \mathcal{O}_X)$ is called a *pair* and is often written as (X, B). A pair (X, B) is *effective* if B is effective.

2.2.6 Log resolutions

A *log resolution* of a triple $(X, B, \mathfrak{a}^c)$ is a projective birational morphism $\pi : Y \to X$ satisfying the following conditions:

- Y is smooth and $\operatorname{Exc}(\pi)$ has pure codimension one;
- $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, where F is a Cartier divisor on Y;
- $\operatorname{Exc}(\pi) + \tilde{B} + F$ is a simple normal crossing divisor.

As a consequence of [Hir64], if $(X, B, \mathfrak{a})$ is a triple, then there exists a log resolution $\pi : Y \to X$ of $(X, B, \mathfrak{a}^c)$. Furthermore, if $\rho : Z \to X$ is an additional proper birational morphism, we may choose π so that it factors through ρ.

2.2.7 Singularities of the MMP

Let (X, B) be a pair. Given a proper birational morphism $\pi : Y \to X$ of normal varieties, fix a canonical divisor K_Y on Y and set $K_X = \pi_* K_Y$. We define a $\mathbf{Q}$-divisor B_Y on Y by requiring

$$K_Y + B_Y = \pi^*(K_X + B).$$

It is straightforward to show that B_Y is independent of the choice of a canonical divisor on Y. When $B = 0$, we write $K_{Y/X}$ for $-B_Y$, since it equals $K_Y - \pi^* K_X$. The divisor $K_{Y/X}$ is called the *relative canonical divisor* of π. Note that $\mathrm{Supp}(K_{Y/X}) \subset \mathrm{Exc}(\pi)$.

Let $\pi : Y \to X$ and $\rho : Z \to Y$ be proper birational morphisms of normal varieties. If (X, B) is a pair, then (Y, B_Y) and (Z, B_Z) are pairs. Furthermore, $B_Z = (B_Y)_Z$ and $\rho_* B_Z = B_Y$. In the case when K_Y is $\mathbf{Q}$-Cartier, $B_Z = \rho^*(B_Y) - K_{Z/Y}$.

Let $Y \to X$ be a log resolution of a pair (X, B). We say (X, B) is *klt* (resp., *log canonical*) if the coefficients of $-B_Y$ are strictly > -1 (resp., ≥ -1). The above definition is independent of the choice of a log resolution. See [KM98, Section 2.3] for further details.

The previous definitions make sense for triples. Let $(X, B, \mathfrak{a}^c)$ be an effective triple with log resolution $Y \to X$. We say $(X, B, \mathfrak{a}^c)$ is *klt* (resp., *log canonical*) if for all log resolutions of the triple $\pi : Y \to X$, with $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, the coefficients of $-B_Y - cF$ are strictly > -1. (resp., ≥ -1). As before, the definition is independent of the choice of a log resolution.

2.2.8 Graded sequences of ideals

A *graded sequence of ideals* is a sequence $\mathfrak{a}_\bullet = (\mathfrak{a}_m)_{m \in \mathbf{N}^*}$ of ideals on X satisfying $\mathfrak{a}_p \cdot \mathfrak{a}_q \subset \mathfrak{a}_{p+q}$ for all $p, q \in \mathbf{N}^*$. We will always assume $\mathfrak{a}_m \neq (0)$ for some $m \in \mathbf{N}^*$. We write $M(\mathfrak{a}_\bullet) := \{m \in \mathbf{N}^* \mid \mathfrak{a}_m \neq (0)\}$. By convention, $\mathfrak{a}_0 := \mathcal{O}_X$.

Given a valuation $v \in \mathrm{Val}_X$, it follows from Fekete's Lemma that the limit

$$v(\mathfrak{a}_\bullet) := \lim_{M(\mathfrak{a}_\bullet) \ni m \to \infty} \frac{v(\mathfrak{a}_m)}{m}$$

exists, and equals $\inf_m v(\mathfrak{a}_m)/m$; see [JM12]. Note that if v is a non-trivial valuation, then $\mathfrak{a}_\bullet(v)$ is a graded sequence of ideals.

A graded sequence $\mathfrak{a}_\bullet$ of ideals will be called *nontrivial* if there exists a divisorial valuation v such that $v(\mathfrak{a}_\bullet) > 0$. By Izumi's inequality, this is equivalent to the existence of a point $\xi \in X$ and $\delta > 0$ such that $\mathfrak{a}_m \subset \mathfrak{m}_\xi^{\lceil \delta m \rceil}$ for all $m \in \mathbf{N}$.

Lemma 2.2.1. *Let $v \in \mathrm{Val}_X$ and $\mathfrak{a}_\bullet$ a graded sequence of ideals on X. If $v(\mathfrak{a}_\bullet) \geq 1$, then $\mathfrak{a}_m \subset \mathfrak{a}_m(v)$ for all $m \in \mathbf{N}$.*

Proof. Since $1 \leq v(\mathfrak{a}_\bullet) = \inf_m v(\mathfrak{a}_m)/m$, we see that $m \leq v(\mathfrak{a}_m)$. Therefore, $\mathfrak{a}_m \subset \mathfrak{a}_m(v)$. □

2.2.9 Volume and multiplicities

Let X be a variety and $x \in X$ a closed point. An ideal $\mathfrak{a} \subset \mathcal{O}_X$ is $\mathfrak{m}_x$-*primary* if $\operatorname{Supp}(\mathcal{O}_X/\mathfrak{a}) = \{x\}$. Equivalently, $\mathfrak{m}_x^d \subset \mathfrak{a} \subset \mathfrak{m}_x$ for some $d \in \mathbf{N}^*$. If $\mathfrak{a}$ is $\mathfrak{m}_x$-primary, then the *multiplicity* of $\mathfrak{a}$ is

$$\mathrm{e}(\mathfrak{a}) = \lim_{m\to\infty} \frac{\ell(\mathcal{O}_X/\mathfrak{a}^m)}{m^n/n!},$$

where $\ell(\mathcal{O}_X/\mathfrak{a}^m)$ denotes the length of $\mathcal{O}_X/\mathfrak{a}^m$ as a $\mathcal{O}_X$-module and n is the dimension of X.

If $\mathfrak{a}_\bullet$ is a graded sequence of $\mathfrak{m}_x$-primary ideals, then the *multiplicity* and *volume* of $\mathfrak{a}_\bullet$, as defined in [ELS03], are given by

$$\mathrm{e}(\mathfrak{a}_\bullet) = \lim_{m\to\infty} \frac{\mathrm{e}(\mathfrak{a}_m)}{m^n} \text{ and } \operatorname{vol}(\mathfrak{a}_\bullet) = \limsup_{m\to\infty} \frac{\ell(\mathcal{O}_X/\mathfrak{a}_m)}{m^n/n!}.$$

It follows from Teissier's Minkowski inequality that $\mathrm{e}(\mathfrak{a}_\bullet) = \inf_{m\in M(\mathfrak{a}_\bullet)}$ and the limit in the definition of $e(\mathfrak{a}_\bullet)$ exists [Mus02, Corollary 1.5]. As a result of [ELS03, Mus02, LM09, Cut13], $\mathrm{e}(\mathfrak{a}_\bullet) = \operatorname{vol}(\mathfrak{a}_\bullet)$ and $\lim_{m\to\infty} \frac{\ell(\mathcal{O}_{X,x}/\mathfrak{a}_m)}{m^n/n!}$ exists.

Let v be a valuation centered at a closed point $x \in X$. The *volume* of v, as defined in [ELS03], is given by

$$\operatorname{vol}(v) := \operatorname{vol}(\mathfrak{a}_\bullet(v)) = \mathrm{e}(\mathfrak{a}_\bullet(v)).$$

The volume function is homogeneous of order $-n$, i.e. $\operatorname{vol}(c \cdot v) = c^{-n} \operatorname{vol}(v)$ for $c > 0$.

2.2.10 Integral closure

Let X be a normal variety and $\mathfrak{a} \subset \mathcal{O}_X$ a nonzero ideal. Write $\nu : X^+ \to X$ for the normalized blowup up X along $\mathfrak{a}$ and $\mathfrak{a} \cdot \mathcal{O}_{X^+} = \mathcal{O}_{X^+}(-F)$, where F is a Cartier divisor on X^+. The ideal $\overline{\mathfrak{a}} := \nu_*(\mathcal{O}_{X^+}(-F))$ is called the *integral closure* of $\mathfrak{a}$. Note that $\mathfrak{a} \subset \overline{\mathfrak{a}}$. If $\mathfrak{a}$ is $\mathfrak{m}_x$-primary for some closed point $x \in X$, then $\mathrm{e}(\mathfrak{a}) = \mathrm{e}(\overline{\mathfrak{a}})$.

The integral closure may also be expressed using valuations [Laz04, Example 9.6.8]. Indeed, if $U \subset X$ is an open affine set, then

$$\overline{\mathfrak{a}}(U) = \{f \in \mathcal{O}_X(U) \,|\, v(f) \geq v(\mathfrak{a}) \text{ for all } v \in \mathrm{DivVal}_U\}.$$

Chapter 3

Valuation space and singularities

In this section, we work in the setting of log pairs and collect results on log discrepancies, multiplier ideals, and log canonical thresholds. While this material is known to experts, much of these results are not written for log pairs. The results for smooth varieties and normal varieties can be found in [JM12, BdFFU15].

3.1 Structure of the valuation space

We (roughly) follow [BdFFU15, Section 2] in recalling information from [JM12] on the structure of the valuation space.

Definition 3.1.1. A *normalizing subscheme* of a variety X is a non-trivial closed subscheme $N \subset X$ that contains X_{sing}. We write

$$\mathrm{Val}_X^N = \{v \in \mathrm{Val}_X \mid v(\mathcal{I}_N) = 1\},$$

for the *normalized valuation space* of X defined by N.

Note that

$$\mathbf{R}_+^* \cdot \mathrm{Val}_X^N = \{v \in \mathrm{Val}_X \mid v(\mathcal{I}_N) > 0\}$$

and

$$\mathrm{Val}_X^* = \bigcup_{N \subset X} \mathbf{R}_+^* \cdot \mathrm{Val}_X^N$$

where the union runs through all normalizing subschemes of X.

Definition 3.1.2. Let N be a normalizing subscheme of a variety X. We say $\pi : Y \to X$ is a *good resolution* of X if π is a proper birational morphism satisfying

- π is an isomorphism over $X \setminus N$,
- Y is smooth,
- $\mathrm{Exc}(\pi)$ and $\pi^{-1}(N)$ are both pure codimension 1 in Y, and
- the divisor $\sum_{i \in I} E_i$ given by $\pi^{-1}(N)_{\mathrm{red}}$ has simple normal crossing and the intersection $E_J := \cap_{j \in J} E_J$ is irreducible (or empty) for each subset $J \subset I$.

Let N be a normalizing subscheme of X and $\pi : Y \to X$ a good resolution. To the simple normal crossing divisor $\sum_{i \in I} E_i = \pi^{-1}(N)_{\mathrm{red}}$, we associate a set of quasimonomial valuations. Fix $J \subset I$ such that E_J is nonempty and write η_J for its generic point in Y. Next, choose local coordinates $\{y_j \mid \ j \in J\}$ for $\mathcal{O}_{Y,\eta_J}$ such that y_j locally defines E_j for each $j \in J$. To each vector $(\alpha_i \,|\, i \in I) \in \mathbf{R}_+^I$ with $\alpha_i = 0$ for each $i \in I \setminus J$, we associate the quasimonomial valuation v_α, as in Section 2.2.3, satisfying $v(y_j) = \alpha_j$ for each $j \in J$. Note that

$$v_\alpha(\mathcal{I}_N) = \sum_{j \in J} \alpha_j \operatorname{ord}_{E_j}(\mathcal{I}_N).$$

We write $\Delta_\pi^N \subset \mathrm{Val}_X^N$ for the set of quasimonomial valuations v_α, as above, satisfying $v_\alpha(\mathcal{I}_N) = 1$.

Viewing Δ_π^N as a subset of $\mathbf{R}_+^I$ gives a geometric realization of the dual complex of $\sum_{i \in I} E_i$. Recall that the dual complex of $\sum_{i \in I} E_i$ is the simplicial complex whose vertices are in bijection with elements of I and m-simplices are in bijections with subsets $J \subset I$ of cardinality m such that $E_J \neq \emptyset$.

Given $v \in \mathrm{Val}_X^N$, we write $r_\pi^N(v)$ for the valuation in Δ_π^N that takes value $v(E_i)$ on E_i. See [JM12, Section 4.3] for further details. This gives a continuous *retraction* map

$$r_\pi^N : \mathrm{Val}_X^N \to \Delta_\pi^N.$$

If π and π' are good resolutions of N, we write $\pi \leq \pi'$ when π' factors through π. In this case $r_\pi^N \circ r_{\pi'}^N = r_\pi^N$ [JM12, Lemma 4.6].

Theorem 3.1.3. *[JM12] If N is a normalizing subscheme of X, then*

$$\mathrm{Val}_X^N = \overline{\bigcup_\pi \Delta_\pi^N},$$

where the union runs through all good resolutions of N. Furthermore, if $v \in \mathrm{Val}_X^N$, then $v = \sup_\pi r_\pi^N(v)$.

Definition 3.1.4. We call σ a *face* of Val_X^N if there exists a good resolution π such that σ is a face of Δ_π^N. We say that a function $\mathrm{Val}_X^N \to \mathbf{R} \cup \{+\infty\}$ is *affine* (resp., convex) on σ if it is affine (resp., convex) on the previously described embedding of σ into $\mathbf{R}^I$.

Remark 3.1.5. If $\pi : Y \to X$ is a good resolution of N and D is a $\mathbf{Q}$-divisor supported on $\pi^{-1}(N)_{\mathrm{red}}$, then $v \mapsto v(D)$ is affine on Δ_π^N. Indeed, if $D = \sum_{i \in I} c_i E_i$ and $v_\alpha \in \Delta_\pi^N$, then

$$v_\alpha(D) = \sum_{i \in I} \alpha_i \operatorname{ord}_{E_i}(D) = \sum_{i \in I} (\alpha_i \cdot c_i) .$$

Furthermore, it follows from the definition of r_π^N that $v(D) = r_\pi^N(D)$ for all $v \in \mathrm{Val}_X^N$.

Proposition 3.1.6. *Let N be a normalizing subscheme of X and $\mathfrak{a} \subset \mathcal{O}_X$ an ideal. If π is a good resolution of N, then $r_\pi^N(v)(\mathfrak{a}) \leq v(\mathfrak{a})$ for all $v \in \mathrm{Val}_X^N$. Furthermore, if N contains the zero locus of $\mathfrak{a}$ and π dominates the blowup of $\mathfrak{a}$, then v is affine on the faces of Δ_π^N and $r_\pi^N(v)(\mathfrak{a}) = v(\mathfrak{a})$.*

Proof. See [BdFFU15, Proposition 2.4] for a proof of the statement. □

Proposition 3.1.7. *Let N be a normalizing subscheme of X and $\mathfrak{a}_\bullet$ a graded sequence of ideals on X.*

(i) The function $v \mapsto v(\mathfrak{a}_\bullet)$ is upper semicontinuous on Val_X.

(ii) If π is a good resolution of N and $v \in \mathrm{Val}_X^N$, then $r_\pi^N(v)(\mathfrak{a}_\bullet) \leq v(\mathfrak{a}_\bullet)$.

(iii) If N contains the zero locus of $\mathfrak{a}_{m'}$ for some $m' \in M(\mathfrak{a}_\bullet)$, then $v \mapsto v(\mathfrak{a}_\bullet)$ is bounded on Val_X^N.

Proof. The statements follow from the previous proposition and the equation $v(\mathfrak{a}_\bullet) = \inf_m \frac{v(\mathfrak{a}_m)}{m}$. □

Definition 3.1.8. In further sections, we will work in the settings of pairs. If (X, B) is a pair, we say N is a *normalizing subscheme* of (X, B) if N is a normalizing subscheme of X and N contains $\mathrm{Supp}(B)$. Note that if N is normalizing subscheme of (X, B) and $\pi : Y \to X$ is a good resolution of N, then π is also a log resolution of the triple $(X, B, \mathcal{I}_N)$ and B_Y is supported on $\pi^{-1}(N)_{\mathrm{red}}$.

3.2 Log discrepancies

In this section we define the *log discrepancy* function associated to a pair (X, B). Note that such a function was constructed for smooth varieties in [JM12] and normal varieties in [BdFFU15].

In order to define the *log discrepancy* function on the space of valuations, we first define the function for divisorial valuations. Let (X, B) be a log pair, and consider a valuation $v = c \cdot \mathrm{ord}_E \in \mathrm{Val}_X$, where $Y \to X$ is a proper birational morphism with Y normal, $E \subset Y$ a prime divisor, and $c \in \mathbf{Q}_+^*$. We set

$$A_{X,B}(v) = c\,(1 + \text{the coefficient of } E \text{ in } -B_Y)\,.$$

It is straightforward to check that $A_{X,B}(v)$ is independent of the morphism $Y \to X$. Therefore, this gives a function

$$A_{X,B} : \mathrm{DivVal}_X \to \mathbf{R}\,.$$

Theorem 3.2.1. *There is a unique extension of $A_{X,B}$ to a homogeneous lower semicontinuous function $A_{X,B} : \mathrm{Val}_X \to \mathbf{R} \cup \{+\infty\}$ such that for each normalizing subscheme N of (X, B) the following hold:*

(i) The function $A_{X,B}$ is affine on the faces of Val_X^N.

(ii) If $v \in \mathrm{Val}_X^N$, then $A_{X,B}(v) = \sup_\pi A_{X,B}(r_\pi^N(v))$, where the supremum runs through all good resolutions of N.

Following the argument in [BdFFU15], we prove the existence of such a function by reducing to the smooth case. The following result was proved in [JM12].

Proposition 3.2.2. *If X is smooth and $B = 0$, then Theorem 3.2.1 holds. Additionally, if $\pi : Y \to X$ is a proper birational morphism of smooth varieties, then $A_{X,0}(v) = A_{Y,0}(v) + v(K_{Y/X})$ for all $v \in \mathrm{Val}_X$.*

Proof. See [JM12, Proposition 5.1]. □

Proposition 3.2.3. *Let (X, B) be a log pair and $\pi : Y \to X$ be a log resolution of (X, B). The function $\mathrm{Val}_X \to \mathbf{R} \cup \{+\infty\}$ defined by $v \mapsto A_{Y,0}(v) - v(B_Y)$ is independent of the choice of log resolution $\pi : Y \to X$. Furthermore, the function is homogeneous, lower semicontinuous, and agrees with $A_{X,B} : \mathrm{DivVal}_X \to \mathbf{R}$ on DivVal_X.*

Proof. We first prove that the function is independent of the choice of π. Since any two log resolutions of (X, B) may be dominated by a third log resolution, it is sufficient to consider the case when there are morphisms $Z \xrightarrow{\rho} Y \xrightarrow{\pi} X$ such that π and $\rho \circ \pi$ are both log resolutions of (X, B) and then show

$$A_{Y,0}(v) - v(B_Y) = A_{Z,0}(v) - v(B_Z) \tag{3.1}$$

for all $v \in \mathrm{Val}_X$. Applying the previous theorem gives $A_{Y,0}(v) = A_{Z,0}(v) + v(K_{Z/Y})$. Since $B_Z = \rho^* B_Y - K_{Z/Y}$, we have $v(B_Z) = v(B_Y) - v(K_{Z/Y})$ for all $v \in \mathrm{Val}_X$. Thus, (3.1) holds.

Next, we show the second statement. It is clear that the function is homogeneous and lower semicontinuous, since both $v \mapsto A_{Y,0}(v)$ and $v \mapsto -v(B_Y)$ satisfy these properties. To show the function agrees with $A_{X,B}$ on DivVal_X, fix $v \in \mathrm{DivVal}_X$. We may find a log resolution $W \to X$ of (X, B) and a prime divisor E on W such that $v = c \cdot \mathrm{ord}_E$ for some $c \in \mathbf{Q}_+^*$. Now,

$$A_{X,B}(c \cdot \mathrm{ord}_E) = c(1 - \mathrm{ord}_E(B_W)) = A_{W,0}(v) - v(B_W).$$

By the previous paragraph, $A_{W,0}(v) - v(B_W) = A_{Y,0}(v) - v(B_Y)$, and the proof is complete. $\square$

Proof of Theorem 3.2.1. The uniqueness of such a function is clear. Indeed, property (i) combined with the definition of $A_{X,B}$ on DivVal_X uniquely characterizes $A_{X,B}$ on the faces of Val_X^N. After $A_{X,B}$ is determined on the faces of Val_X^N, property (ii) determines the function on the remaining valuations in Val_X^N.

To prove the existence of $A_{X,B}$, we look at the function considered in Proposition 3.2.3. Write $\tilde{A} : \mathrm{Val}_X \to \mathbf{R} \cup \{+\infty\}$ for this function. In light of Proposition 3.2.3, we are left to show $\tilde{A}$ satisfies properties (i) and (ii).

Fix a normalizing subscheme N of (X, B) and $\pi : Y \to X$, a good resolution of N. By Proposition 3.2.2, $v \mapsto A_{Y,0}(v)$ is affine on Δ_π^N. Since B_Y is supported on $\pi^{-1}(N)_{\mathrm{red}}$, $v \mapsto v(B_Y)$ is affine on Val_X^N. Hence, $\tilde{A}$ satisfies (i). For (ii), we note that Proposition 3.2.2 implies $A_{Y,0}(v) = \sup_{\pi' \geq \pi} A_{Y,0}(r_\pi^N(v))$, where the supremum runs through all good resolutions π' of N that factor through π. Since $v(B_Y) = r_{\pi'}^N(v)(B_Y)$ for such π', we conclude $\tilde{A}(v) = \sup_{\pi' \geq \pi} A_{Y,0}(v)$. $\square$

In the following propositions we describe various properties of the log discrepancy function.

Proposition 3.2.4. *Let (X,B) be a pair.*

(i) If $\pi : Y \to X$ is a proper birational morphism with Y normal, then $A_{X,B}(v) = A_{Y,B_Y}(v)$ for all $v \in \mathrm{Val}_X$.

(ii) If B' is a $\mathbf{Q}$-Cartier divisor, then $A_{X,B+B'}(v) = A_{X,B}(v) - v(B') =$ for all $v \in \mathrm{Val}_X$.

Proof. It is straightforward to show that the above formulas hold for all $v \in \mathrm{DivVal}_X$. Applying Theorem 3.2.1 gives that the formulas hold for all $v \in \mathrm{Val}_X$. □

Proposition 3.2.5. *Let (X,B) be a pair, N a normalizing subscheme of (X,B), and $\pi : Y \to X$ a good resolution of N. For any $v \in \mathrm{Val}_X^N$, we have $A_{X,B}(r_\pi^N(v)) \leq A_{X,B}(v)$ and equality holds if and only if $v \in \Delta_\pi^N$.*

Proof. The statement reduces to a result in [JM12]. Indeed, combining Proposition 3.2.4 with the equality $v(B_Y) = r_\pi^N(v)(B_Y)$ yields

$$A_{X,B}(v) - A_{X,B}(r_\pi^N(v)) = A_{Y,0}(v) - A_{Y,0}(r_\pi^N(v)).$$

Since Y is smooth, we may apply [JM12, Corollary 5.4] to see $A_{Y,0}(v) \geq A_{Y,0}(r_\pi^N(v))$ and equality holds if and only if $v \in \Delta_\pi^N$. Hence, the result follows. □

Proposition 3.2.6. *Let (X,B) be a pair and N a normalizing subscheme of (X,B).*

(ii) If (X,B) is klt (resp., log canonical), then $A_{X,B}(v) > 0$ (resp., ≥ 0) for all $v \in \mathrm{Val}_X^N$.

(iii) For each $a \in \mathbf{R}$, the set $\mathrm{Val}_X^N \cap \{A_{X,B} \leq a\}$ is compact.

Proof. For (i), assume (X,B) is klt (resp., log canonical). Thus, $A_{X,B} > 0$ (resp., ≥ 0) on DivVal_X. Now, let π be a good resolution of N. Since $A_{X,B}(v) \geq A_{X,B}(r_\pi^N(v))$ for all $v \in \mathrm{Val}_X^N$, in order to show $A_{X,B} > 0$ (resp., ≥ 0) on Val_X^N, it is sufficient to show the inequality on Δ_π^N. We know that inequality holds on the vertices of Δ_π^N (the vertices correspond to divisorial valuations). Since $A_{X,B}$ is affine on Δ_π^N, the inequality holds on Δ_π^N and (i) is complete.

For (ii), fix a good resolution $\pi : Y \to X$ of N. By [BdFFU15, Lemma 3.4], the set $\{A_{Y,0} \leq a'\} \cap \mathrm{Val}_X^N$ is compact for each $a' \in \mathbf{R}$. Note that $A_{X,B}(v) = A_{Y,0}(v) + v(B_Y)$ and $v(B_Y)$ is bounded on Val_X^N. Therefore, given $a \in \mathbf{R}$, we may find a' so that $\{A_{X,B} \leq a\} \cap \mathrm{Val}_X^N$ is contained in $\{A_{Y,0} \leq a'\} \cap \mathrm{Val}_X^N$. Since $A_{X,B}$ is lower semicontinuous, $\{A_{X,B} \leq a\} \cap \mathrm{Val}_X^N$ is closed in $\{A_{Y,0} \leq a'\} \cap \mathrm{Val}_X^N$, and, hence, compact. □

3.3 Multiplier ideals

In this section, we recall basic properties of multiplier ideals. See [Laz04] for further details and applications.

Definition 3.3.1. Let (X, B) be an effective pair and $\mathfrak{a} \subset \mathcal{O}_X$ a nonzero ideal. Fix a log resolution $\pi : Y \to X$ of the triple $(X, B, \mathfrak{a})$ with $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. For $c \in \mathbf{Q}_+^*$, the *multiplier ideal* $\mathcal{J}((X, B), \mathfrak{a}^c)$ is defined as

$$\mathcal{J}((X, B), \mathfrak{a}^c) := \pi_* \mathcal{O}_Y(\lceil -B_Y - cF \rceil).$$

It is a basic fact that $\mathcal{J}((X, B), \mathfrak{a}^c) \subset \mathcal{O}_X$ and is an ideal. Furthermore, the ideal is independent of the choice of π.

We use the convention that if $\mathfrak{a} = (0)$ is the zero ideal, then $\mathcal{J}((X, B), \mathfrak{a}^c) = (0)$. When $c = 1$, we simply write $\mathcal{J}((X, B), \mathfrak{a})$ for $\mathcal{J}((X, B), \mathfrak{a}^c)$. Note that if $m \in \mathbf{N}^*$, then $\mathcal{J}((X, B), \mathfrak{a}^m) = \mathcal{J}((X, B), \mathfrak{b})$, where $\mathfrak{b}$ is the standard m-th power of $\mathfrak{a}$.

If $\mathfrak{a}_1, \ldots, \mathfrak{a}_r \subset \mathcal{O}_X$ are nonzero ideals and $c_1, \ldots, c_r \in \mathbf{Q}_+^*$, we may define the *mixed multiplier ideal* $\mathcal{J}((X, B), \mathfrak{a}_1^{c_1} \cdot \cdots \cdot \mathfrak{a}_r^{c_r})$ in a similar manner. Fix a morphism $\pi\colon Y \to X$ that is a resolution of the triple $(X, B, \mathfrak{a}_1 \cdot \cdots \cdot \mathfrak{a}_r)$ with $\mathfrak{a}_i \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_i)$. We set

$$\mathcal{J}((X, B), {\mathfrak{a}_1}^{c_1} \cdot \cdots \cdot \mathfrak{a}_r^{c_r}) := \pi_* \mathcal{O}_Y(\lceil -B_Y - c_1 F_1 - \cdots - c_r F_r \rceil).$$

Note that if we choose $b, a_1, \ldots, a_r \in \mathbf{Q}_+^*$ such that $c_i = a_i/b$, then the above mixed multiplier ideal is equal to the usual multiplier ideal

$$\mathcal{J}\left((X, B), (\mathfrak{a}_1^{a_1} \cdot \cdots \cdot \mathfrak{a}_r^{a_r})^{1/b}\right).$$

Proposition 3.3.2. *An effective triple $(X, B, \mathfrak{a}^c)$ is klt if and only if $\mathcal{J}((X, B), \mathfrak{a}^c) = \mathcal{O}_X$.*

Proof. Let $\pi : Y \to X$ be a log resolution of $(X, B, \mathfrak{a}^c)$ with $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. Now, $\mathcal{J}((X, B), \mathfrak{a}^c) = \mathcal{O}_X$ if and only if $\lceil -B_Y - cF \rceil$ is effective. The latter is equivalent to $-B_Y - cF$ having coefficients > -1, which is precisely the condition for $(X, B, \mathfrak{a}^c)$ to be klt. □

Multiplier ideals satisfy the following containment relations. See [Laz04, Proposition 9.2.32] for variants of these statements in the case when X is smooth and $B = 0$.

Lemma 3.3.3. *Let (X, B) be an effective pair, $\mathfrak{a}_1, \mathfrak{a}_2 \subset \mathcal{O}_X$ nonzero ideals, and $c_1, c_2 \in \mathbf{Q}_+^*$.*

(i) If $\mathfrak{a}_i \subset \mathfrak{b}_i$ *, then*

$$\mathcal{J}((X,B), \mathfrak{a}_1^{c_1} \cdot \mathfrak{a}_2^{c_2}) \subset \mathcal{J}((X,B), \mathfrak{b}_1^{c_1} \cdot \mathfrak{b}_2^{c_2}).$$

(ii) If $c_i > d_i > 0$ *are rational numbers, then*

$$\mathcal{J}((X,B), \mathfrak{a}_1^{c_1} \cdot \mathfrak{a}_2^{c_2}) \subset \mathcal{J}((X,B), \mathfrak{a}_1^{d_1} \cdot \mathfrak{a}_2^{d_2}).$$

(iii) If (X,B) *is klt, then*

$$\mathfrak{a}_1 \subset \mathcal{J}((X,B), \mathfrak{a}_1).$$

(iv) We have

$$\mathfrak{a}_1 \cdot \mathcal{J}((X,B), \mathfrak{a}_2^{c_2}) \subset \mathcal{J}((X,B), \mathfrak{a}_1 \cdot \mathfrak{a}_2^{c_2}).$$

An essential property of multiplier ideals is the following *subadditivity* property. The version we state appears in [Tak13] and generalizes [DEL00, Tak06, Eis11].

Theorem 3.3.4. *Let* (X,B) *be an effective pair and* $r \in \mathbf{N}^*$ *such that* $r(K_X + B)$ *is Cartier. If* $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_X$ *are nonzero ideals and* $c, d \in \mathbf{Q}_+^*$*, then*

$$\mathrm{Jac}_X \cdot \mathcal{J}((X,B), \mathfrak{a}^c \cdot \mathfrak{b}^d \cdot \mathcal{O}_X(-rB)^{1/r}) \subset \mathcal{J}((X,B), \mathfrak{a}^c) \cdot \mathcal{J}((X,B), \mathfrak{b}^d),$$

where Jac_X *denotes the Jacobian ideal of* X*.*

The previous result implies the following statement, which will be more useful for our purposes.

Corollary 3.3.5. *Let* (X,B) *be an effective pair and* $r \in \mathbf{N}^*$ *such that* $r(K_X + B)$ *is Cartier. If* $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_X$ *are nonzero ideals and* $c, d \in \mathbf{Q}_+^*$*, then*

$$\mathrm{Jac}_X \cdot \mathcal{O}_X(-rB) \cdot \mathcal{J}((X,B), \mathfrak{a}^c \cdot \mathfrak{b}^d) \subset \mathcal{J}((X,B), \mathfrak{a}^c) \cdot \mathcal{J}((X,B), \mathfrak{b}^d),$$

where Jac_X *denotes the Jacobian ideal of* X*.*

Proof. Lemma 3.3.3 implies

$$\begin{aligned} \mathcal{O}_X(-rB) \cdot \mathcal{J}((X,B), \mathfrak{a}^c \cdot \mathfrak{b}^d) &\subset \mathcal{J}((X,B), \mathfrak{a}^c \cdot \mathfrak{b}^d \cdot \mathcal{O}_X(-rB)) \\ &\subset \mathcal{J}((X,B), \mathfrak{a}^c \cdot \mathfrak{b}^d \cdot \mathcal{O}_X(-rB)^{1/r}). \end{aligned}$$

Applying the previous theorem yields the desired formula. $\square$

3.3.1 Asymptotic multiplier ideals

Definition 3.3.6. Let (X,B) be an effective pair, $\mathfrak{a}_\bullet$ a graded sequence of ideals on X, and $c \in \mathbf{Q}_+^*$. By Lemma 3.3.3, we have

$$\mathcal{J}\left((X,B),\mathfrak{a}_p^{c/p}\right) \subset \mathcal{J}\left((X,B),\mathfrak{a}_{pq}^{c/(pq)}\right)$$

for all positive integers p,q. This, together with the Noetherianity of X, implies

$$\left\{\mathcal{J}\left((X,B),\mathfrak{a}_p^{c/p}\right)\right\}_{p\in\mathbf{N}}$$

has a unique maximal element, which we call the *c-th asymptotic multiplier ideal* and denote by $\mathcal{J}\left((X,B),\mathfrak{a}_\bullet^c\right)$. Note that $\mathcal{J}\left((X,B),\mathfrak{a}_\bullet^c\right) = \mathcal{J}\left((X,B),\mathfrak{a}_p^{c/p}\right)$ for all p divisible enough.

Corollary 3.3.7. *Let (X,B) be a klt pair, and fix $r \in \mathbf{N}^*$ such that $r(K_X+B)$ is Cartier. If $\mathfrak{a}_\bullet$ is a graded sequence of ideals on X, then*

$$(\mathrm{Jac}_X \cdot \mathcal{O}_X(-rB))^{m-1} \cdot \mathcal{J}((X,B),\mathfrak{a}_\bullet^{mc}) \subset \mathcal{J}((X,B),\mathfrak{a}_\bullet^c)^m.$$

for all $m,\ell \in \mathbf{N}^*$.

Proof. The theorem can be deduced from Corollary 3.3.5 and the definition of the asymptotic multiplier ideal. See [Laz04, Theorem 11.2.3] for a similar argument in the case when X is smooth and $B=0$. □

3.3.2 Multiplier ideals and valuations

Following [BdFFU15], we explain that (asymptotic) multiplier ideals may be defined valuatively.

Proposition 3.3.8. *Let (X,B) be an effective pair, $\mathfrak{a} \subset \mathcal{O}_X$ a nonzero ideal, and $c \in \mathbf{Q}_+^*$. Fix an open affine subset $U \subset X$ and $f \in \mathcal{O}_X(U)$. The following conditions are equivalent:*

(i) $f \in \mathcal{J}((X,B),\mathfrak{a}^c)(U)$.

(ii) $v(f) > cv(\mathfrak{a}) - A_{X,B}(v)$ *for all* $v \in \mathrm{DivVal}_U$.

(iii) $v(f) > cv(\mathfrak{a}) - A_{X,B}(v)$ *for all* $v \in {\mathrm{Val}_U}^*$.

Proof. It is sufficient to consider the case when X is affine and $U = X$. Assume this is the case, and choose a log resolution of $\pi : Y \to X$ of $(X, B, \mathfrak{a})$ with $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. By the definition of the multiplier ideal, $f \in \mathcal{J}((X, B), \mathfrak{a}^c)(X)$ if and only if

$$\mathrm{ord}_E(f) + \lceil \mathrm{ord}_E(-B_Y - cF) \rceil \geq 0$$

for all prime divisors E on Y. Since

$$\mathrm{ord}_E(-B_Y - cF) = A_{X,B}(\mathrm{ord}_E) - 1 - c \cdot \mathrm{ord}_E(\mathfrak{a})$$

and $\mathrm{ord}_E(f) \in \mathbf{Z}$, we see $f \in \mathcal{J}((X, B), c \cdot \mathfrak{a})(X)$ if and only if

$$\mathrm{ord}_E(f) > c \cdot \mathrm{ord}_E(\mathfrak{a}) - A_{X,B}(\mathrm{ord}_E)$$

for all such E. Since any divisorial valuation may be realized as coming from a log resolution of $(X, B, \mathfrak{a})$, we conclude (i) and (ii) are equivalent.

Since the implication (iii) $\implies$ (ii) is trivial, we are left to show (ii) $\implies$ (iii). Assume (ii) holds and fix a normalizing subscheme N of (X, B) that contains the zero loci of both $\mathfrak{a}$ and (f). We will show that the function $\phi : \mathrm{Val}_X^N \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\phi(v) = v(f) - c \cdot v(\mathfrak{a}) + A_{X,B}(v)$$

is > 0.

To show the previous inequality, first choose a good resolution $\pi : Y \to X$ of N that factors through the blowup of $\mathfrak{a}$. By our assumption that (ii) holds, we see $\phi > 0$ on the vertices of Δ_π^N. Since ϕ is affine on the faces of Δ_π^N (see Proposition 3.1.6 and Theorem 3.2.1), we see $\phi > 0$ on Δ_π^N. Since $\phi \geq \phi \circ r_\pi^N$ (see Proposition 3.1.6 and Theorem 3.2.1), we conclude that $\phi > 0$ on Val_X^N. $\square$

Proposition 3.3.9. *Let (X, B) be an effective pair, $\mathfrak{a}_\bullet$ a graded sequence of ideals on X, and $c \in \mathbf{Q}_+^*$ Fix an affine open subset $U \subset X$ and $f \in \mathcal{O}_X(U)$. The following conditions are equivalent:*

(i) $f \in \mathcal{J}((X, B), \mathfrak{a}_\bullet^c)(U)$.

(ii) $v(f) > cv(\mathfrak{a}_\bullet) - A_{X,B}(v)$ *for all* $v \in \mathrm{DivVal}_U$.

(iii) $v(f) > cv(\mathfrak{a}_\bullet) - A_{X,B}(v)$ *for all* $v \in {\mathrm{Val}_U}^*$.

Proof. Since $\mathcal{J}((X,B),\mathfrak{a}_\bullet^c) = \mathcal{J}((X,B),\mathfrak{a}_m^{c/m})$ for all m divisible enough, we apply the previous theorem to see i) is equivalent to the following two statements:

(ii') If $m \in \mathbf{N}^*$ is divisible enough, then $v(f) > (c/m)v(\mathfrak{a}_m) - A_{X,B}(v)$ for all $v \in \mathrm{DivVal}_U$

(iii') If $m \in \mathbf{N}^*$ is divisible enough, then $v(f) > (c/m)v(\mathfrak{a}_m) - A_{X,B}(v)$ for all $v \in \mathrm{Val}_U$

Since $v(\mathfrak{a}_\bullet) = \inf_m v(\mathfrak{a}_m)/m = \lim_m v(\mathfrak{a}_m)/m$, we see (ii) $\iff$ (ii') and (iii) $\iff$ (iii') hold. □

3.3.3 Approximation of valuation ideals

In this section, we consider the following generalization of [ELS03, Theorem A] regarding the asymptotic behavior of valuation ideals.

Proposition 3.3.10. *Let (X,B) be a klt pair and r a positive integer so that $r(K_X+B)$ is Cartier. If $v \in \mathrm{Val}_X^*$ satisfies $A_{X,B}(v) < +\infty$, then*

$$(\mathrm{Jac}_X \cdot \mathcal{O}_X(-rB))^{\ell-1}\mathfrak{a}_m(v)^\ell \subseteq (\mathrm{Jac}_X \cdot \mathcal{O}_X(-rB))^{\ell-1}\mathfrak{a}_{m\cdot\ell}(v) \subseteq (\mathfrak{a}_{m-A_{X,B}(v)}(v))^\ell$$

for all $m,\ell \in \mathbf{N}^*$.

The proposition is a consequence of properties of asymptotic multiplier ideals. Before beginning the proof, we prove the following statements.

Lemma 3.3.11. *If X is variety and $v \in \mathrm{Val}_X^*$, then $v(\mathfrak{a}_\bullet(v)) = 1$.*

Proof. By the definition of $\mathfrak{a}_m(v)$, we have $v(\mathfrak{a}_m(v)) \geq m$. Next, set $\alpha := v(\mathfrak{a}_1(v))$. We have $\mathfrak{a}_1(v)^{\lceil m/\alpha\rceil} \subseteq \mathfrak{a}_m(v)$, since $v(\mathfrak{a}_1(v)^{\lceil m/\alpha\rceil}) = \alpha\lceil m/\alpha\rceil \geq m$. Thus,

$$v(\mathfrak{a}_m(v)) \leq v(\mathfrak{a}_1(v)^{\lceil m/\alpha\rceil}) = \alpha\lceil m/\alpha\rceil.$$

The previous two bounds combine to show

$$1 \leq \frac{v(\mathfrak{a}_m(v))}{m} \leq \frac{\alpha\cdot\lceil m/\alpha\rceil}{m},$$

and the result follows. □

Proposition 3.3.12. *Let (X,B) be an effective pair. If $v \in \mathrm{Val}_X^*$ satisfies $A_{X,B}(v) < +\infty$ and $c \in \mathbf{Q}_+^*$, then*

$$\mathcal{J}((X,B),\mathfrak{a}_\bullet(v)^c) \subset \mathfrak{a}_{c-A_{X,B}(v)}(v).$$

Proof. By Proposition 3.3.9, we see

$$\mathcal{J}((X,B),\mathfrak{a}_\bullet(v)^c)\subset \mathfrak{a}_m(v),$$

where $m = c\cdot v(\mathfrak{a}_\bullet(v)) - A_{X,B}(v)$. Applying the previous lemma completes the proof. $\square$

Proof of Proposition 3.3.10. Since $\mathfrak{a}_\bullet(v)$ is a graded sequence of ideals, $(\mathfrak{a}_m(v))^\ell \subset \mathfrak{a}_{m\cdot\ell}(v)$. This proves the first inclusion.

To prove the second inclusion, we observe

$$\begin{aligned}(\mathrm{Jac}_X\cdot\mathcal{O}_X(-rB))^{\ell-1}\mathfrak{a}_{m\cdot\ell}(v) &\subset (\mathrm{Jac}_X\cdot\mathcal{O}_X(-rB))^{\ell-1}\mathcal{J}((X,B),\mathfrak{a}_{m\cdot\ell})\\ &\subset (\mathrm{Jac}_X\cdot\mathcal{O}_X(-rB))^{\ell-1}\mathcal{J}((X,B),\mathfrak{a}_\bullet^{m\cdot\ell})\\ &\subset \mathcal{J}((X,B),\mathfrak{a}_\bullet^{m})^\ell,\end{aligned}$$

where the first inclusion follows from Lemma 3.3.3, the second from the definition of the asymptotic multiplier ideal, and the third from Corollary 3.3.7. Applying Proposition 3.3.12 completes the proof. $\square$

3.4 Log canonical thresholds

Definition 3.4.1. Let (X,B) be a klt pair. For a nonzero ideal $\mathfrak{a}\subset\mathcal{O}_X$, the *log canonical threshold* of $\mathfrak{a}$ with respect to (X,B) is given by

$$\mathrm{lct}(X,B;\mathfrak{a}) = \sup\{c\in\mathbf{Q}_+^* \mid \mathcal{J}((X,B),\mathfrak{a}^c)=\mathcal{O}_X\}.$$

This invariant measures the singularities of the subscheme cut out by $\mathfrak{a}$. Note that $\mathrm{lct}(X,B;\mathcal{O}_X)=+\infty$. By convention, we set $\mathrm{lct}(X,B;\mathfrak{a})=0$, when $\mathfrak{a}$ is the zero ideal.

Similarly, if $\mathfrak{a}_\bullet$ is a graded sequence of ideals on X, then *log canonical threshold* of $\mathfrak{a}_\bullet$ is given by

$$\mathrm{lct}(X,B;\mathfrak{a}_\bullet) := \sup\{c\in\mathbf{Q}_+^* \mid \mathcal{J}((X,B),\mathfrak{a}_\bullet^c)=\mathcal{O}_X\}.$$

When the choice of the pair (X,B) is clear, we will often simply write $\mathrm{lct}(\mathfrak{a})$ and $\mathrm{lct}(\mathfrak{a}_\bullet)$.

Proposition 3.4.2. *If (X,B) is a klt pair and $\mathfrak{a}\subset\mathcal{O}_X$ is a nonzero ideal, then*

$$\mathrm{lct}(X,B;\mathfrak{a}) = \inf_{v\in\mathrm{DivVal}_X}\frac{A_{X,B}(v)}{v(\mathfrak{a})} = \inf_{v\in\mathrm{Val}_X^*}\frac{A_{X,B}(v)}{v(\mathfrak{a})}.[1]$$

[1]We use the convention that if either $A_{X,B}(v)=+\infty$ or $v(\mathfrak{a})=0$, then $A_{X,B}(v)/v(\mathfrak{a})=+\infty$. Similarly, if $\mathfrak{a}_\bullet$ is graded sequence of ideals, we set $A_{X,B}(v)/v(\mathfrak{a}_\bullet)=+\infty$ if either $A_{X,B}(v)=+\infty$ or $v(\mathfrak{a}_\bullet)=0$.

Furthermore, if $\pi : Y \to X$ is a log resolution of $(X, B, \mathfrak{a})$ and $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, then the above infimum is achieved by a valuation of the form ord_E where E is a prime divisor in the support of F.

Proof. By Proposition 3.3.12, $\mathcal{J}((X,B),\mathfrak{a}^c) = \mathcal{O}_X$ if and only if $v(1) > cv(\mathfrak{a}) - A_{X,B}(v)$ for all $v \in \mathrm{DivVal}_X$ (resp., $v \in \mathrm{Val}_X^*$). Noting that $v(1) = 0$ for all $v \in \mathrm{Val}_X$ yields the desired formulas. Using a similar argument and the definition of $\mathcal{J}((X,B),\mathfrak{a}^c)$ in terms of a log resolution implies the last statement of the proposition. □

Proposition 3.4.3. *If (X, B) is a klt pair and $\mathfrak{a}_\bullet$ a graded sequence of ideals on X, then*

$$\mathrm{lct}(X, B; \mathfrak{a}_\bullet) = \inf_{v \in \mathrm{DivVal}_X} \frac{A_X(v)}{v(\mathfrak{a}_\bullet)} = \inf_{v \in \mathrm{Val}_X^*} \frac{A_{X,B}(v)}{v(\mathfrak{a}_\bullet)}.$$

Proof. The proof is the same as the proof for the similar statement in Proposition 3.4.2, but uses Proposition 3.3.9 rather than 3.3.8. □

Proposition 3.4.4. *If $\mathfrak{a}_\bullet$ is a graded sequence of ideals on X, then*

$$\mathrm{lct}(X, B; \mathfrak{a}_\bullet) = \lim_{M(\mathfrak{a}_\bullet) \ni m \to \infty} m \cdot \mathrm{lct}(X, B; \mathfrak{a}_m) = \sup_{m \in M(\mathfrak{a}_\bullet)} m \cdot \mathrm{lct}(X, B; \mathfrak{a}_m).$$

Proof. Following the argument in [Mus02], we begin by showing $\lim_{M(\mathfrak{a}_\bullet) \ni m \to \infty} m \cdot \mathrm{lct}(X, B; \mathfrak{a}_m)$ exists and equals $\sup_{m \in M(\mathfrak{a}_\bullet)} m \cdot \mathrm{lct}(\mathfrak{a}_m)$. Since $\mathfrak{a}_m \cdot \mathfrak{a}_p \subset \mathfrak{a}_{m+p}$, we see

$$\frac{v(\mathfrak{a}_{m+p})}{A_{X,B}(v)} \leq \frac{v(\mathfrak{a}_m)}{A_{X,B}(v)} + \frac{v(\mathfrak{a}_p)}{A_{X,B}(v)}.$$

for all $v \in \mathrm{DivVal}_X$. By Proposition 3.4.3, it follows that

$$\frac{1}{\mathrm{lct}(\mathfrak{a}_{m+p})} \leq \frac{1}{\mathrm{lct}(\mathfrak{a}_m)} + \frac{1}{\mathrm{lct}(\mathfrak{a}_p)}.$$

for all $m, p \in M(\mathfrak{a}_\bullet)$. Applying [JM12, Lemma 2.3] yields the desired statement.

We now move on to show $\mathrm{lct}(X, B; \mathfrak{a}_\bullet) = \lim_{M(\mathfrak{a}_\bullet) \ni m \to \infty} m \cdot \mathrm{lct}(X, B; \mathfrak{a}_m)$. Fix $c \in \mathbf{Q}_+^*$, and note that $c < \mathrm{lct}(\mathfrak{a}_\bullet)$ if and only if $\mathcal{J}((X,B),\mathfrak{a}_\bullet^c) = \mathcal{O}_X$. Since $\mathcal{J}((X,B),\mathfrak{a}_\bullet^c) = \mathcal{J}((X,B),\mathfrak{a}_m^{c/m})$ for all m-divisible enough, the latter condition is equivalent to $c < m \cdot \mathrm{lct}(X, B; \mathfrak{a}_m)$ for all m divisible enough. Thus, $\mathrm{lct}(\mathfrak{a}_\bullet) = \lim_{M(\mathfrak{a}_\bullet) \ni m \to \infty} m \cdot \mathrm{lct}(X, B; \mathfrak{a}_m)$ and the proof is complete. □

Proposition 3.4.5. *Let (X, B) be a klt pair and $\mathfrak{a}_\bullet$ a graded sequence of ideals. The graded sequence $\mathfrak{a}_\bullet$ is nontrivial if and only if $\mathrm{lct}(X, B; \mathfrak{a}_\bullet) < +\infty$.*

Proof. Recall that $\mathfrak{a}_\bullet$ is nontrivial if and only if there exists a divisorial valuation v on X such that $v(\mathfrak{a}_\bullet) > 0$. Therefore, the statement is an immediate consequence of Proposition 3.4.3. □

Definition 3.4.6. If (X, B) is klt pair and D a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor on X, we can also make sense of *log canonical threshold* of D. Pick $m \geq 1$ so that mD is a Cartier divisor, and set $\operatorname{lct}(X, B; D) = m \cdot \operatorname{lct}(X, B; \mathcal{O}_X(-mD))$. It is straightforward to check that this definition is independent of the choice of m and $\operatorname{lct}(X, B; D) = \sup\{\lambda \in \mathbf{Q}_+^* \mid (X, B + cD) \text{ is klt}\}$.

The following statement follows immediately from the above definition and Proposition 3.4.2.

Proposition 3.4.7. *If (X, B) is a klt pair and D is a $\mathbf{Q}$-Cartier divisor on X, then*

$$\operatorname{lct}(X, B; \mathfrak{a}) = \inf_{v \in \operatorname{DivVal}_X} \frac{A_{X,B}(v)}{v(D)} = \inf_{v \in \operatorname{Val}_X^*} \frac{A_{X,B}(v)}{v(D)}.$$

Furthermore, if $\pi : Y \to X$ is a log resolution of $(X, B + D)$, then the above infimum is achieved by a valuation of the form ord_E where E is a prime divisor in the support of $\pi^* D$.

3.4.1 Valuations computing the log canonical threshold

Definition 3.4.8. Let (X, B) be a klt pair and $\mathfrak{a}_\bullet$ a graded sequence of ideals on X. In light of Proposition 3.4.3, we say that a valuation $v_* \in \operatorname{Val}_X^*$ computes $\operatorname{lct}(X, B; \mathfrak{a}_\bullet)$ if $\operatorname{lct}(\mathfrak{a}_\bullet) = A_{X,B}(v_*)/v_*(\mathfrak{a}_\bullet)$.

Lemma 3.4.9. *If (X, B) is a klt pair and $v \in \operatorname{Val}_X^*$, then $\operatorname{lct}(X, B; \mathfrak{a}_\bullet(v)) \leq A_{X,B}(v)$ and equality holds if and only if v computes $\operatorname{lct}(X, B; \mathfrak{a}_\bullet(v))$.*

Proof. The statement is an immediate consequence of Proposition 3.4.3 and the fact that $v(\mathfrak{a}_\bullet(v)) = 1$ (Lemma 3.3.11). □

The following theorem generalizes [JM12, Theorem A] to klt pairs. Our proof is similar in technique to the proof in [JM12].

Theorem 3.4.10. *If (X, B) is a klt pair and $\mathfrak{a}_\bullet$ a graded sequence of ideals on X, then there exists a valuation $v_* \in \operatorname{Val}_X^*$ computing $\operatorname{lct}(X, B; \mathfrak{a}_\bullet)$.*

Proof. If $\operatorname{lct}(X, B; \mathfrak{a}_\bullet) = +\infty$, then any valuation $v \in \operatorname{Val}_X^*$ computes $\operatorname{lct}(X, B; \mathfrak{a}_\bullet)$. Thus, we may assume $\operatorname{lct}(X, B; \mathfrak{a}_\bullet) < +\infty$ and set $c := \operatorname{lct}(X, B; \mathfrak{a}_\bullet)$.

Fix a normalizing subscheme N of (X, B) such that N contains the zero locus of $\mathfrak{a}_{m'}$ for some $m' \in M(\mathfrak{a}_\bullet)$. We claim

$$\operatorname{lct}(X, B; \mathfrak{a}_\bullet) = \inf_{v \in \operatorname{Val}_X^N} \frac{A_{X,B}(v)}{v(\mathfrak{a}_\bullet)}.$$

Indeed, since $v \mapsto \frac{A_{X,B}(v)}{v(\mathfrak{a}_\bullet)}$ is invariant under scaling, it is sufficient to show that if $v(\mathfrak{a}_\bullet) > 0$, then $v \in \mathbf{R}_+^* \cdot \operatorname{Val}_X^N$. Now, if $v(\mathfrak{a}_\bullet) > 0$, then $v(\mathfrak{a}_{m'}) > 0$. Since N was chosen to contain the zero locus of $\mathfrak{a}_{m'}$, $v(\mathfrak{a}_{m'}) > 0$ implies $v(\mathcal{I}_N) > 0$. This completes the claim.

Next, by Proposition 3.1.7, we may choose $B \in \mathbf{R}_+$ so that $v(\mathfrak{a}_\bullet) \leq B$ for all $v \in \operatorname{Val}_X^N$. Fix $\varepsilon > 0$, and note that if $A_{X,B}(v)/v(\mathfrak{a}_\bullet) < c + \varepsilon$, then $A_{X,B}(v) \leq (c+\varepsilon)B$. Therefore,

$$\operatorname{lct}(X, B; \mathfrak{a}_\bullet) = \inf_{v \in W} \frac{A_{X,B}(v)}{v(\mathfrak{a}_\bullet)}, \tag{3.2}$$

where $W = \operatorname{Val}_X^N \cap \{A_{X,B} \leq (c+\varepsilon)B\}$.

Let $\phi : \operatorname{Val}_X^N \to \mathbf{R} \cup \{+\infty\}$ be the function defined by

$$\phi(v) = A_{X,B}(v) - c \cdot v(\mathfrak{a}_\bullet).$$

By (3.2), $\inf_{v \in W} \phi(v) = 0$. Since W is compact (Proposition 3.2.6) and ϕ is lower semicontinous (Proposition 3.1.7 and Theorem 3.2.1), there exists $v_* \in W$ such that $\phi(v_*) = A_{X,B}(v_*) - c \cdot v_*(\mathfrak{a}_\bullet) = 0$. The latter implies v_* computes $\operatorname{lct}(X, B; \mathfrak{a}_\bullet)$. □

3.4.2 Log canonical thresholds in families

In this section we prove well known results on the behavior of the log canonical threshold along a family of ideals. See [Kol96] and [Amb16] for related statements.

We consider the following setup. Let (X, B) be a klt pair and T a variety. Write $p : X \times T \to T$ for the second projection map. Set $X_t := p^{-1}(t)$ and $B_t = B \times \{t\}$. If $\mathfrak{a}$ is an ideal on $\mathcal{O}_{X \times T}$, we write $\mathfrak{a}_t := \mathfrak{a} \cdot \mathcal{O}_{X \times \{t\}}$ for each $t \in T$.

Proposition 3.4.11. *If $\mathfrak{a} \subset \mathcal{O}_{X \times T}$ is a nonzero ideal, then there exists a nonempty open set $U \subset T$ such that U is smooth and*

$$\operatorname{lct}(X \times U, B \times U; \mathfrak{a}|_{X \times U}) = \operatorname{lct}(X_t, B_t; \mathfrak{a}_t)$$

for all closed points $t \in U$.

Proof. Since we may shrink T, we may assume T is smooth. Hence, $(X \times T, B \times T)$ is a pair. Let $\pi : Y \to X \times T$ be a log resolution of $(X \times T, B \times T, \mathfrak{a})$. For $t \in T$, we set Y_t equal to the fiber of $Y \to X \times T \to T$. Write

$$K_Y - \mu^*(K_{X\times T} + B \times T) = \sum_{i=1}^{r} a_i E_i \quad \text{ and } \quad \mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y\left(-\sum_{i=1}^{r} b_i E_i\right),$$

where each E_i is a distinct prime divisor on Y. Shrinking T further, we may also assume each E_i dominates T.

By generic smoothness, there exists a nonempty open set $U \subset T$ such that $Y \to T$ is smooth over U and $\mathrm{Exc}(\pi) + \widetilde{B \times T} + \sum E_i$ has relative simple normal crossing over U. Now, for each $t \in U$, Y_t is a log resolution of $(X_t, B_t, \mathfrak{a}_t)$ and

$$K_{Y_t} - \pi^*(K_{X_t} + B_t) = \sum_{i=1}^{r} a_i E|_t \quad \text{ and } \quad \mathfrak{a}_t \cdot \mathcal{O}_{Y_t} = \mathcal{O}_Y\left(-\sum_{i=1}^{r} b_i E_i|_t\right).$$

Thus, $\mathrm{lct}(X_t, B_t; \mathfrak{a}_t) = \min_{i=1,\dots,r} \frac{a_i+1}{b_i}$ for all $t \in U$. Since the latter minimum is precisely $\mathrm{lct}(X \times U, B \times U; \mathfrak{a}|_{X\times U})$, the proof is complete. $\square$

Proposition 3.4.12. *Assume T is a smooth curve and $0 \in T$ is a closed point. If $\mathfrak{a} \subset \mathcal{O}_{X\times T}$ is a nonzero ideal such that $V(\mathfrak{a})$ is proper over T, then there exists an open set $0 \in U \subset T$ such that*

$$\mathrm{lct}(X_0, B_0; \mathfrak{a}_0) \leq \mathrm{lct}(X_t, B_t; \mathfrak{a}_t)$$

for all closed points $t \in U$.

Proof. If $\mathfrak{a}_0$ is the zero ideal, then $\mathrm{lct}((X_0, B_0), \mathfrak{a}_0) = +\infty$ and the statement holds with $U = X$. Thus, we may assume $\mathfrak{a}_0$ is nonzero and set $c = \mathrm{lct}(X_0, B_0; \mathfrak{a}_0)$. Since $(X_0, B_0, \mathfrak{a}_0^c)$ is log canonical, [KM98, Theorem 5.50] implies there exists an open set $W \subset X$ such that $X_0 \subset W$ and the restriction of $(X, B + X_0, \mathfrak{a}_0^c)$ to W is log canonical. Therefore, $(X, B, \mathfrak{a}_0^c)$ restricted to W is log canonical as well.

Now, set

$$V := \{t \in T \mid V(\mathfrak{a}) \cap X_t \subset W\}.$$

The set V is nonempty and open in T. Indeed, V contains 0. Additionally, V is the complement of $\pi(V(\mathfrak{a}) \setminus W)$ and $V(\mathfrak{a})$ is proper over T.

Next, note that $(X \times V, B \times V, \mathfrak{a}|^c_{X\times V})$ is log canonical. By Proposition 3.4.11, we may find a nonempty open set $V' \subset V$ such that

$$\mathrm{lct}(X \times V', B \times V'; \mathfrak{a}|_{X\times V'}) = \mathrm{lct}(X_t, B_t : \mathfrak{a}_t)$$

for all $t \in V'$. Therefore, $\mathrm{lct}(X_t, B_t; \mathfrak{a}_t) \geq c$ for all $t \in V'$. Setting $U := V' \cup \{0\}$ completes the proof. □

3.5 The toric case

In this section, we will describe log discrepancies and log canonical thresholds in the toric setting. Throughout, we will freely use notation and results found in [Ful93].

Let $N \simeq \mathbf{Z}^n$ be a lattice and $M = \mathrm{Hom}(N, \mathbf{Z})$ the corresponding dual lattice. We consider a pair (X, B), where $X = X(\Delta)$ is given by a rational fan $\Delta \subset N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$ and B is a torus invariant $\mathbf{Q}$-divisor on X.

The open torus of X is denoted by $T \subset X$. Let $v_1, \ldots, v_d$ denote the primitive generators of the one-dimensional cones in Δ and $D_1, \ldots, D_d$ be the corresponding torus invariant divisors on X. Hence, there exist integers $b_i \in \mathbf{Q}_+$ so that $B = \sum_{i=1}^d b_i D_i$. Since $K_X = -\sum_{i=1}^d D_i$, the condition that $K_X + B$ is $\mathbf{Q}$-Cartier translates to the following statement: for each cone $\sigma \in \Delta$, there exists $b(\sigma) \in M_{\mathbf{Q}}$ such that $\langle b(\sigma), v_i\rangle = b_i - 1$ for all $v_i \in \sigma$.

3.5.1 Toric valuations

Given $v \in |\Delta|$ (where $|\Delta|$ denotes the support of Δ), let σ be the unique cone in Δ containing v in its interior. The map

$$k[\sigma^\vee \cap M] = \bigoplus_{u\in\sigma^\vee\cap M} k \cdot \chi^u \to \mathbf{R}_+$$

defined by

$$\sum_{u\in\sigma^\vee\cap M} \alpha_u \chi^u \mapsto \min\{\langle u, v\rangle \mid \alpha_u \neq 0\} \tag{3.3}$$

gives rise to a valuation on X that we slightly abusively also denote by v. Its center on X is the generic point of $V(\sigma)$.

This induces in embedding $|\Delta| \hookrightarrow \mathrm{Val}_X$, and we shall simply view $|\Delta|$ as a subset of Val_X. These are called *toric valuations*, and we will sometimes refer to this set as Val_X^T.

The valuation associated to the point $v_i \in N_{\mathbf{R}}$ is ord_{D_i} for $1 \le i \le d$, and the valuation associated to $0 \in N_{\mathbf{R}}$ is the trivial valuation on X.

Proposition 3.5.1. *The restriction of the log discrepancy function $A_{X,B}$ to $|\Delta| \subset \mathrm{Val}_X$ is the unique function that is linear on the cones in Δ and satisfies $A(v_i) = 1 - b_i$ for $1 \le i \le d$.*

Proof. We first consider the case when X is a smooth and $B = 0$. In this case, the effective divisor $-K_Y = D_1 + \cdots + D_r$ is Cartier, and we write $N \subset X$ for the corresponding subscheme. Note that N is a normalizing subscheme of $(X, 0)$ and $\mathrm{id} : X \to X$ is a good resolution of N. Furthermore, $\Delta^N_{\mathrm{id}} = \{v \in \Delta \,|\, v(-K_Y) = 1\}$. Since $A_{X,0}$ is affine on Δ^N_{id} and takes value 1 on the vertices, we conclude $A_{X,0}(v) = v(-K_Y)$ for all $v \in |\Delta|$.

We move on to the general case. Consider any cone $\sigma \in \Delta$. Let $v_i \in N$, $1 \le i \le r$, be the generators of the 1-dimensional cones contained in σ. Since $K_X + B$ is $\mathbf{Q}$-Cartier, there exists $b(\sigma) \in M_{\mathbf{Q}}$ such that $\langle b(\sigma), v_i \rangle = -1 + b_i$ for $1 \le i \le r$. To complete the proof, we will show that $A_{X,B}(v) = -\langle b(\sigma), v \rangle$ for all $v \in \sigma$.

Pick a refinement Δ' of Δ so that $X' := X(\Delta')$ is smooth. Hence, $\pi : X' \to X$ is a proper birational morphism and $|\Delta'| = |\Delta|$. Proposition 3.2.4 implies

$$A_{X,B}(v) = A_{X',0}(v) - v(B_{X'}) = A_{X',0}(v) + v(K_{X'}) - v(\pi^*(K_X + B))$$

for all $v \in \mathrm{Val}_X$. From the smooth case, we know $A_{X',0}(v) = -v(K_{X'})$. Since $v(\pi^*(K_X + B)) = \langle b(\sigma), v \rangle$ for all $v \in \sigma$, the desired statement follows. □

3.5.2 Log canonical thresholds

The next proposition follows from the proof of [JM12, Proposition 8.1]. We say that an ideal $\mathfrak{a}$ on X is *T-invariant* if it is invariant with respect to the torus action on X. Equivalently, for each $\sigma \in \Delta$, the ideal $\mathfrak{a}(U_\sigma) \subset k[\sigma^\vee \cap M]$ is generated by monomials.

Proposition 3.5.2. *If $\mathfrak{a}_\bullet$ is a nontrivial graded sequence of T-invariant ideals on X, then there exists a toric valuation computing $\mathrm{lct}(X, B; \mathfrak{a}_\bullet)$. Further, any valuation that computes $\mathrm{lct}(X, B; \mathfrak{a}_\bullet)$ is toric.*

Proof. Pick a refinement Δ' of Δ such that $X' := X(\Delta')$ is smooth. This induces a proper birational morphism $X' \to X$. Let D' be the sum of the torus invariant divisors on X'.

By Theorem 3.4.10, there exists a valuation $w \in \mathrm{Val}_X$ computing $\mathrm{lct}(X, B; \mathfrak{a}_\bullet)$. Since $\mathrm{Val}_{X'} = \mathrm{Val}_X$, we may also view w as a valuation on X'. We now follow [JM12, §8].

Let $r_{X',D'} : \mathrm{Val}_{X'} \to \mathrm{QM}(X', D') = |\Delta|$ denote the retraction map defined in *loc. cit*, and set $v := r_{X',D'}(w) \in N_{\mathbf{R}}$. Then $v(\mathfrak{a}_\bullet) = w(\mathfrak{a}_\bullet) > 0$. Further, $A_{X',0}(v) \le A_{X',0}(w)$, with equality if and only if $w = v$. Note that $A_{X,B}(v) = A_{X'}(v) - v(B_{X'})$. Additionally, $v(B_{X'}) = w(B_{X'})$, since $\mathrm{Supp}(B_{X'}) \subset \mathrm{Supp}(D')$.

This implies $A_{X,B}(v) \le A_{X,B}(w)$, with equality if and only if $w = v$. Thus, $\mathrm{lct}(X, B; \mathfrak{a}_\bullet) \le A_X(v)/v(\mathfrak{a}_\bullet) \le A_X(w)/w(\mathfrak{a}_\bullet) = \mathrm{lct}(X, B; \mathfrak{a}_\bullet)$. Hence, $w = v$ and the proof is complete. □

Proposition 3.5.3. *If (X, B) is a klt pair and $D = c_1D_1 + \cdots c_dD_d$ is an effective $\mathbf{Q}$-Cartier divisor on X, then*

$$\mathrm{lct}(X, B; D) = \inf_{v \in N_{\mathbf{R}} \setminus \{0\}} \frac{A_{X,B}(v)}{v(D)} = \min_{i=1,\dots,d} \frac{1 - b_i}{c_i}.$$

Proof. Pick a refinement Δ' of Δ such that $X' := X(\Delta')$ is smooth. This induces a log resolution $\pi : X' \to X$ of $(X, B + D)$. Hence, there exists a prime divisor in the support of π^*D that computes $\mathrm{lct}(X, B; D)$. Since such a valuation is toric, the first inequality follows. (Alternatively, it is straightforward to deduce this equality from the previous proposition.)

Next, the functions $v \to A_{X,B}(v)$ and $v \to v(D)$ on $N_{\mathbf{R}}$ are both linear on the cones of Δ, so the function $v \to A_{X,B}(v)/v(D)$ on $|\Delta|$ attains its infimum at some v_i, $1 \le i \le d$. Since $A_{X,B}(v_i) = 1 - b_i$ and $v_i(D) = c_i$ we are done. □

Chapter 4

Minimizing the normalized volume function

4.1 Normalized volume of a valuation

Fix a klt pair (X, B) of dimension n and $x \in X$ a closed point. Throughout this chapter we will use our abbreviated notation for log discrepancies and log canonical thresholds. Specifically, we write $A(-)$ and $\mathrm{lct}(-)$ for $A_{X,B}(-)$ and $\mathrm{lct}(X, B; -)$.

As introduced in [Li15a], the *normalized volume* of a valuation $v \in \mathrm{Val}_{X,x}$ is defined as

$$\widehat{\mathrm{vol}}(v) := A(v)^n \,\mathrm{vol}(v).$$

In the case when $A(v) = +\infty$, we set $\widehat{\mathrm{vol}}(v) := +\infty$. The word *normalized* refers to the property that $\widehat{\mathrm{vol}}(c \cdot v) = \widehat{\mathrm{vol}}(v)$ for $c \in \mathbf{R}_+^*$.

Given a graded sequence $\mathfrak{a}_\bullet$ of $\mathfrak{m}_x$-primary ideals on X, we consider a related invariant. We call

$$\mathrm{lct}(\mathfrak{a}_\bullet)^n \,\mathrm{e}(\mathfrak{a}_\bullet)$$

the *normalized multiplicity* of $\mathfrak{a}_\bullet$. Similar to the normalized volume, when $\mathrm{lct}(\mathfrak{a}_\bullet) = +\infty$, we set $\mathrm{lct}(\mathfrak{a}_\bullet)^n \,\mathrm{e}(\mathfrak{a}_\bullet) := +\infty$. The above invariant was previously studied in [dFEM04] and [Mus02].

The following lemma provides elementary information on the normalized multiplicity. The proof is left to the reader.

Lemma 4.1.1. *Let $\mathfrak{a}$ be an $\mathfrak{m}_x$-primary ideal and $\mathfrak{a}_\bullet$ a graded sequence of $\mathfrak{m}_x$-primary ideals on X.*

1. *If* $\mathrm{lct}(\mathfrak{a}_\bullet)^n \,\mathrm{e}(\mathfrak{a}_\bullet) < +\infty$, *then*

$$\mathrm{lct}(\mathfrak{a}_\bullet)^n \,\mathrm{e}(\mathfrak{a}_\bullet) = \lim_{m\to\infty} \mathrm{lct}(\mathfrak{a}_m)^n \,\mathrm{e}(\mathfrak{a}_m).$$

2. *If* $\mathfrak{b}_\bullet$ *is the graded sequence of ideals given by* $\mathfrak{b}_m := \mathfrak{a}^m$, *then*

$$\operatorname{lct}(\mathfrak{a})^n \operatorname{e}(\mathfrak{a}) = \operatorname{lct}(\mathfrak{b}_\bullet)^n \operatorname{e}(\mathfrak{b}_\bullet).$$

3. *If* $\mathfrak{a}_{N\bullet}$ *is the graded sequence whose* m*-th term is* $\mathfrak{a}_{N\cdot m}$, *then*

$$\operatorname{lct}(\mathfrak{a}_\bullet)^n \operatorname{e}(\mathfrak{a}_\bullet) = \operatorname{lct}(\mathfrak{a}_{N\bullet})^n \operatorname{e}(\mathfrak{a}_{N\bullet}).$$

Lemma 4.1.2. *Let* $\mathfrak{a}_\bullet$ *be a graded sequence of* $\mathfrak{m}_x$ *primary ideals on* X. *If* $\operatorname{lct}(\mathfrak{a}_\bullet) < +\infty$, *then* $\operatorname{lct}(\mathfrak{a}_\bullet)^n \operatorname{e}(\mathfrak{a}_\bullet)$ *is finite and nonzero.*

Proof. Assume $\operatorname{lct}(\mathfrak{a}_\bullet) < +\infty$. By Proposition 3.4.5, $\mathfrak{a}_\bullet$ is nontrivial (as defined in Section 2.2.8). This means there exists $\delta > 0$ such that $\mathfrak{a}_m \subset \mathfrak{m}_x^{\lceil m\delta \rceil}$ for all $m \in \mathbf{N}$. Therefore, $\operatorname{e}(\mathfrak{a}_\bullet) \geq \delta \operatorname{e}(\mathfrak{m}_x) > 0$.

We are left to show $\operatorname{e}(\mathfrak{a}_\bullet) < +\infty$ and $\operatorname{lct}(\mathfrak{a}_\bullet) > 0$. Pick m' with $\mathfrak{a}_{m'}$ not equal to the zero ideal. Since

$$\operatorname{e}(\mathfrak{a}_\bullet) \leq \frac{\operatorname{e}(\mathfrak{a}_{m'})}{m'^n} < +\infty \qquad \text{and} \qquad \operatorname{lct}(\mathfrak{a}_\bullet) \geq m' \operatorname{lct}(\mathfrak{a}_{m'}) > 0,$$

the proof is complete. □

The following proposition of Liu relates the normalized volume, an invariant of valuations, to the normalized multiplicity, an invariant of graded sequences of ideals.

Proposition 4.1.3. *[Liu16, Theorem 27] The following equality holds:*

$$\inf_{v \in \operatorname{Val}_{X,x}} \widehat{\operatorname{vol}}(v) = \inf_{\mathfrak{a}_\bullet \ \mathfrak{m}_x\text{-primary}} \operatorname{lct}(\mathfrak{a}_\bullet)^n \operatorname{e}(\mathfrak{a}_\bullet) = \inf_{\mathfrak{a} \ \mathfrak{m}_x\text{-primary}} \operatorname{lct}(\mathfrak{a})^n \operatorname{e}(\mathfrak{a}).$$

The previous statement first appeared in [Liu16]. In the case when $x \in X$ is a smooth point, one inequlity was given in [Li15a, Example 3.7]. We provide Liu's proof, since the argument will be useful to us. The proposition is a consequence of the following lemma.

Lemma 4.1.4. *[Liu16] The following statements hold.*

1. *If* $\mathfrak{a}_\bullet$ *is a graded sequence of* $\mathfrak{m}_x$*-primary ideals and* $v \in \operatorname{Val}_{X,x}$ *computes* $\operatorname{lct}(\mathfrak{a}_\bullet)$, *then* $\widehat{\operatorname{vol}}(v) \leq \operatorname{lct}(\mathfrak{a}_\bullet)^n \operatorname{e}(\mathfrak{a}_\bullet)$.

2. *If* $v \in \operatorname{Val}_{X,x}$, *then* $\operatorname{lct}(\mathfrak{a}_\bullet(v))^n \operatorname{e}(\mathfrak{a}_\bullet(v)) \leq \widehat{\operatorname{vol}}(v)$.

Proof. To prove (1), we first rescale v so that $v(\mathfrak{a}_\bullet) = 1$. Thus, $A(v) = A(v)/v(\mathfrak{a}_\bullet) = \operatorname{lct}(\mathfrak{a}_\bullet)$. By Lemma 3.3.11, we have $\mathfrak{a}_m \subseteq \mathfrak{a}_m(v)$ for all $m \in \mathbf{N}$. Therefore $\mathrm{e}(\mathfrak{a}_\bullet(v)) \leq \mathrm{e}(\mathfrak{a}_\bullet)$, and the desired inequality follows.

For (2), we apply Lemma 3.4.9 to see $\operatorname{lct}(\mathfrak{a}_\bullet) \leq A(v)$. Thus,

$$\operatorname{lct}(\mathfrak{a}_\bullet(v))^n \,\mathrm{e}(\mathfrak{a}_\bullet(v)) \leq A(v)^n \,\mathrm{e}(\mathfrak{a}_\bullet(v)) = \widehat{\operatorname{vol}}(v).$$

□

Proof of Proposition 4.1.3. The first equality follows immediately from the previous proposition and the fact that given a graded sequence $\mathfrak{a}_\bullet$ of $\mathfrak{m}_x$-primary ideals, there exists a valuation $v_* \in \operatorname{Val}_{X,x}$ that computes $\operatorname{lct}(\mathfrak{a}_\bullet)$. The last equality follows from Lemma 4.1.1. □

Remark 4.1.5. Above, we provided a dictionary between the normalized volume of a valuation and the normalized multiplicity of a graded sequence of ideals. The normalized multiplicity also extends to a functional on the set of (formal) plurisubharmonic functions in the sense of [BFJ08]. In a slightly different setting, similar functionals, arising from non-Archimedean analogues of functionals in Kähler geometry, were explored in [BHJ15].

4.1.1 Normalized volume minimizers

Proposition 4.1.6. *If there exists a graded sequence of $\mathfrak{m}_x$-primary ideals $\tilde{\mathfrak{a}}_\bullet$ such that*

$$\operatorname{lct}(\tilde{\mathfrak{a}}_\bullet)^n \,\mathrm{e}(\tilde{\mathfrak{a}}_\bullet) = \inf_{v \in \operatorname{Val}_{X,x}} \widehat{\operatorname{vol}}(v),$$

then there exists $v_* \in \operatorname{Val}_{X,x}$ that is a minimizer of $\widehat{\operatorname{vol}} : \operatorname{Val}_{X,x} \to \mathbf{R} \cup \{+\infty\}$. Furthermore, if there exists an $\mathfrak{m}_x$-primary ideal $\tilde{\mathfrak{a}}$ such that

$$\operatorname{lct}(\tilde{\mathfrak{a}})^n \,\mathrm{e}(\tilde{\mathfrak{a}}) = \inf_{v \in \operatorname{Val}_{X,x}} \widehat{\operatorname{vol}}(v), ,$$

then we may choose v_* to be divisorial.

Proof. Assume there exists such a graded sequence $\tilde{\mathfrak{a}}_\bullet$. By Theorem 3.4.10, there exists a valuation v_* that computes $\operatorname{lct}(\tilde{\mathfrak{a}}_\bullet)$. Since $\widehat{\operatorname{vol}}(v_*) \leq \operatorname{lct}(\tilde{\mathfrak{a}}_\bullet)^n \,\mathrm{e}(\tilde{\mathfrak{a}}_\bullet)$ (Lemma 4.1.4), v_* minimizes $\widehat{\operatorname{vol}}$.

For the second statement, apply Proposition 3.4.2 to find a divisorial valuation v_* that computes $\operatorname{lct}(\mathfrak{a})$. By the above argument, we see v_* minimizes $\widehat{\operatorname{vol}}$. □

Proposition 4.1.7. *If v_* minimizes $\widehat{\mathrm{vol}} : \mathrm{Val}_{X,x} \to \mathbf{R} \cup \{+\infty\}$, then v_* computes $\mathrm{lct}(\mathfrak{a}_\bullet(v_*))$.*

Proof. Note that

$$\mathrm{lct}(\mathfrak{a}_\bullet(v_*))^n \,\mathrm{e}(\mathfrak{a}_\bullet(v)) \leq A(v_*)^n \,\mathrm{e}(\mathfrak{a}_\bullet(v_*)) = \widehat{\mathrm{vol}}(v_*),$$

where the inequality follows from Lemma 3.4.9. Since v_* minimizes $\widehat{\mathrm{vol}}$, we see that $\mathrm{lct}(\mathfrak{a}_\bullet(v_*)) = A(v_*)$. Now, Lemma 3.4.9 implies v_* computes $\mathrm{lct}(\mathfrak{a}_\bullet(v_*))$. □

Remark 4.1.8. A conjecture of Jonsson and Mustaţă states that valuations computing log canonical thresholds of graded sequences on smooth varieties are always quasimonomial [JM12, Conjecture B]. Their conjecture in the klt case implies [Li15a, Conjecture 6.1.3], which says that normalized volume minimizers are quasimonomial.

4.1.2 Uniform approximation of volume

Given a valuation $v \in \mathrm{Val}_X$ centered at a closed point on a n-dimensional variety X, we have

$$\mathrm{vol}(v) = \lim_{m\to\infty} \frac{\mathrm{e}(\mathfrak{a}_m(v))}{m^n}.$$

The following theorem provides a uniform rate of convergence for the terms in the above limit.

Theorem 4.1.9. *Let (X, B) be a n-dimensional klt pair and $x \in X$ a closed point. For $\varepsilon > 0$ and constants $B, s \in \mathbf{N}^*$, there exists $N = N(\varepsilon, B, s)$ such that the following holds: If $v \in \mathrm{Val}_{X,x}$ satisfies $A(v) \leq B$ and $v(\mathfrak{m}_x) \geq \frac{1}{s}$, then*

$$\mathrm{vol}(v) \leq \frac{\mathrm{e}(\mathfrak{a}_m(v))}{m^n} < \mathrm{vol}(v) + \varepsilon$$

for all $m \geq N$.

Remark 4.1.10. In an early version of [Blu16b], the previous theorem was proved with the additional assumption that $x \in X$ is an isolated singularity. We are grateful to Mircea Mustaţă for noticing that a clever modification of the original proof results in the more general statement.

Proof. For any valuation $v \in \mathrm{Val}_{X,x}$, the first inequality is well known. Indeed, the inclusion $\mathfrak{a}_m(v)^p \subset \mathfrak{a}_{mp}(v)$ implies

$$\frac{\mathrm{e}(\mathfrak{a}_{mp}(v))}{(mp)^n} \leq \frac{\mathrm{e}(\mathfrak{a}_m(v))}{(m)^n}.$$

Fixing m and sending $p \to \infty$ gives $\mathrm{vol}(v) \leq \mathrm{e}(\mathfrak{a}_m(v))/m^n$.

To prove the second inequality it is sufficient to show that for each $\varepsilon > 0$, there exists N so that the following holds: if $v \in \mathrm{Val}_{X,x}$ satisfies $A(v) \leq B$ and $v(\mathfrak{m}_x) \geq 1/s$, then

$$\frac{\mathrm{e}(\mathfrak{a}_m(v))^{1/n}}{m} < \mathrm{vol}(v)^{1/n} + \varepsilon \tag{4.1}$$

for all $m \geq N$. Indeed, if $v(\mathfrak{m}_x) \geq \frac{1}{s}$, then

$$\mathfrak{m}_x^{ms} \subseteq \mathfrak{a}_m(v) \tag{4.2}$$

for all $m \in \mathbf{N}^*$. Thus,

$$\mathrm{vol}(\mathfrak{a}_\bullet(v)) \leq \lim_{m \to \infty} \frac{\mathrm{e}(\mathfrak{m}_x^{ms})}{m^n} = s^n \cdot \mathrm{e}(\mathfrak{m}_x).$$

We proceed to show there exists an N so that (4.1) holds.

Fix $\varepsilon > 0$ and choose $r \in \mathbf{N}^*$ such that $r(K_X + B)$ is Cartier. Now, consider $v \in \mathrm{Val}_{X,x}$ satisfying $A(v) \leq E$ and $v(\mathfrak{m}_x) \geq \frac{1}{s}$. By Proposition 3.3.10 and the inequality $A(v) \leq E$, we have

$$(\mathrm{Jac}_X \cdot \mathcal{O}_X(-rB))^\ell \, \mathfrak{a}_{m \cdot \ell}(v) \subset (\mathfrak{a}_{m - A(v)}(v))^\ell \subset (\mathfrak{a}_{m-E}(v))^\ell$$

for all $m \geq E$ and $\ell \in \mathbf{N}^*$. After replacing m by $m + E$, we see

$$(\mathrm{Jac}_X \, \mathcal{O}_X(-rB))^\ell \cdot \mathfrak{a}_{(m+E)\ell}(v) \subseteq \mathfrak{a}_m(v)^\ell. \tag{4.3}$$

for all $m, \ell \in \mathbf{N}^*$. If $x \notin X_{\mathrm{sing}} \cup \mathrm{Supp}(B)$, then $\mathrm{Jac}_X \, \mathcal{O}_X(-rB)$ is trivial in a neighborhood of x. Therefore, (4.3) implies

$$\ell \cdot \mathrm{e}(\mathfrak{a}_m(v))^{1/n} = \mathrm{e}(\mathfrak{a}_m(v)^\ell)^{1/n} \leq \mathrm{e}(\mathfrak{a}_{(m+E)\ell}(v))^{1/n}$$

for all $m, \ell \in \mathbf{N}^*$. Dividing by $m \cdot \ell$ gives

$$\frac{\mathrm{e}(\mathfrak{a}_m(v))^{1/n}}{m} \leq \frac{m+E}{m} \cdot \frac{\mathrm{e}\left(\mathfrak{a}_{(m+E)\ell}(v)\right)^{1/n}}{(m+E)\ell}.$$

After letting $\ell \to \infty$, we see

$$\frac{\mathrm{e}(\mathfrak{a}_m(v))^{1/n}}{m} \le \mathrm{vol}(v)^{1/n} + \frac{E \cdot \mathrm{vol}(v)^{1/n}}{m}.$$

Since $\mathrm{vol}(v) \le s^n \,\mathrm{e}(\mathfrak{m}_x)$, we conlude that (4.1) holds when N is chosen so that $E \cdot s \cdot \mathrm{e}(\mathfrak{m}_x)^{1/n}/N < \epsilon$.

We move onto the case when $x \in X_{\mathrm{sing}} \cup \mathrm{Supp}(B)$. It follows from (4.3) and (4.2) and the valuative criterion for integral closure (see Section 2.2.10) that

$$(\mathrm{Jac}_X \cdot \mathcal{O}_X(-rB) + \mathfrak{m}_x^{ms})^\ell \mathfrak{a}_{(m+E)\ell}(v) \subseteq \overline{\mathfrak{a}_m(v)^\ell}. \tag{4.4}$$

Indeed, let w be a discrete valuation of the function field of X and f and g be local sections of $(\mathrm{Jac}_X \cdot \mathcal{O}_X(-rB))^i$ and $\mathfrak{m}_x^{msj}$, respectively, with $i + j = \ell$. We have $\ell \cdot w(f) + i \cdot w(\mathfrak{a}_{(m+E)\ell}(v)) \ge i \cdot w(\mathfrak{a}_m(v)^\ell)$ and $w(g) \ge j \cdot w(\mathfrak{a}_m(v))$ by the two inclusions. Thus,

$$\begin{aligned} w(fg) = w(f) + w(g) &\ge \frac{i}{\ell}\left(w(\mathfrak{a}_m(v)^\ell) - w(\mathfrak{a}_{(m+E)\ell}(v))\right) + \frac{j}{\ell} w(\mathfrak{a}_m(v)^\ell) \\ &= w(\mathfrak{a}_m(v)^\ell) - w(\mathfrak{a}_{(m+E)\ell}(v)). \end{aligned}$$

From (4.4) and Teissier's Minkowski Inequality [Laz04, Example 1.6.9], we see

$$\ell \cdot \mathrm{e}\left(\mathfrak{a}_m(v)\right)^{1/n} \le \ell \cdot \mathrm{e}\left(\mathrm{Jac}_X \cdot \mathcal{O}_X(-rB) + \mathfrak{m}_x^{ms}\right)^{1/n} + \mathrm{e}\left(\mathfrak{a}_{(m+E)\ell}(v)\right)^{1/n}.$$

Dividing by $m \cdot \ell$ gives

$$\frac{\mathrm{e}\left(\mathfrak{a}_m(v)\right)^{1/n}}{m} \le \frac{\mathrm{e}\left(\mathrm{Jac}_X \cdot \mathcal{O}_X(-rB) + \mathfrak{m}_x^{ms}\right)^{1/n}}{m} + \frac{m+E}{m} \cdot \frac{\mathrm{e}\left(\mathfrak{a}_{(m+E)\ell}(v)\right)^{1/n}}{(m+E)\ell}.$$

After sending $\ell \to \infty$, we obtain

$$\frac{\mathrm{e}\left(\mathfrak{a}_m(v)\right)^{1/n}}{m} \le \frac{\mathrm{e}\left(\mathrm{Jac}_X \cdot \mathcal{O}_X(-rB) + \mathfrak{m}_x^{ms}\right)^{1/n}}{m} + \frac{m+E}{m} \mathrm{vol}(v)^{1/n}.$$

We will be able to find $N := N(B, s, \varepsilon)$ so that (4.1) holds, if we show that

$$\lim_{m \to \infty} \frac{\mathrm{e}\left(\mathrm{Jac}_X \cdot \mathcal{O}_X(-rB) + \mathfrak{m}_x^{ms}\right)^{1/n}}{m} = 0. \tag{4.5}$$

Choose a nonzero element $h \in \mathrm{Jac}_X \cdot \mathcal{O}_X(-rB) \cdot \mathcal{O}_{X,x}$, and set $R := \mathcal{O}_{X,x}/(h)$ and $\tilde{\mathfrak{m}}_x := \mathfrak{m}_x \cdot R$. Now, Lech's inequality [Lec60, Theorem 3] implies

$$\mathrm{e}(\mathrm{Jac}_X \cdot \mathcal{O}_X(-rB) + \mathfrak{m}_x^{ms}) \le n! \cdot \mathrm{e}(\mathfrak{m}_x)\ell(\mathcal{O}_X/\,\mathrm{Jac}_X \cdot \mathcal{O}_X(-rB) + \mathfrak{m}_x^{ms}).$$

By our choice of h, the latter term is less than

$$n! \cdot \mathrm{e}(\mathfrak{m}_x)\ell(\mathcal{O}_{X,x}/h + \mathfrak{m}_x^{ms}) = n! \cdot \mathrm{e}(\mathfrak{m}_x)\ell(R/\tilde{\mathfrak{m}}_x^{ms}).$$

Since R has dimension $n-1$, $\lim_{m\to\infty} \frac{\ell(R/\tilde{\mathfrak{m}}_x^{ms})}{m^n} = 0$. Therefore, (4.5) holds. $\square$

4.1.3 Multiplicity in families

We proceed to prove a result on the behavior of the Hilbert-Samuel multiplicity in a family of ideals. The result is well known to commutative algebraists.

First, we recall an interpretation of the Hilber-Samuel multiplicity in terms of intersection numbers [Ram73]. Let X be a projective variety of dimension n, $x \in X$ a closed point, and $\mathfrak{a} \subset \mathcal{O}_X$ a $\mathfrak{m}_x$-primary ideal. If $Y \to X$ is a proper birational morphism such that Y nonsingular and $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, where F is a Cartier divisor on Y, then $\mathrm{e}(\mathfrak{a}) = (-1)^{n-1}F^n$.

Proposition 4.1.11. *Let X, T be varieties, $x \in X$ a closed point, and $\mathfrak{a} \subset \mathcal{O}_{X\times T}$ an ideal. If $\mathfrak{a}_t := \mathfrak{a} \cdot \mathcal{O}_{X\times\{t\}}$ is $\mathfrak{m}_{x\times t}$-primary for each $t \in T$, then there exists a nonempty open set $U \subset T$ such that the function $U \ni t \mapsto \mathrm{e}(\mathfrak{a}_t)$ is constant.*

Proof. It is sufficient to consider the case when X is projective. Indeed, replace X with a projective compactification of an open affine neighborhood of x.

Fix a projective birational morphism $Y \to X \times T$ such that Y is smooth and $\mathfrak{a}\cdot\mathcal{O}_Y(-F)$ where F is a Cartier divisor on Y. We may find a nonempty open set $U \subset T$ such that $Y \to T$ is flat over U and $Y_t \to X_t$ is a resolution of singularities for all $t \in U$.

Now, we have that $\mathfrak{a}_t \cdot \mathcal{O}_{Y_t} = \mathcal{O}_{Y_t}(-F_t)$ for each $t \in U$. Since $Y \to T$ is flat over U, [Kol96, Proposition IV.2.9] implies $U \ni t \mapsto (-1)^{n-1}F_t^n$ is constant. $\square$

4.2 Limit points of collections of graded sequences of ideals

In this section we construct a space that parameterizes graded sequences of ideals on a fixed variety X. We use this parameter space to find "limit points" of a collection of graded sequences of ideals on X. The ideas behind this construction arise from the work of de Fernex-Mustaţă [dFM09], Kollár [Kol08], and de Fernex-Ein-Mustaţă [dFEM10] [dFEM11].

4.2.1 Parameterizing graded sequences of ideals

Fix an affine variety $X = \mathrm{Spec}(R)$ and a closed point $x \in X$ with corresponding maximal ideal $\mathfrak{m} \subset R$. We seek to parameterize graded sequence of ideals $\mathfrak{a}_\bullet$ on R satisfying

$$(\dagger)\ \mathfrak{m}^m \subseteq \mathfrak{a}_m \subseteq \mathfrak{m} \text{ for all } m \in \mathbf{N}.$$

Note that there is a correspondence between ideals $\mathfrak{a} \subset R$ such that $\mathfrak{m}^m \subset \mathfrak{a}$ and ideals of $R/\mathfrak{m}^m$. For each $m > 0$, let $\mathrm{Hilb}(R/\mathfrak{m}^m)$ denote the Hilbert scheme parameterizing non-unit ideals of $R/\mathfrak{m}^m$. Since $\dim_k R/\mathfrak{m}^m < \infty$, $\mathrm{Hilb}(R/\mathfrak{m}^m)$ is a finite type k-scheme. Set

$$J_m = \mathrm{Hilb}(R/\mathfrak{m}^1)_{\mathrm{red}} \times \mathrm{Hilb}(R/\mathfrak{m}^2)_{\mathrm{red}} \times \cdots \times \mathrm{Hilb}(R/\mathfrak{m}^m)_{\mathrm{red}}.$$

Note that the irreducible components of J_m are varieties.

For each $p \geq m$, let $\pi_{p,m} : J_p \to J_m$ denote the natural projection map. Our desired set is the projective limit

$$J = \varprojlim J_m(k),$$

where $J_m(k)$ denotes the k-valued points of J_m. For each $m > 0$, let $\pi_m : J \to J_m$ denote the natural projection map.

By construction, there is a surjective map

$$J \longrightarrow \{\text{sequence of ideals } \mathfrak{b}_\bullet \text{ of } R \text{ satisfying } (\dagger)\}.$$

Note that the sequences of ideals on the right hand side are not necessarily graded.

Given a sequence of ideals $\mathfrak{b}_\bullet$, we can construct a graded sequence $\mathfrak{a}_\bullet$ inductively by setting $\mathfrak{a}_1 := \mathfrak{b}_1$ and

$$\mathfrak{a}_m := \mathfrak{b}_m + \sum_{\substack{p+q=m \\ p,q>0}} \mathfrak{a}_p \cdot \mathfrak{a}_q.$$

It is clear that if $\mathfrak{b}_\bullet$ satisfies $(\dagger)$ then so does $\mathfrak{a}_\bullet$. Furthermore, if $\mathfrak{b}_\bullet$ was graded to begin with, then $\mathfrak{a}_\bullet = \mathfrak{b}_\bullet$. Thus, we have a map

$$J \longrightarrow \{\text{graded sequences of ideals } \mathfrak{a}_\bullet \text{ of } R \text{ satisfying } (\dagger)\}.$$

For each $m > 0$, we have universal ideals $(\mathcal{A}_{i,m})_{i=1}^m$ on $X \times J_m$ such that for each $z \in J$, the ideals $(\mathfrak{a}_{i,z})_{i=1}^m$ correspond to the restriction of the ideals $(\mathcal{A}_{i,m})_{i=1}^m$ to $X \times \{z_m\}$, where $z_m = \pi_m(z)$. We will simply write $\mathcal{A}_m$ for the ideal $\mathcal{A}_{m,m}$ on $X \times J_m$.

The following technical lemma will be used in the following section. The proof of the lemma relies on the fact that every descending sequence of non-empty constructible subsets of a variety over an uncountable field has nonempty intersection.

Lemma 4.2.1. *For each $m \in \mathbf{N}^*$, let $W_m \subset J_m$ be a nonempty constructible subset. If $W_{m+1} \subset \pi^{-1}_{m+1,m}(W_m)$ for all $m \in \mathbf{N}^*$, then there exists $\tilde{z} \in J$ such that $\pi_m(\tilde{z}) \in W_m$ for each $m \in \mathbf{N}^*$.*

Proof. Note that a point $\tilde{z}$ as above is equivalent to a sequence of closed points $(z_m \in W_m)_{m \in \mathbf{N}^*}$ such that $\pi_{m+1,m}(z_{m+1}) = z_m$. We proceed to construct such a sequence.

We first look to find a candidate for z_1. We observe that

$$W_1 \supseteq \pi_{2,1}(W_2) \supseteq \pi_{3,1}(W_3) \supseteq \cdots$$

is a descending sequence of nonempty sets. Note that W_1 is constructible by assumption and so are $\pi_{d,1}(W_d)$ for all d by Chevalley's Theorem [Har77, Exercise II.3.9]. Therefore,

$$W_1 \cap \pi_{2,1}(W_2) \cap \pi_{3,1}(W_3) \cap \cdots$$

is non-empty and we may choose a point z_1 lying in the set.

Next, we look at

$$W_2 \cap \pi^{-1}_{2,1}(z_1) \supseteq \pi_{3,2}(W_3) \cap \pi^{-1}_{2,1}(z_1) \supseteq \pi_{4,2}(W_4) \cap \pi^{-1}_{2,1}(z_1),$$

and note that for $m \geq 2$ each $\pi_{m,2}(W_m) \cap \pi^{-1}_{2,1}(z_1)$ is nonempty by our choice of z_1. By the same argument as before, we see

$$\pi^{-1}_{2,1}(z_1) \cap W_2 \cap \pi_{3,2}(W_3) \cap \pi_{4,2}(W_4) \cap \cdots$$

is non-empty and contains a closed point z_2. Continuing in this manner, we construct a desired sequence. □

4.2.2 Finding Limit Points

The proof of the following proposition relies on the parameterization of graded sequences of ideals described in the previous section. Our proof is inspired by arguments in [dFEM04, Kol08, dFEM10, dFEM11].

Proposition 4.2.2. *Let (X, B) be a klt pair and $x \in X$ a closed point. Assume there exists a sequence of graded sequences of $\mathfrak{m}_x$-primary ideals $(\mathfrak{a}_\bullet^{(i)})_{i \in \mathbf{N}}$ such that $\mathfrak{m}_x^m \subset \mathfrak{a}_m^{(i)}$ for all $m, i \in \mathbf{N}$. If*

(i) $L := \limsup_{i \to \infty} \operatorname{lct}(\mathfrak{a}_\bullet^{(i)})$ *and*

(ii) $E := \limsup_{m \to \infty} \left(\limsup_{i \to \infty} \operatorname{e}(\mathfrak{a}_m^{(i)})/m^n \right)$.

are both finite, then there exists a graded sequence of $\mathfrak{m}_x$-primary ideals $\tilde{\mathfrak{a}}_\bullet$ on X such that $\mathfrak{m}_x^m \subset \tilde{\mathfrak{a}}_m$ for all $m \in \mathbf{N}$, $\operatorname{lct}(\tilde{\mathfrak{a}}_\bullet) \leq L$, and $\operatorname{vol}(\tilde{\mathfrak{a}}_\bullet) \leq E$.

Proof. It is sufficient to consider the case when X is affine. We recall that Section 4.2.1 constructs a set J parameterizing graded sequences of ideals on X satisfying (†). Additionally, for each $m \in \mathbf{N}^*$, the variety J_m parameterize the first m elements of such a sequence.

Since each graded sequence $\mathfrak{a}_\bullet^{(i)}$ satisfies (†), we may choose a point $z_i \in J$ corresponding to $\mathfrak{a}_\bullet^{(i)}$. Note that $\pi_m(z_i) \in J_m$ corresponds to the first m-terms of $\mathfrak{a}_\bullet^{(i)}$.

Claim 1: We may choose infinite subsets $\mathbf{N} \supset I_1 \supset I_2 \supset \cdots$ and set

$$Z_m := \overline{\{\pi_m(z_i) | i \in I_m\}}$$

such that $(**)$ holds.

$(**)$ If $Y \subsetneq Z_m$ is a closed set, there are only finitely many $i \in I_m$ such that $\pi_m(z_i) \in Y$.

To prove Claim 1, we construct such a sequence inductively. First, we set $I_1 = \mathbf{N}$. Since J_1 is a point, $(**)$ is trivially satisfied for $m = 1$. After having chosen I_m, choose $I_{m+1} \subset I_m$ so that $(**)$ is satisfied for Z_{m+1}. By the Noetherianity of J_m, such a choice is possible.

Claim 2: For each $m \in \mathbf{N}$, there exists a nonempty open set $U_m \subset Z_m$ and constants λ_m and μ_m such that if $z \in U_m$, then

(i) $m \cdot \operatorname{lct}(\mathcal{A}_m|_z) = \lambda_m$ and

(ii) $\frac{\operatorname{e}(\mathcal{A}_m|_z)}{m^n} = \mu_m$.

Furthermore, $\sup_m \lambda_m \leq L$ and $\limsup_{m\to\infty} \mu_m \leq E$.

By Propositions 3.4.11 and 4.1.11, for each $m \in \mathbf{N}^*$ we may find a nonempty open set $U_m \subset Z_m$ and constants λ_m and μ_m such that

$$\lambda_m = m \cdot \operatorname{lct}(\mathcal{A}_m|_z) \text{ and } \mu_m = \mathrm{e}(\mathcal{A}_m|_z)/m^n$$

for all $z \in U_m$. Now, we let

$$I_m^\circ = \{i \in I_m \,|\, \pi_m(z_i) \in U_m\} \subseteq I_m.$$

By $(**)$, the set $I_m \setminus I_m^\circ$ is finite; hence, I_m° is infinite. Since $\lambda_m = m \cdot \operatorname{lct}(\mathfrak{a}_m^{(i)})$ and $\mu_m = \frac{\mathrm{e}(\mathfrak{a}_m^{(i)})}{m^n}$ for all $i \in I_m^\circ$, we see

$$\lambda_m \leq \limsup_{i\to\infty} m \cdot \operatorname{lct}(\mathfrak{a}_m^{(i)}) \qquad \text{and} \qquad \mu_m \leq \limsup_{i\to\infty} \frac{\mathrm{e}(\mathfrak{a}_m^{(i)})}{m^n}$$

for each $m \in \mathbf{N}^*$. From the definition of L and the inequality $\operatorname{lct}(\mathfrak{a}_m^{(i)}) \leq \operatorname{lct}(\mathfrak{a}_\bullet^{(i)})$ (see Proposition 3.4.4), we see $\lambda_m \leq L$. Similarly, the definition of E implies $\limsup_{m\to\infty} \mu_m \leq E$.

Claim 3: There exists a point $\tilde{z} \in J$ such that $\pi_m(\tilde{z}) \in U_m$ for all $m \in \mathbf{N}^*$.

Granted this claim, the graded sequence of ideals associated to $\tilde{z} \in J$ satisfies the conclusion of our proposition. Indeed, this follows from Claim 2 and the fact that for a graded sequence of $\mathfrak{m}_x$-primary ideals $\mathfrak{a}_\bullet$, we have

$$\operatorname{lct}(\mathfrak{a}_\bullet) = \sup_m m \cdot \operatorname{lct}(\mathfrak{a}_m) \qquad \text{and} \qquad \operatorname{vol}(\mathfrak{a}_\bullet) = \lim_{m\to\infty} \frac{\mathrm{e}(\mathfrak{a}_m)}{m^n}.$$

We are let to prove Claim 3. To this end, we seek to apply Lemma 4.2.1. For each $m \in \mathbf{N}^*$, set

$$W_m := U_m \cap \pi_{m,m-1}^{-1}(U_{m-1}) \cap \pi_{m,m-2}^{-1}(U_{m-2}) \cap \cdots \cap \pi_{m,0}^{-1}(U_0).$$

Clearly, $W_m \subset J_m$ is constructible and $W_{m+1} \subset \pi_{m+1,m}^{-1}(W_m)$. We are left to show W_m is nonempty. Note that

$$\pi_m(z_i) \in W_m \text{ for all } i \in I_m^\circ \cap I_{m-1}^\circ \cap \cdots \cap I_0^m,$$

and the latter index set is nonempty, since it can be written as $I_m \setminus \cup_{j=0}^m (I_j \setminus I_j^\circ)$, where I_m is infinite and each $I_j \setminus I_j^\circ$ is finite. Applying Lemma 4.2.1 yields a point $\tilde{z} \in J$ such that $\pi_m(\tilde{z}) \in W_m \subset U_m$. This completes the proof of Claim 3, and the proof the of the proposition. $\square$

4.3 Existence of minimizers

In this section we prove Theorem A. The proof relies on the results in 4.2 and also uses estimates from [Li15a].

Proposition 4.3.1. *Let (X, B) be a klt pair and $x \in X$ a closed point. Assume $(v_i)_{i\in\mathbf{N}}$ is a sequence of valuations such that $v_i(\mathfrak{m}_x) = 1$ for each $i \in \mathbf{N}$ and the limits*

$$A := \lim_{i\to\infty} A(v_i) \text{ and } V := \lim_{i\to\infty} \operatorname{vol}(v_i)$$

both exist and take values in $\mathbf{R}_+$. Then, there exists $v_* \in \operatorname{Val}_{X,x}$ such that

$$A(v_*) \leq A \quad \text{and} \quad \operatorname{vol}(v_*) \leq V.$$

This will follow from Proposition 4.2.2 and the following lemma.

Lemma 4.3.2. *Keep the notation and hypotheses of Proposition 4.3.1. The following hold:*

(i) $A \geq \limsup_{i\to\infty} \operatorname{lct}(\mathfrak{a}_\bullet(v_i))$;

(ii) $V \geq \limsup_{m\to\infty} \left(\limsup_{i\to\infty} \mathrm{e}(\mathfrak{a}_m(v_i))/m^n \right)$.

Proof. Since $A(v_i) \geq \operatorname{lct}(\mathfrak{a}_\bullet(v_i))$, we see

$$A = \lim_{i\to\infty} A(v_i) \geq \limsup_{i\to\infty} \operatorname{lct}(\mathfrak{a}_\bullet^{(i)}).$$

The second inequality is more subtle. Fix $\varepsilon > 0$. Since $(A(v_i))_{i\in\mathbf{N}}$ is bounded from above, we may apply Proposition 4.1.9 to find N so that

$$\frac{\mathrm{e}(\mathfrak{a}_m(v_i))}{m^n} < \operatorname{vol}(v_i) + \varepsilon.$$

for all $m \geq N$ and $i \in \mathbf{N}$. Therefore,

$$\limsup_{i\to\infty} \frac{\mathrm{e}(\mathfrak{a}_m(v_i))}{m^n} \leq V + \varepsilon$$

for all $m \geq N$. Thus,

$$\limsup_{m\to\infty}\left(\limsup_{i\to\infty}\frac{\mathrm{e}(\mathfrak{a}_m(v_i))}{m^n}\right)\leq V+\epsilon,$$

and the proof is complete. □

Proof of Proposition 4.3.1. We consider the sequence $(\mathfrak{a}_\bullet(v_i))_{i\in\mathbf{N}}$. Combining the previous lemma and Proposition 4.2.2, we see that there exists a graded sequence of ideals $\tilde{\mathfrak{a}}_\bullet$ such that $\mathfrak{m}_x^m \subset \tilde{\mathfrak{a}}_m$, $\mathrm{lct}(\tilde{\mathfrak{a}}_\bullet) \leq A$, and $\mathrm{vol}(\tilde{\mathfrak{a}}_\bullet) \leq V$. By Theorem 3.4.10, there exists $v_* \in \mathrm{Val}_{X,x}$ computing $\mathrm{lct}(\tilde{\mathfrak{a}}_\bullet)$. After rescaling, we may assume $v_*(\tilde{\mathfrak{a}}_\bullet) = 1$. Therefore,

$$A(v_*) = \frac{A(v_*)}{v_*(\tilde{\mathfrak{a}}_\bullet)} = \mathrm{lct}(\tilde{\mathfrak{a}}_\bullet) \leq A.$$

By Lemma 2.2.1, $\tilde{\mathfrak{a}}_m \subset \mathfrak{a}_m(v_*)$ for each $m \in \mathbf{N}$. Thus, $\mathrm{vol}(v_*) \leq \mathrm{vol}(\tilde{\mathfrak{a}}_\bullet) \leq V$, and the proof is complete. □

Proof of Theorem A. We fix a klt pair (X, B) and a closed point $x \in X$. Next, choose a sequence of valuations $(v_i)_{i\in\mathbf{N}}$ in $\mathrm{Val}_{X,x}$ such that

$$\lim_i \widehat{\mathrm{vol}}(v_i) = \inf_{v\in\mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}(v).$$

The latter infimum is > 0 by [Li15a, Theorem 1.2].

After scaling our valuations, we may assume that $v_i(\mathfrak{m}_x) = 1$ for all $i \in \mathbf{N}$. Since $\mathfrak{m}_x^m \subset \mathfrak{a}_m(v_i)$ for all $m, i \in \mathbf{N}$, $0 \leq \mathrm{vol}(v_i) \leq \mathrm{e}(\mathfrak{m}_x)$ for all $i \in \mathbf{N}$. Therefore, the sequence $(\mathrm{vol}(v_i))_{i\in\mathbf{N}}$ is bounded from above. After passing to a subsequence, we may assume $V := \lim_{i\to\infty}\mathrm{vol}(v_i)$ exists, and is finite. Thus, the limit $A := \lim_{i\to\infty} A(v_i)$ exists, but may take the value $+\infty$. By [Li15a, Theorem 4.3], there exists a constant c_0 so that

$$\widehat{\mathrm{vol}}(v_i) \geq c_0 \cdot A(v_i)$$

for all i. Hence, A must be finite. Now, it follows that $A, V \in \mathbf{R}_+$ and $A^n \cdot V = \inf_{v\in\mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}(v)$.

By Proposition 4.3.1, there exists $v_* \in \mathrm{Val}_{X,x}$ with $A(v_*) \leq A$ and $\mathrm{vol}(v_*) \leq V$. Therefore, $\widehat{\mathrm{vol}}(v_*) \leq \inf_v \widehat{\mathrm{vol}}(v)$ and, hence, minimizes $\widehat{\mathrm{vol}} : \mathrm{Val}_{X,x} \to \mathbf{R}\cup\{+\infty\}$. □

4.4 Toric case

Let $N \simeq \mathbf{Z}^n$ be a lattice and $M = \mathrm{Hom}(N, \mathbf{Z})$ the corresponding dual lattice. We consider an affine toric klt pair (X, B) where $X = X(\sigma)$ is given by a maximal dimension, strongly convex, rational, polyhedral cone $\sigma \subset N_{\mathbf{R}}$ and B is a torus invariant $\mathbf{Q}$-divisor on X. Let $x \in X$ denote the unique torus invariant closed point.

As in Section 3.5, let $v_1, \ldots, v_d$ denote the primitive generators of the one-dimensional face of σ and $D_1, \ldots, D_d$ the corresponding torus invariant valuations. Hence, we may choose coefficients b_i so that $B = b_1 D_1 + \cdots + b_d D_d$. A vector $v \in \sigma$ gives rise to a valuation on X that we slightly abusively also denote by v (see Section 3.5.1). Recall that a valuation $v \in \sigma \subset \mathrm{Val}_X$ has center x if and only if the vector v lies in $\mathrm{Int}(\sigma)$.

We proceed to describe the normalized volume function of a toric valuation on X with center x. Since $K_X + B$ is klt, there exists $a \in M_{\mathbf{Q}}$ such that $\langle a, v_i \rangle = 1 - b_i$ for $1 \leq i \leq d$. By Proposition 3.5.1,

$$A_{X,B}(v) = \langle a, v \rangle$$

for all $v \in \sigma \subset \mathrm{Val}_X$. For $v \in \sigma$ and $m \in \mathbf{N}$, we set $H_v(m) = \{u \in M_{\mathbf{R}} \,|\, \langle u, v \rangle \geq m\}$. Note that

$$\mathfrak{a}_m(v) = (\chi^u \,|\, u \in H_v(m) \cap \sigma^\vee \cap M)\,.$$

In the case when $u \in \mathrm{Int}(\sigma)$,

$$\mathrm{vol}(v) = n! \cdot \mathrm{vol}(\sigma^\vee \setminus H_v(1)),$$

where the latter vol denotes the Euclidean volume.

4.4.1 Deformation to the Initial Ideal

As explained in [Eis95], when $X_\sigma \simeq \mathbb{A}^n$ and $I \subset R_\sigma$, there exists a deformation of I to a monomial ideal. A similar argument works in our setting.

Following the approach of [KK12, Section 6], we put a $\mathbf{Z}^n_{\geq 0}$ order on the monomials of R_σ. Fix $y_1, \ldots, y_n \in N \cap \sigma$ that are linearly independent in $M_{\mathbf{R}}$. This gives an injective map $\rho : M \to \mathbf{Z}^n$ defined by

$$u \longmapsto (\langle y_1, u \rangle, \ldots, \langle y_n, u \rangle)\,.$$

Since each $y_i \in \sigma$, we have $\rho(M \cap \sigma^\vee) \subseteq \mathbf{Z}^n_{\geq 0}$. After endowing $\mathbf{Z}^n_{\geq 0}$ with the lexicographic order, we get an order $>$ on the monomials of R_σ.

An element $f \in R_\sigma$ may be written as a sum of scalar multiples of distinct monomials. The *initial term* of f, denoted $\mathrm{in}_> f$, is the greatest term of f with respect to the order $>$.

For an ideal $I \subset R_\sigma$, the *initial ideal* of I is

$$\mathrm{in}_> I = (\mathrm{in}_> f \mid f \in I).$$

Note that if I is $\mathfrak{m}$-primary, then so is $\mathrm{in}_> I$. Also, if $(I_m)_{m \in \mathbf{N}}$ is a graded sequence of ideals of R_σ, then so is $\{\mathrm{in}_> I_m\}_{m \in \mathbf{N}}$. This follows from the fact that $\mathrm{in}_> f \cdot \mathrm{in}_> g = \mathrm{in}_> fg$.

Lemma 4.4.1. *If $I \subset R_\sigma$ is an $\mathfrak{m}$-primary ideal, then*

$$\ell(R_\sigma / I) = \ell(R_\sigma / \mathrm{in}_> I).$$

Proof. The proof is similar to the proof of [Eis95, Theorem 15.3]. □

As in [Eis95], we construct a deformation of I to $\mathrm{in}_> I$. Since R_σ is Noetherian, we may choose elements $g_1, \ldots, g_s \in I$ such that

$$I = (g_1, \ldots, g_s) \quad \text{and} \quad \mathrm{in}_> I = (\mathrm{in}_> g_1, \ldots, \mathrm{in}_> g_s).$$

Fix an integral weight $\lambda : M \cap \sigma^\vee \to \mathbf{Z}_{\geq 0}$ such that

$$\mathrm{in}_{>_\lambda}(g_i) = \mathrm{in}_>(g_i)$$

for all i. Note that $>_\lambda$ denotes the order on the monomials induced by the weight function λ.

Let $R_\sigma[t]$ denote the polynomial ring in one variable over R_σ. For $g = \sum \alpha_m \chi^m$, we write $b := \max\{\lambda(m) \mid \alpha_m \neq 0\}$ and set

$$\tilde{g} := t^b \sum \alpha_m t^{-\lambda(m)} \chi^m.$$

Next, let

$$\tilde{I} = (\tilde{g}_1 \ldots \tilde{g}_s) \subset R_\sigma[t].$$

For $c \in k$, we write I_c for the image of $\tilde{I}$ under the map $R_\sigma[t] \to R_\sigma$ defined by $t \mapsto c$. It is clear that $I_1 = I$ and $I_0 = \mathrm{in}_> I$.

Proposition 4.4.2. *If $I \subset R_\sigma$ is an $\mathfrak{m}_x$ primary ideal, then* $\mathrm{lct}(\mathrm{in}_<(I)) \leq \mathrm{lct}(I)$.

Proof. We consider the automorphism of $\varphi : R_\sigma[t, t^{-1}] \to R_\sigma[t, t^{-1}]$ that sends χ^m to $t^{\lambda(m)} \chi^m$. Note that φ sends $\tilde{I} R_\sigma[t, t^{-1}]$ to $I R_\sigma[t, t^{-1}]$. Therefore, for each $c \in k^*$, we get an automorphism $\varphi_c : R_\sigma \to R_\sigma$ such that $\varphi_c(I_c) = I$. Thus, $\mathrm{lct}(I_c) = \mathrm{lct}(I)$ for all $c \in k^*$.

Since I_c is $\mathfrak{m}_x$-primary for all $c \in k$, we may apply Proposition 3.4.12 to see $\operatorname{lct}(I_0) \leq \operatorname{lct}(I)$. Since $\operatorname{in}_>(I) = I_0$, we are done. $\square$

The following proposition and theorem generalize arguments of [Mus02] to the singular toric case.

Proposition 4.4.3. *Let $\mathfrak{a}_\bullet$ be a graded sequence of $\mathfrak{m}_x$-primary ideals on X_σ. We have that*

$$\operatorname{lct}(\operatorname{in}_>(\mathfrak{a}_\bullet))^n \operatorname{e}(\operatorname{in}(\mathfrak{a}_\bullet)) \leq \operatorname{lct}(\mathfrak{a}_\bullet)^n \operatorname{e}(\mathfrak{a}_\bullet).$$

Proof. Note that

$$\operatorname{e}(\operatorname{in}_>(\mathfrak{a}_\bullet)) := \limsup_{m\to\infty} \frac{\ell(\mathcal{O}_{X_\sigma,x}/\operatorname{in}_>(\mathfrak{a}_m))}{m^n/n!} = \limsup_{m\to\infty} \frac{\ell(\mathcal{O}_{X_\sigma,x}/\mathfrak{a}_m)}{m^n/n!} =: \operatorname{e}(\mathfrak{a}_\bullet),$$

where the second equality follows from Proposition 4.4.1. By Proposition 4.4.2,

$$\operatorname{lct}(\operatorname{in}_>(\mathfrak{a}_\bullet)) \leq \operatorname{lct}(\mathfrak{a}_\bullet).$$

The result follows. $\square$

Theorem 4.4.4. *We have the following equality*

$$\inf_{v\in \operatorname{Val}_{X,x}} \widehat{\operatorname{vol}}(v) = \inf_{v\in \operatorname{Int}(\sigma)\subset \operatorname{Val}_{X,x}} \widehat{\operatorname{vol}}(v).$$

Furthermore, there exists $v \in \operatorname{Int}(\sigma) \subset \operatorname{Val}_{X,x}$ computing the infimum.

Proof. To prove the theorem, it is suffices to show the second statement. By Theorem A, we may find $w \in \operatorname{Val}_{X,x}$ minimizing $\widehat{\operatorname{vol}} : \operatorname{Val}_{X,x} \to \mathbf{R} \cup \{+\infty\}$. Now,

$$\operatorname{lct}(\operatorname{in}_>(\mathfrak{a}_\bullet(w)))^n \operatorname{e}(\operatorname{in}_>(\mathfrak{a}_\bullet(w))) \leq \operatorname{lct}(\mathfrak{a}_\bullet(w))^n \operatorname{e}(\mathfrak{a}_\bullet(w)) \leq A_X(w)^n \operatorname{vol}(w),$$

where the first inequality follows from Proposition 4.4.3 and the second from the inequality $\operatorname{lct}(\mathfrak{a}_\bullet(w)) \leq A(w)$ (see Lemma 3.4.9). Since $\operatorname{in}_>(\mathfrak{a}_\bullet(w))$ is a graded sequence of T-invariant ideals, we may apply Proposition 3.5.2 to find a toric valuation $v \in \operatorname{Int}(\sigma) \subset \operatorname{Val}_{X,x}$ that computes $\operatorname{lct}(\operatorname{in}_>(\mathfrak{a}_\bullet(w))$. Now, Lemma 4.1.4 implies $\widehat{\operatorname{vol}}(v) \leq \operatorname{lct}(\operatorname{in}_>(\mathfrak{a}_\bullet(w)))^n \operatorname{e}(\operatorname{in}(\mathfrak{a}_\bullet(w)))$. Thus, $\widehat{\operatorname{vol}}(v) = \widehat{\operatorname{vol}}(w)$, and we conclude v is a normalized volume minimizer. $\square$

Chapter 5

Thresholds, valuations, and K-stability

This exposition and results in this section are the result of joint work with Mattias Jonsson. We thank him for allowing these results to be reproduced in this thesis.

5.1 Linear series, filtrations, and Okounkov bodies

In this section we recall facts about linear series, filtrations, and Okounkov bodies, following [LM09, KK12, BC11, Bou14]. The new results are Lemma 5.1.2 and Corollary 5.1.10.

Let X be a normal projective variety of dimension n and L a big line bundle on X. Set

$$R_m := H^0(X, mL) \quad \text{and} \quad N_m := \dim_k R_m$$

for $m \in \mathbf{N}$, and write $M(L) \subset \mathbf{N}$ for the semigroup of $m \in \mathbf{N}$ for which $N_m > 0$. Since L is big, we have $m \in M(L)$ for $m \gg 1$. Write

$$R = R(X, L) = \bigoplus_m R_m = \bigoplus_m H^0(X, mL)$$

for the section ring of L.

5.1.1 Graded linear series

A *graded linear series* of L is a graded k-subalgebra

$$V_\bullet = \bigoplus_m V_m \subset \bigoplus_m R_m = R.$$

We say $V_\bullet$ *contains an ample series* if $V_m \neq 0$ for $m \gg 0$, and there exists a decomposition $L = A + E$ with A an ample $\mathbf{Q}$-line bundle and E an effective $\mathbf{Q}$-divisor such that

$$H^0(X, mA) \subset V_m \subset H^0(X, mL) = R_m$$

for all sufficiently divisible m.

5.1.2 Okounkov bodies

Fix a system $z = (z_1, \dots, z_n)$ of parameters centered at a regular closed point ξ of X. This defines a real rank-n valuation

$$\mathrm{ord}_z \colon \mathcal{O}_{X,\xi} \setminus \{0\} \to \mathbf{N}^n,$$

where $\mathbf{N}^n$ is equipped with the lexicographic ordering. As in §2.2.1 we also define $\mathrm{ord}_z(s)$ for any nonzero section $s \in R_m$.

Now consider a nonzero graded linear series $V_\bullet \subset R(X, L)$. For $m \in \mathbf{N}$, the subset

$$\Gamma_m := \Gamma_m(V_\bullet) := \mathrm{ord}_z(V_m \setminus \{0\}) \subset \mathbf{N}^n$$

has cardinality $\dim_k V_m$, since ord_z has transcendence degree 0. Hence

$$\Gamma := \Gamma(V_\bullet) := \{(m, \alpha) \in \mathbf{N}^{n+1} \mid \alpha \in \Gamma_m\}$$

is a subsemigroup of $\mathbf{N}^{n+1}$. Let $\Sigma = \Sigma(V_\bullet) \subset \mathbf{R}^{n+1}$ be the closed convex cone generated by Γ. The *Okounkov body* of $V_\bullet$ with respect to z is given by

$$\Delta = \Delta_z(V_\bullet) = \{\alpha \in \mathbf{R}^n \mid (1, \alpha) \in \Sigma\}.$$

This is a compact convex subset of $\mathbf{R}^n$. For $m \geq 1$, let ρ_m be the atomic positive measure on Δ given by

$$\rho_m = m^{-n} \sum_{\alpha \in \Gamma_m} \delta_{m^{-1}\alpha}.$$

The following result is a special case of [Bou14, Théorème 1.12].

Theorem 5.1.1. *If $V_\bullet$ contains an ample series, then its Okounkov body $\Delta \subset \mathbf{R}^n$ has nonempty interior, and we have $\lim_{m\to\infty} \rho_m = \rho$ in the weak topology of measures, where ρ denotes Lebesgue measure on $\Delta \subset \mathbf{R}^n$. In particular, the limit*

$$\mathrm{vol}(V_\bullet) = \lim_{m\to\infty} \frac{n!}{m^n} \dim_k V_m \in (0, \mathrm{vol}(L)] \tag{5.1}$$

exists, and equals $n! \,\mathrm{vol}(\Delta)$.

In fact, the limit in (5.1) always exists, but may be zero in general; see [Bou14, Théorème 3.7] for a much more precise result due to Kaveh and Khovanskii [KK12].

For the proof of Theorem A we will need the following estimate.

Lemma 5.1.2. *For every $\varepsilon > 0$ there exists $m_0 = m_0(\varepsilon) > 0$ such that*

$$\int_\Delta g\, d\rho_m \le \int_\Delta g\, d\rho + \varepsilon$$

for every $m \ge m_0$ and every concave function $g\colon \Delta \to \mathbf{R}$ satisfying $0 \le g \le 1$.

The main point here is the uniformity in g.

Proof. We essentially follow the proof of [Bou14, Théorème 1.12]. The sets

$$\Delta_\gamma := \{\alpha \in \mathbf{R}^n \mid \alpha + [-\gamma, \gamma]^n \subset \Delta\},$$

for $\gamma > 0$, form a decreasing family of relatively compact subsets of Δ whose union equals the interior of Δ. Since $\partial\Delta$ has zero Lebesgue measure, we can pick $\gamma > 0$ such that $\rho(\Delta \setminus \Delta_{2\gamma}) \le \varepsilon/2$. Since $\lim_m \rho_m = \rho$ weakly on Δ, we get $\overline{\lim}\, \rho_m(\Delta \setminus \Delta_\gamma) \le \rho(\Delta \setminus \Delta_{2\gamma})$, so we can pick m_1 large enough so that $\rho_m(\Delta \setminus \Delta_\gamma) \le \varepsilon$ for $m \ge m_1$. By [Bou14, Lemme 1.13] there exists m_2 such that

$$m^{-1}\Gamma_m \cap \Delta_\gamma = m^{-1}\mathbf{Z}^n \cap \Delta_\gamma \tag{5.2}$$

for $m \ge m_2$. Now set $m_0 = \max\{m_1, m_2, \gamma^{-1}\}$. For $m \ge m_0$ we set

$$A'_m = \{\alpha \in \tfrac{1}{m}\mathbf{Z}^n \mid \alpha + [0, \tfrac{1}{m}]^n \subset \Delta\}$$

and

$$A_m = \{\alpha \in \tfrac{1}{m}\mathbf{Z}^n \mid \alpha + [-\tfrac{1}{m}, \tfrac{1}{m}]^n \subset \Delta\}.$$

If λ denotes Lebesgue measure on the unit cube $[0,1]^n \subset \mathbf{R}^n$, we see that

$$\begin{aligned}
\int_\Delta g\, d\rho \ge \sum_{\alpha \in A'_m} \int_{\alpha + [0, \frac{1}{m}]^n} g\, d\rho &= m^{-n} \sum_{\alpha \in A'_m} \int_{[0,1]^n} g(\alpha + m^{-1}w) d\lambda(w) \\
&\ge m^{-n} \sum_{\alpha \in A'_m} 2^{-n} \sum_{w \in \{0,1\}^n} g(\alpha + m^{-1}w) \ge m^{-n} \sum_{\alpha \in A_m} g(\alpha) \\
&\ge \int_{\Delta_\gamma} g\, d\rho_m \ge \int_\Delta g\, d\rho_m - \rho_m(\Delta \setminus \Delta_\gamma) \ge \int_\Delta g\, d\rho_m - \varepsilon.
\end{aligned}$$

Here the second inequality follows from the concavity of g, the fourth inequality from (5.2) together with $A_m \subset \Delta_\gamma$, and the fifth inequality from $\gamma \le 1$. This completes the proof. $\square$

5.1.3 Filtrations

By a *filtration* $\mathcal{F}$ on $R(X, L) = \bigoplus_m R_m$ we mean the data of a family

$$\mathcal{F}^\lambda R_m \subset R_m$$

of k-vector subspaces of R_m for $m \in \mathbf{N}$ and $\lambda \in \mathbf{R}_+$, satisfying

(F1) $\mathcal{F}^\lambda R_m \subset \mathcal{F}^{\lambda'} R_m$ when $\lambda \geq \lambda'$;

(F2) $\mathcal{F}^\lambda R_m = \bigcap_{\lambda' < \lambda} \mathcal{F}^{\lambda'} R_m$ for $\lambda > 0$;

(F3) $\mathcal{F}^0 R_m = R_m$ and $\mathcal{F}^\lambda R_m = 0$ for $\lambda \gg 0$;

(F4) $\mathcal{F}^\lambda R_m \cdot \mathcal{F}^{\lambda'} R_{m'} \subset \mathcal{F}^{\lambda+\lambda'} R_{m+m'}$.

The main example for us will be filtrations defined by valuations, see §5.2.1.

5.1.4 Induced graded linear series

Any filtration $\mathcal{F}$ on $R(X, L)$ defines a family

$$V_\bullet^t = V_\bullet^{\mathcal{F},t} = \bigoplus_m V_m^t$$

of graded linear series of L, indexed by $t \in \mathbf{R}_+$, and defined by

$$V_m^t := \mathcal{F}^{mt} R_m$$

for $m \in \mathbf{N}$. Set

$$T_m := T_m(\mathcal{F}) := \sup\{t \geq 0 \mid V_m^t \neq 0\},$$

with the convention $T_m = 0$ if $R_m = 0$. By (F4) above, $T_{m+m'} \geq \frac{m}{m+m'} T_m + \frac{m'}{m+m'} T_{m'}$, so Fekete's Lemma implies that the limit

$$T(\mathcal{F}) := \lim_{m\to\infty} T_m(\mathcal{F}) \in [0, +\infty]$$

exists, and equals $\sup_m T_m(\mathcal{F})$. By [BC11, Lemma 1.6], $V_\bullet^t$ contains an ample linear series for any $t < T(\mathcal{F})$. It follows that

$$T(\mathcal{F}) = \sup\{t \geq 0 \mid \mathrm{vol}(V_\bullet^t) > 0\}. \tag{5.3}$$

We say that the filtration $\mathcal{F}$ is *linearly bounded* if $T(\mathcal{F}) < \infty$.

5.1.5 Concave transform and limit measure

Let $\Delta = \Delta(L) \subset \mathbf{R}^n$ be the Okounkov body of $R(X, L)$. The filtration $\mathcal{F}$ of $R(X, L)$ induces a *concave transform*

$$G = G^{\mathcal{F}} : \Delta \to \mathbf{R}_+$$

defined as follows. For $t \geq 0$, consider the graded linear series $V_\bullet^t \subset R(X, L)$ and the associated Okounkov body $\Delta^t = \Delta(V_\bullet^t) \subset \mathbf{R}^n$. We have $\Delta^t \supset \Delta^{t'}$ for $t < t'$, $\Delta^0 = \Delta$ and $\Delta^t = \emptyset$ for $t > T(\mathcal{F})$. The function G is now defined on Δ by

$$G(\alpha) = \sup\{t \in \mathbf{R}_+ \mid \alpha \in \Delta^t\}. \tag{5.4}$$

In other words, $\{G \geq t\} = \Delta^t$ for $0 \leq t \leq T(\mathcal{F})$. Thus G is a concave, upper semicontinuous function on Δ with values in $[0, T(\mathcal{F})]$.

As noted in the proof of [BKMS16, Lemma 2.22], the Brunn-Minkowski inequality implies

Proposition 5.1.3. *The function $t \to \mathrm{vol}(V_\bullet^t)^{1/n}$ is non-increasing and concave on $[0, T(\mathcal{F}))$. As a consequence, it is continuous on $\mathbf{R}_+$, except possibly at $t = T(\mathcal{F})$.*

We define the *limit measure* $\mu = \mu^{\mathcal{F}}$ of the filtration $\mathcal{F}$ as the pushforward

$$\mu = G_* \rho.$$

Thus μ is a positive measure on $\mathbf{R}_+$ of mass $\mathrm{vol}(\Delta) = \frac{1}{n!}\mathrm{vol}(L)$, with support in $[0, T(\mathcal{F})]$.

Corollary 5.1.4. *The limit measure μ satisfies*

$$\mu = -\frac{1}{n!}\frac{d}{dt}\mathrm{vol}(V_\bullet^t) = -\frac{d}{dt}\mathrm{vol}(\Delta^t)$$

and is absolutely continuous with respect to Lebesgue measure, except possibly at $t = T(\mathcal{F})$, where $\mu\{T(\mathcal{F})\} = \lim_{t \to T(\mathcal{F})-}\mathrm{vol}(V_\bullet^t)$.

As a companion to $T(\mathcal{F})$ we now define another invariant of $\mathcal{F}$:

$$S(\mathcal{F}) := \frac{1}{\mathrm{vol}(L)}\int_0^\infty \mathrm{vol}(V_\bullet^t)\, dt = \frac{n!}{\mathrm{vol}(L)}\int_0^\infty t\, d\mu(t) = \frac{1}{\mathrm{vol}(\Delta)}\int_\Delta G\, d\rho.$$

Note that $\mu^{\mathcal{F}}$, $S(\mathcal{F})$, and $T(\mathcal{F})$ do not depend on the choice of the auxiliary valuation z.

Remark 5.1.5. The invariant $S(\mathcal{F})$ can also be interpreted as the (suitably normalized) volume of the filtered Okounkov body associated to $\mathcal{F}$, see [BC11, Corollary 1.13].

Lemma 5.1.6. *We have* $\frac{1}{n+1}T(\mathcal{F}) \le S(\mathcal{F}) \le T(\mathcal{F})$.

Proof. The second inequality is clear since $\mathrm{vol}(V_\bullet^t) \le \mathrm{vol}(L)$ and $\mathrm{vol}(V_\bullet^t) = 0$ for $t > T(\mathcal{F})$. The first follows from the concavity of $t \mapsto \mathrm{vol}(V_\bullet^t)^{1/n}$, which yields $\mathrm{vol}(V_\bullet^t) \ge \mathrm{vol}(L)(1 - \frac{t}{T(\mathcal{F})})^n$. □

Remark 5.1.7. At least when L is ample, a filtration on $R(X, L)$ induces a metric on the Berkovich analytification of L with respect to the trivial absolute value on k. It is shown in [BoJ18] that S and T extend as "energy-like" functionals on the space of such metrics. As a special case of that analysis, it is shown that $S(\mathcal{F}) \le \frac{n}{n+1}T(\mathcal{F})$. The case when the filtration is associated to a test configuration is treated in [BHJ15].

5.1.6 Jumping numbers

Given a filtration $\mathcal{F}$ as above, consider the *jumping numbers*

$$0 \le a_{m,1} \le \cdots \le a_{m,N_m} = mT_m(\mathcal{F}),$$

defined for $m \in M(L)$ by

$$a_{m,j} = a_{m,j}(\mathcal{F}) = \inf\{\lambda \in \mathbf{R}_+ \mid \operatorname{codim} \mathcal{F}^\lambda R_m \ge j\}$$

for $1 \le j \le N_m$. Define a positive measure $\mu_m = \mu_m^{\mathcal{F}}$ on $\mathbf{R}_+$ by

$$\mu_m = \frac{1}{m^n}\sum_j \delta_{m^{-1}a_{m,j}} = -\frac{1}{m^n}\frac{d}{dt}\dim \mathcal{F}^{mt}R_m.$$

The following result is [BC11, Theorem 1.11].

Theorem 5.1.8. *If $\mathcal{F}$ is linearly bounded,* i.e. $T(\mathcal{F}) < +\infty$*, then we have*

$$\lim_{m\to\infty} \mu_m = \mu$$

in the weak sense of measures on $\mathbf{R}_+$.

For $m \in M(L)$, consider the rescaled sum of the jumping numbers:

$$S_m(\mathcal{F}) = \frac{1}{mN_m}\sum_j a_{m,j} = \frac{m^n}{N_m}\int_0^\infty t\,d\mu_m(t).$$

Clearly $0 \le S_m(\mathcal{F}) \le T_m(\mathcal{F})$.

Lemma 5.1.9. *For any linearly bounded filtration $\mathcal{F}$ on $R(X,L)$ we have*

$$S_m(\mathcal{F}) \le \frac{m^n}{N_m}\int_\Delta G\,d\rho_m, \tag{5.5}$$

for any $m \in M(L)$. Further, we have $\lim_{m\to\infty} S_m(\mathcal{F}) = S(\mathcal{F})$.

Proof. The equality $\lim_m S_m(\mathcal{F}) = S(\mathcal{F})$ follows from Theorem 5.1.8. For the inequality, pick a basis $s_1, s_2, \dots, s_{N_m}$ of R_m such that $\sup\{\lambda \mid s_j \in \mathcal{F}^\lambda R_m\} = a_{m,j}$, $1 \le j \le N_m$. Set $\alpha_j := \mathrm{ord}_z(s_j)$. Since ord_z has transcendence degree 0, we have $\Gamma_m = \{\alpha_1, \dots, \alpha_m\}$. Thus the right hand side of (5.5) equals $\frac{1}{N_m}\sum_{j=1}^{N_m} G(m^{-1}\alpha_j)$ whereas the left-hand side is equal to $\frac{1}{N_m}\sum_{j=1}^{N_m} m^{-1}a_{m,j}$, so it suffices to prove $G(m^{-1}\alpha_j) \ge m^{-1}a_{m,j}$ for $1 \le j \le N_m$. But this is clear from (5.4), since $\alpha_j = \mathrm{ord}_z(s_j)$ and $s_j \in \mathcal{F}^{a_{m,j}}R_m$ imply $m^{-1}\alpha_j \in \Delta^{m^{-1}a_{m,j}}$. □

Corollary 5.1.10. *For every $\varepsilon > 0$ there exists $m_0 = m_0(\varepsilon) > 0$ such that*

$$S_m(\mathcal{F}) \le (1+\varepsilon)S(\mathcal{F})$$

for any $m \ge m_0$ and any linearly bounded filtration $\mathcal{F}$ on $R(X,L)$.

Proof. Set $V := \mathrm{vol}(\Delta)$. Pick $\varepsilon' > 0$ with $(V^{-1}+\varepsilon')(V+(n+1)\varepsilon') \le (1+\varepsilon)$. Note that $0 \le G \le T(\mathcal{F})$. Applying Lemma 5.1.2 to $g = G/T(\mathcal{F})$ we pick $m_0 \in M(L)$ such that

$$\int_\Delta G\,d\rho_m \le \int_\Delta G\,d\rho + \varepsilon' T(\mathcal{F}) = VS(\mathcal{F}) + \varepsilon' T(\mathcal{F}) \le (V+(n+1)\varepsilon')S(\mathcal{F})$$

for $M(L) \ni m \ge m_0$, where we have used Lemma 5.1.6 in the last inequality. By Theorem 5.1.1 we may also assume $\frac{m^n}{N_m} \le V^{-1}+\varepsilon'$ for $M(L) \ni m \ge m_0$. Lemma 5.1.9 now yields

$$S_m(\mathcal{F}) \le \frac{m^n}{N_m}\int_\Delta G\,d\rho_m \le (V^{-1}+\varepsilon')(V+(n+1)\varepsilon')S(\mathcal{F}) \le (1+\varepsilon)S(\mathcal{F}),$$

for $M(L) \ni m \ge m_0$, which completes the proof. □

5.1.7 N-filtrations.

A filtration $\mathcal{F}$ of $R(X,L)$ is an $\mathbf{N}$-*filtration* if all its jumping numbers are integers, that is,

$$\mathcal{F}^\lambda R_m = \mathcal{F}^{\lceil\lambda\rceil}R_m$$

for all $\lambda \in \mathbf{R}_+$ and $m \in M(L)$. Any filtration $\mathcal{F}$ induces an $\mathbf{N}$-filtration $\mathcal{F}_\mathbf{N}$ by setting

$$\mathcal{F}_\mathbf{N}^\lambda R_m := \mathcal{F}^{\lceil\lambda\rceil}R_m.$$

Note that $\mathcal{F}_{\mathbf{N}}$ is a filtration of $R(X,L)$. Indeed, conditions (F1)–(F4) in §5.1.3 are trivially satisfied and (F4) follows from $\lceil\lambda\rceil + \lceil\lambda'\rceil \geq \lceil\lambda+\lambda'\rceil$.

The jumping numbers of $\mathcal{F}_{\mathbf{N}}$ and $\mathcal{F}$ are related by $a_{m,j}(\mathcal{F}_{\mathbf{N}}) = \lfloor a_{m,j}(\mathcal{F})\rfloor$. This implies the following statement.

Proposition 5.1.11. *If $\mathcal{F}$ is a filtration of $R(X,L)$, then*

$$T_m(\mathcal{F}_{\mathbf{N}}) = \lfloor m \cdot T_m(\mathcal{F})\lfloor/m \quad \textit{and } S_m(\mathcal{F}) - m^{-1} \leq S_m(\mathcal{F}_{\mathbf{N}}) \leq S_m(\mathcal{F})$$

for $m \in M(L)$. As a consequence, $T(\mathcal{F}_{\mathbf{N}}) = T(\mathcal{F})$, $S(\mathcal{F}_{\mathbf{N}}) = S(\mathcal{F})$, and $\mu^{\mathcal{F}_{\mathbf{N}}} = \mu^{\mathcal{F}}$.

As a consequence, we obtain the following formula for $S(\mathcal{F})$, similar to [FO16, Lemma 2.2].

Corollary 5.1.12. *If $\mathcal{F}$ is a filtration of $R(X,L)$, then*

$$S(\mathcal{F}) = S(\mathcal{F}_{\mathbf{N}}) = \lim_{m\to\infty} \frac{1}{mN_m} \sum_{j\geq 1} \dim \mathcal{F}^j R_m.$$

Proof. Since the jumping numbers of $\mathcal{F}_{\mathbf{N}}$ are integers, we have

$$S_m(\mathcal{F}_{\mathbf{N}}) = \frac{1}{mN_m}\sum_{j\geq 0} j\left(\dim \mathcal{F}^j_{\mathbf{N}} R_m - \dim \mathcal{F}^{j+1}_{\mathbf{N}} R_m\right) = \frac{1}{mN_m}\sum_{j\geq 1} \dim \mathcal{F}^j_{\mathbf{N}} R_m$$

for any $m \in M(L)$. Letting $m \to \infty$ and using Proposition 5.1.11 completes the proof. $\square$

5.2 Global invariants of valuations

As before, X is a normal projective variety of dimension n over k.

Let L be a big line bundle on X. Following [BKMS16] we study invariants of valuations on X defined using the section ring of L. The new results here are Corollary 5.2.6 and the results in §5.2.5.

5.2.1 Induced filtrations

Any valuation $v \in \mathrm{Val}_X$ induces a filtration $\mathcal{F}_v$ on $R(X,L)$ via

$$\mathcal{F}^t_v R_m := \{s \in R_m \mid v(s) \geq t\}$$

for $m \in \mathbf{N}$ and $t \in \mathbf{R}_+$, where we recall that $R_m = H^0(X, mL)$.

We say that v has *linear growth* if $\mathcal{F}_v$ is linearly bounded. By Lemma 2.8 in [BKMS16] this notion depends only on v as a valuation, and not on the pair (X, L). Theorem 2.16 in *loc. cit.* states that if v is centered at a closed point on X, then v has linear growth iff $\mathrm{vol}(v) > 0$.

Lemma 5.2.1. *Any divisorial valuation has linear growth. If (X, B) is a klt pair, then any $v \in \mathrm{Val}_X$ satisfying $A_{X,B}(v) < \infty$ has linear growth.*

Proof. We may assume X is smooth and $B = 0$. By [BKMS16, Proposition 2.12], every divisorial valuation has linear growth. For the second assertion, if $A(v) < \infty$, Izumi's inequality (see [JM12, Proposition 5.10]) implies $v \le A(v)\,\mathrm{ord}_\xi$, where $\xi = c_X(v)$. Since ord_ξ is divisorial, it has linear growth; hence so does v. □

5.2.2 Global invariants

Consider a valuation v of linear growth. We define invariants of v as the corresponding invariants of the induced filtration $\mathcal{F}_v$, namely:

(i) the *limit measure* of v is $\mu_v := \mu^{\mathcal{F}_v}$;

(ii) the *expected vanishing order* of v is $S(v) := S(\mathcal{F}_v) = \int_0^\infty t\, d\mu_v(t)$;

(iii) the *maximal vanishing order* or *pseudo-effective threshold* of v is $T(v) := T(\mathcal{F}_v)$.

Note that $T(v)$ is denoted by $a_{\max}(\|L\|, v)$ in [BKMS16]. It follows from Lemma (5.1.6) (see also Remark 5.1.7) that

$$\frac{1}{n+1} T(v) \le S(v) \le T(v). \tag{5.6}$$

The invariants S and T are homogeneous of order 1: $S(tv) = tS(v)$ and $T(tv) = tT(v)$ for $t > 0$. Similarly, $\mu_{tv} = t_*\mu_v$, where $t\colon \mathbf{R}_+ \to \mathbf{R}_+$ denotes multiplication by t. In particular, if v is the trivial valuation on X, then $S(v) = T(v) = 0$ and $\mu_v = \delta_0$.

Remark 5.2.2. If we think of v as an order of vanishing, then the limit measure μ_v describes the asymptotic distribution of the (normalized) orders of vanishing of v on $R(X, L)$. This explains the chosen name of $S(v)$ and the first name of $T(v)$.

For an alternative description of $S(v)$ and $T(v)$, define, for $t \ge 0$,

$$\mathrm{vol}(L; v \ge t) := \mathrm{vol}(V_\bullet^t) = \lim_{m\to\infty} \frac{n!}{m^n} \dim \mathcal{F}_v^{tm} H^0(X, mL).$$

Theorem 5.2.3. *Let L be a big line bundle and $v \in \mathrm{Val}_X^*$ a valuation of linear growth. Then the limit defining* $\mathrm{vol}(L; v \geq t)$ *exists for every $t \geq 0$. Further:*

(i) $T(v) = \sup\{t \geq 0 \mid \mathrm{vol}(L; v \geq t) > 0\}$*;*

(ii) the function $t \mapsto \mathrm{vol}(L; v \geq t)^{1/n}$ *is decreasing and concave on* $[0, T(v))$*;*

(iii) $\mu_v = -\frac{d}{dt}\mathrm{vol}(L; v \geq t)$*; further,* $\mathrm{supp}\,\mu_v = [0, T(v)]$*, and μ is absolutely continuous with respect to Lebesgue measure, except for a possible point mass at $T(v)$;*

(iv) $S(v) = V^{-1}\int_0^{T(v)} \mathrm{vol}(L; v \geq t)\,dt$*;*

(v) if L is nef, then the function $t \mapsto \mathrm{vol}(L; v \geq t)$ *is strictly decreasing on* $[0, T(v)]$ *and* $\mathrm{supp}\,\mu_v = [0, T(v)]$.

Proof. The assertions (i)–(iv) are special cases of the properties of linearly bounded filtrations in §5.1. If L is nef, the discussion after Remark 2.7 in [BKMS16] shows that $t \mapsto \mathrm{vol}(L; v \geq t)$ is strictly decreasing on $[0, T(v))$. This implies $\mathrm{supp}\,\mu = [0, T(v)]$, so that (v) holds. □

Remark 5.2.4. In fact, the measure μ_v likely has no point mass at $T(v)$. This is true when v is divisorial, or simply quasimonomial, see [BKMS16, Proposition 2.25].

We also define $S_m(v) := S_m(\mathcal{F}_v)$ and $T_m(v) := T_m(\mathcal{F}_v)$ for $m \in M(L)$. These invariants can be concretely described as follows. First,

$$T_m(v) = \max\{m^{-1}v(s) \mid s \in H^0(X, mL)\}. \tag{5.7}$$

A similar description is true for S_m.

Lemma 5.2.5. *For any $m \in M(L)$ and any $v \in \mathrm{Val}_X$ we have*

$$S_m(v) = \max_{s_j} \frac{1}{mN_m}\sum_{j=1}^{N_m} v(s_j), \tag{5.8}$$

where the maximum is over all bases $s_1, \dots, s_{N_m}$ of $H^0(X, mL)$.

Proof. First consider any basis $s_1, \dots, s_{N_m}$ of $H^0(X, mL)$. We may assume $v(s_1) \leq v(s_2) \leq \cdots \leq v(s_{N_m})$. Then $v(s_j) \leq a_{m,j}$, for all j, where $a_{m,j}$ is the jth jumping number of $\mathcal{F}_v H^0(X, mL)$. Thus $(mN_m)^{-1}\sum_j v(s_j) \leq (mN_m)^{-1}\sum_j a_{m,j} = S_m(v)$. On the other hand, we can pick the basis such that $v(s_j) = a_{m,j}$, and then $(mN_m)^{-1}\sum_j v(s_j) = S_m(v)$. □

Corollary 5.1.10 immediately implies

Corollary 5.2.6. *For any $v \in \mathrm{Val}_X$ of linear growth, we have $\lim_{m\to\infty} S_m(v) = S(v)$. Further, given $\varepsilon > 0$ there exists $m_0 = m_0(\varepsilon) > 0$ such that if $m \geq m_0$, then*

$$S_m(v) \leq S(v)(1+\varepsilon)$$

for all $v \in \mathrm{Val}_X$ of linear growth.

5.2.3 Behavior of invariants

The invariants $S(v)$, $T(v)$ and μ_v depend on L (and X). If we need to emphasize this dependence, we write $S(v; L)$, $T(v; L)$ and $\mu_{v;L}$.

Lemma 5.2.7. *Let v be a valuation of linear growth.*

(i) If $r \in \mathbf{N}^*$, then $S(v; rL) = rS(v; L)$, $T(v; rL) = rT(v; L)$ and $\mu_{v;rL} = r_*\mu_{v;L}$.

(ii) If $\rho\colon X' \to X$ is a projective birational morphism, with X' normal, and $L' = \rho^*L$, then $S(v; L') = S(v; L)$, $T(v; L') = T(v; L)$, and $\mu_{v;L'} = \mu_{v;L}$;

(iii) the invariants $S(v; L)$, $T(v; L)$ and $\mu_{v;L}$ only depend on the numerical class of L.

Proof. Properties (i)–(ii) are clear from the definitions. As for (ii), [BKMS16, Proposition 3.1] asserts that the measure $\mu_{v;L}$ only depends on the numerical class of L; hence the same true for $S(v; L)$ and $T(v; L)$. $\square$

Remark 5.2.8. In view of (i) and (iii) we can define $S(v; L)$ for a big class $L \in \mathrm{NS}(X)_{\mathbf{Q}}$ by $S(v; L) := r^{-1}S(v; rL)$ for r sufficiently divisible. The same holds for $T(v; L)$ and $\mu_{v;L}$.

5.2.4 The case of divisorial valuations

We now interpret the invariants $S(v)$ and $T(v)$ in the case when v is a divisorial valuation. By homogeneity in v and by Lemma 5.2.7 (ii) it suffices to consider the case when $v = \mathrm{ord}_E$ for a prime divisor E on X. In this case, $\mathrm{vol}(L; v \geq t) = \mathrm{vol}(L - tE)$, so Theorem 5.2.3 implies

Corollary 5.2.9. *Let $E \subset X$ be a prime divisor. Then we have:*

(i) $T(\mathrm{ord}_E) = \sup\{t > 0 \mid L - tE \text{ is pseudoeffective}\}$;

(ii) $S(\mathrm{ord}_E) = \mathrm{vol}(L)^{-1} \int_0^\infty \mathrm{vol}(L - tE)\, dt.$

Statement (i) explains the name pseudoeffective threshold for $T(v)$.

Remark 5.2.10. The invariants $S(v)$ and $T(v)$ for v divisorial have been explored by K. Fujita [Fuj16b], C. Li [Li15b], and Y. Liu [Liu16]. In the notation of [Fuj16b],

$$T(\mathrm{ord}_E) = \tau(E) \quad \text{and} \quad S(\mathrm{ord}_E) = \tau(E) - \mathrm{vol}(L)^{-1} j(E).$$

The invariant $S(\mathrm{ord}_\xi)$, for $\xi \in X$ a regular closed point, also plays an important role in [MR15] and was used in unpublished work of P. Salberger from 2006.

Proposition 5.2.11. *If L is ample and $v \in \mathrm{Val}_X$ is divisorial, then $\frac{1}{n+1} \le \frac{S(v)}{T(v)} \le \frac{n}{n+1}$.*

Proof. The first inequality follows from the concavity of $t \to \mathrm{vol}(L; v \ge t)^{1/n}$ and is a special case of Lemma 5.1.6. The second inequality is treated in [Fuj17, Proposition 2.1]. (In *loc. cit.* we have $L = -K_X$, but this assumption is not used in the proof.) □

Remark 5.2.12. When L is ample, Proposition 5.2.11 in fact holds for any $v \in \mathrm{Val}_X$ of linear growth; see Remark 5.1.7.

5.2.5 Invariants as functions on valuation space

Proposition 5.2.13. *The invariants S and T define lower semicontinuous functions on Val_X. For any $m \in M(L)$, the functions S_m and T_m are also lower semicontinuous.*

Proof. First consider $m \in M(L)$. For any nonzero $s \in H^0(X, mL)$, the function $v \mapsto v(s)$ is continuous. It therefore follows from (5.7) and (5.8) that S_m and T_m are lower semicontinuous. Hence $T = \sup_m T_m$ is also lower semicontinuous. The lower semicontinuity of S is slightly more subtle. Pick any $t \in \mathbf{R}_+$. We must show that the set $W := \{v \in \mathrm{Val}_X \mid S(v) > t\}$ is open in Val_X. Pick any $v \in w$ and pick $\varepsilon > 0$ such that $S(v) > (1+\varepsilon)t$. By Corollary 5.2.6, there exists $m \gg 0$ such that $S_m(v) > (1+\varepsilon)t$ and $S_m \le (1+\varepsilon)S$ on Val_X. Since S_m is lower semicontinuous, there exists an open neighborhood U of v in Val_X such that $S_m > (1+\varepsilon)t$ on U. Then $U \subset W$, which completes the proof. □

Remark 5.2.14. The functions S and T are not continuous in general. Consider the case $X = \mathbf{P}^1$, $L = \mathcal{O}_X(1)$. If $(\xi_j)_{j=1}^\infty$ is a sequence of distinct closed points, then $v_j = \mathrm{ord}_{\xi_j}$, $j \ge 1$ defines a sequence in Val_X converging to the trivial valuation v on X. Then $S(v_j) = 1/2$ and $T(v_j) = 1$ for all j, whereas $S(v) = T(v) = 0$.

The next result is a global version of [LiX16, Proposition 2.3].

Proposition 5.2.15. *Let $v, w \in \mathrm{Val}_X$ be valuations of linear growth, such that $v \leq w$.*

(i) We have $S(v) \leq S(w)$ and $T(v) \leq T(w)$.

(ii) If L is ample and $S(v) = S(w)$, then $v = w$.

Remark 5.2.16. The assertion in (ii) is false for T in general. Indeed, let $X = \mathbf{P}^2$ and $L = \mathcal{O}_X(1)$. Consider an affine toric chart $\mathbf{A}^2 \subset \mathbf{P}^2$ with affine coordinates (z_1, z_2). Let v and w be monomial valuations in these coordinates with $v(z_1) = w(z_1) = 1$ and $0 < w(z_2) < w(z_1) \leq 1$. Then $w \leq v$ and $T(v) = T(w) = 1$, but $w \neq v$.

Proof of Proposition 5.2.15. The assertion in (i) is trivial. To establish (ii) we follow the proof of [LiX16, Proposition 2.3]. Note that by Lemma 5.2.7 we may replace L by a positive multiple.

Suppose $v \leq w$ but $v \neq w$. We must prove $S(v) < S(w)$. We may assume there exists $s \in H^0(X, L)$ with $v(s) < w(s)$. Indeed, there exists $\lambda \in \mathbf{R}_+^*$ such that $\mathfrak{a}_\lambda(v) \subsetneq \mathfrak{a}_\lambda(w)$. Replacing L by a multiple, we may assume $L \otimes \mathfrak{a}_\lambda(w)$ is globally generated, and then

$$\mathcal{F}_v^\lambda H^0(X, L) = H^0(X, L \otimes \mathfrak{a}_\lambda(v)) \subsetneq H^0(X, L \otimes \mathfrak{a}_\lambda(w)) = \mathcal{F}_w^\lambda H^0(X, L),$$

so that there exists $s \in H^0(X, L)$ with $v(s) < w(s) = \lambda$. After rescaling v and w, we may assume $w(s) = p \in \mathbf{N}^*$ and $v(s) \leq p - 1$.

We claim that for $m, j \in \mathbf{N}$, we have

$$\dim(\mathcal{F}_w^j R_m / \mathcal{F}_v^j R_m) \geq \sum_{1 \leq i \leq \min\{j/p, m\}} \dim\left(\mathcal{F}_v^{j-ip} R_{m-i} / \mathcal{F}_v^{j-ip+1} R_{m-i}\right). \tag{5.9}$$

To prove the claim, pick, for any i with $1 \leq i \leq \min\{j/p, m\}$, elements

$$s_{i,1}, \ldots, s_{i,b_i} \in \mathcal{F}_v^{j-ip} R_{m-i}$$

whose images form a basis for $\mathcal{F}_v^{j-ip} R_{m-i} / \mathcal{F}_v^{j-ip+1} R_{m-i}$. As in [LiX16, Proposition 2.3], the elements

$$\{s^i s_{i,l} \mid 1 \leq i \leq \min\{j/p, m\}, 1 \leq l \leq b_i\}$$

are then linearly independent in $\mathcal{F}_v^k R_m / \mathcal{F}_w^k R_m$. This completes the proof of the claim.

By Corollary 5.1.12 we have

$$S(v) - S(w) = \lim_{m\to\infty} \frac{1}{mN_m} \sum_{j\geq 1} \left(\dim \mathcal{F}_w^j R_m - \dim \mathcal{F}_v^j R_m\right)$$

Now (5.9) gives

$$\begin{aligned}
\sum_{j\geq 1} \left(\dim \mathcal{F}_w^j R_m - \dim \mathcal{F}_v^j R_m\right) &\geq \sum_{j\geq 1} \sum_{1\leq i\leq \min\{\frac{j}{p}, m\}} \left(\dim \mathcal{F}_v^{j-ip} R_{m-i} - \dim \mathcal{F}_v^{j-ip+1} R_{m-i}\right) \\
&= \sum_{1\leq i\leq m} \sum_{j\geq pi} \left(\dim \mathcal{F}_v^{j-ip} R_{m-i} - \dim \mathcal{F}_v^{j-ip+1} R_{m-i}\right) \\
&= \sum_{1\leq i\leq m} \dim R_{m-i}
\end{aligned}$$

We conclude that

$$S(v) - S(w) \geq \limsup_{m\to\infty} \frac{1}{mN_m} \sum_{1\leq i\leq m} \dim(R_{m-i}) > 0,$$

since $\dim R_m = N_m \sim m^n(L^n)$ as $m \to \infty$. This completes the proof. □

5.2.6 Base ideals of filtrations

In this section we assume L is *ample*. To an arbitrary filtration $\mathcal{F}$ of $R(X, L)$ we associate *base ideals* as follows. For $\lambda \in \mathbf{R}_+$ and $m \in M(L)$, set

$$\mathfrak{b}_{\lambda,m}(\mathcal{F}) := \mathfrak{b}\left(|\mathcal{F}^\lambda H^0(X, mL)|\right).$$

Lemma 5.2.17. *For $\lambda \in \mathbf{R}_+$ the sequence $(\mathfrak{b}_{\lambda,m}(\mathcal{F}))_m$ is stationary, with limit $\sum_{m\in M(L)} \mathfrak{b}_{\lambda,m}$.*

Proof. It follows from (F4) that if $m_1, m_2 \in M(L)$ and $\lambda_1, \lambda_2 \in \mathbf{R}_+$, then

$$\mathfrak{b}_{\lambda_1,m_1}(\mathcal{F}) \cdot \mathfrak{b}_{\lambda_2,m_2}(\mathcal{F}) \subset \mathfrak{b}_{\lambda_1+\lambda_2,m_1+m_2}(\mathcal{F}) \tag{5.10}$$

Since L is ample, there exists $m_0 \in \mathbf{N}^*$ such that mL is globally generated for $m \geq m_0$. In particular, $\mathfrak{b}_{0,m} = \mathcal{O}_X$ for $m \geq m_0$. As a consequence of (5.10), if $m \in M(L)$ and $m' \geq m_0$, then $\mathfrak{b}_{\lambda,m+m'}(\mathcal{F}) \supset \mathfrak{b}_{\lambda,m}(\mathcal{F}) \cdot \mathfrak{b}_{0,m'}(\mathcal{F}) = \mathfrak{b}_{\lambda,m}(\mathcal{F})$. The lemma follows. □

Using the lemma, set $\mathfrak{b}_\lambda(\mathcal{F}) := \mathfrak{b}_{\lambda,m}(\mathcal{F})$ for $m \gg 0$. Thus $\mathfrak{b}_{\lambda,m}(\mathcal{F}) \subset \mathfrak{b}_\lambda(\mathcal{F})$ for $m \in M(L)$.

Corollary 5.2.18. *We have $\mathfrak{b}_0(\mathcal{F}) = \mathcal{O}_X$ and $\mathfrak{b}_\lambda(\mathcal{F}) \cdot \mathfrak{b}_{\lambda'}(\mathcal{F}) \subset \mathfrak{b}_{\lambda+\lambda'}(\mathcal{F})$ for $\lambda, \lambda' \in \mathbf{R}_+$.*

Lemma 5.2.19. *If v is a valuation on X, then $\mathfrak{b}_\lambda(\mathcal{F}_v) = \mathfrak{a}_\lambda(v)$ for all $\lambda \in \mathbf{R}_+$.*

Proof. Given λ, $mL \otimes \mathfrak{a}_\lambda(v)$ is globally generated for $m \gg 0$; hence $\mathfrak{b}_{\lambda,m}(\mathcal{F}_v) = \mathfrak{a}_\lambda(v)$. □

Using base ideals, we can relate the invariants of a filtration to those of a valuation.

Lemma 5.2.20. *If $v(\mathfrak{b}_\bullet(\mathcal{F})) \geq 1$, then $\mathcal{F}^p R_m \subset \mathcal{F}_v^p R_m$ for all $m \in M(L)$ and $p \in \mathbf{N}^*$.*

Proof. By Lemma 2.2.1, we have $\mathfrak{b}_p(\mathcal{F}) \subset \mathfrak{a}_p(v)$. Since we also have $\mathfrak{b}_{\lambda,m}(\mathcal{F}) \subset \mathfrak{b}_\lambda(\mathcal{F})$ for all $m \in M(L)$, this implies

$$\mathcal{F}^p R_m \subset H^0(X, mL \otimes \mathfrak{b}_{p,m}(\mathcal{F})) \subset H^0(X, mL \otimes \mathfrak{a}_p(v)) = \mathcal{F}_v^p R_m.$$

which completes the proof. □

Corollary 5.2.21. *Let $\mathcal{F}$ be a linearly bounded filtration of $R(X, L)$. Then*

$$S(v) \geq v(\mathfrak{b}_\bullet(\mathcal{F}))S(\mathcal{F}) \quad \text{and} \quad T(v) \geq v(\mathfrak{b}_\bullet(\mathcal{F}))T(\mathcal{F}),$$

for any valuation $v \in \mathrm{Val}_X$.

Proof. The assertions are trivial when $v(\mathfrak{b}_\bullet(\mathcal{F})) = 0$, so we may assume $v(\mathfrak{b}_\bullet(\mathcal{F})) = 1$ after scaling v. In this case, Lemma 5.2.20 shows that $\mathcal{F}^p R_m \subset \mathcal{F}_v^p R_m$ for $p \in \mathbf{N}^*$ and $m \in M(L)$. Using Proposition 5.1.11 and Corollary 5.1.12, this implies

$$S(\mathcal{F}) = S(\mathcal{F}_{\mathbf{N}}) \leq S(\mathcal{F}_{v,\mathbf{N}}) = S(\mathcal{F}_v) = S(v),$$

and similarly $T(\mathcal{F}) \leq T(v)$. The proof is complete. □

5.3 Thresholds

Let (X, B) be a klt pair, and L a big line bundle on X. In this section we study the log canonical threshold of L, and introduce a new related invariant, the stability threshold of L. Both are defined in terms of the asymptotic behavior of the singularities of the members of the linear system $|mL|$ as $m \to \infty$.

Throughout this chapter we will use our abbreviated notation for log discrepancies and log canonical thresholds. Specifically, we write $A(-)$ and $\mathrm{lct}(-)$ for $A_{X,B}(-)$ and $\mathrm{lct}(X, B; -)$.

5.3.1 The log canonical threshold

Following [CS08] the *log canonical threshold* $\alpha(L)$ of L is the infimum of $\operatorname{lct}(D)$ with D an effective $\mathbf{Q}$-divisor $\mathbf{Q}$-linearly equivalent to L. As explained by Demailly (see [CS08, Theorem A.3]), this can be interpreted analytically as a generalization of the α-invariant introduced by Tian [Tia97].

For $m \in M(L)$, we also set

$$\alpha_m(L) := \inf\{m \operatorname{lct}(D) \mid D \in |mL|\}.$$

It is then clear that $\alpha(L) = \inf_{m\in M(L)} \alpha_m(L)$. The invariants α_m and α can be computed using invariants of valuations, as follows:

Proposition 5.3.1. *For $m \in M(L)$, we have*

$$\alpha_m(L) = \inf_v \frac{A(v)}{T_m(v)} = \inf_E \frac{A(\operatorname{ord}_E)}{T_m(\operatorname{ord}_E)}, \tag{5.11}$$

where v runs through nontrivial valuations on X with $A(v) < \infty$, and E through prime divisors over X.

Proof. Writing out the definition of $\operatorname{lct}(D)$, we see that

$$\alpha_m(L) = m \cdot \inf_{D\in|mL|} \left(\inf_v \frac{A(v)}{v(D)} \right),$$

where the second infimum may be taken over nontrivial valuations with finite log discrepancy, or only divisorial valuations. Switching the order of the two infima and noting $\sup_{D\in|mL|} v(D) = m \cdot T_m(v)$ yields (5.11). □

Corollary 5.3.2. *We have*

$$\alpha(L) = \inf_v \frac{A(v)}{T(v)} = \inf_E \frac{A(\operatorname{ord}_E)}{T(\operatorname{ord}_E)}, \tag{5.12}$$

where v runs through valuations on X with $A(v) < \infty$ and E over prime divisors over X.

Proof. Since $T(v) = \sup_{m\in M(L)} T_m(v)$, (5.12) follows from (5.11). □

5.3.2 The stability threshold

Given $m \in M(L)$, we say, following [FO16], that an effective $\mathbf{Q}$-divisor $D \sim_{\mathbf{Q}} L$ is of *m-basis type* if there exists a basis $s_1, \dots, s_{N_m}$ of $H^0(X, mL)$ with

$$D = \frac{1}{mN_m} \sum_{j=1}^{N_m} \{s_j = 0\}. \tag{5.13}$$

Set

$$\delta_m(L) := \inf\{\operatorname{lct}(D) \mid \ D \text{ of } m\text{-basis type}\}, \tag{5.14}$$

and define the *stability threshold* of L as

$$\delta(L) := \limsup_{m\to\infty} \delta_m(L).$$

We shall see shortly that this limsup is in fact a limit.

Proposition 5.3.3. *For $m \in M(L)$, we have*

$$\delta_m(L) = \inf_v \frac{A(v)}{S_m(v)} = \inf_E \frac{A(\operatorname{ord}_E)}{S_m(\operatorname{ord}_E)},$$

where v runs through nontrivial valuations on X with $A(v) < \infty$ and E through prime divisors over X.

Proof. Note that

$$\delta_m(L) = \inf_{D \text{ of } m\text{-basis type}} \left(\inf_v \frac{A(v)}{v(D)} \right),$$

where the second infimum runs through all valuations with $A(v) < \infty$ or only divisorial valuations of the form $v = \operatorname{ord}_E$. Switching the order of the two infima and applying Lemma 5.2.5 yields the desired equality. □

Theorem 5.3.4. *We have $\delta(L) = \lim_{m\to\infty} \delta_m(L)$. Further,*

$$\delta(L) = \inf_v \frac{A(v)}{S(v)} = \inf_E \frac{A(\operatorname{ord}_E)}{S(\operatorname{ord}_E)},$$

where v runs through nontrivial valuations on X with $A(v) < \infty$ and E through prime divisors over X.

Proof. We will only prove the first equality; the proof of the second being essentially identical. Let us use Proposition 5.3.3 and Corollary 5.2.6. The fact that $\lim_{m\to\infty} S_m = S$

pointwise on Val_X directly shows that

$$\delta(L) = \limsup_m \delta_m(L) \leq \inf_v \frac{A(v)}{S(v)}. \tag{5.15}$$

On the other hand, given $\varepsilon > 0$ there exists $m_0 = m_0(\varepsilon)$ such that $S_m(v) \leq (1+\varepsilon)S(v)$ for all $v \in \mathrm{Val}_X$ and $m \geq m_0$. Thus

$$\delta(L) = \limsup_m \delta_m(L) = \limsup_m \inf_v \frac{A(v)}{S_m(v)} \geq (1+\varepsilon)^{-1} \inf_v \frac{A(v)}{S(v)}.$$

Letting $\varepsilon > 0$ and combining this inequality with (5.15) completes the proof. □

Remark 5.3.5. It is clear that $\alpha(rL) = r^{-1}\alpha(L)$ and $\delta(rL) = r^{-1}\delta(L)$ for any $r \in \mathbf{N}^*$. This allows us to define $\alpha(L)$ and $\delta(L)$ for any big $\mathbf{Q}$-line bundle L, by setting $\alpha(L) := r^{-1}\alpha(rL)$ and $\delta(L) := r^{-1}\delta(rL)$ for r sufficiently divisible.

5.3.3 Proof of Theorems C, D and E

We are now ready to prove Theorems C, D, and E from the introduction.

We start with Theorems C and E. Theorem E follows from Corollary 5.3.2 and Theorem 5.3.4. In addition, the latter statement implies the limit $\delta(L) = \lim_m \delta_m(L)$ exists. Let us prove the remaining assertions in Theorem C.

The estimate $\alpha(L) \leq \delta(L) \leq (n+1)\alpha(L)$ follows from the corresponding inequalities in (5.6) between $T(v)$ and $S(v)$ together with Theorem C. When L is ample, we obtain the stronger inequality $\delta(L) \geq \frac{n+1}{n}\alpha(L)$ using Proposition 5.2.11. The fact that $\alpha(L)$ and $\delta(L)$ only depend on the numerical equivalence class of L follows from the corresponding properties of the invariants $S(v)$ and $T(v)$, see Lemma 5.2.7 (iii). Finally we prove that $\alpha(L)$ and $\delta(L)$ are strictly positive. It suffices to consider $\alpha(L)$. The case when L is ample is handled in [BHJ15, Theorem 9.14] using Seshadri constants, and the general case follows from Lemma 5.3.6 below by choosing D effective such that $L + D$ is ample.

Lemma 5.3.6. *If L is a big line bundle and D is an effective divisor, then $\alpha(L+D) \leq \alpha(L)$.*

Proof. Given $m \in M(L)$, the assignment $F \mapsto F + mD$ defines an injective map from $|mL|$ to $|m(L+D)|$. Since $\mathrm{lct}(F + mD) \leq \mathrm{lct}(F)$ for all $F \in |mL|$, it follows that $\alpha_m(L+D) \leq \alpha_m(L)$. Letting $m \to \infty$ completes the proof. □

Finally we prove Theorem D. The argument relies heavily on the work by K. Fujita and C. Li, who exploited ideas from the Minimal Model Program, as adapted to K-stability questions by C. Li and C. Xu [LiX14].

For simplicity, assume $K_X + B$ is Cartier. By either [Li15b, Theorem 3.7] or [Fuj17, Theorem 1.4.1], (X, B) is K-semistable iff $\beta(E) \geq 0$ for all prime divisors E over X. In our notation, this reads $A(\mathrm{ord}_E) \geq S(\mathrm{ord}_E)$ for all E, see [Fuj16b, Definition 1.3 (4)] and Remark 5.2.10, and is hence equivalent to $\delta(-(K_X + B)) \geq 1$ in view of Theorem 5.3.4.

Similarly, (X, B) is uniformly K-stable iff there exists $\varepsilon > 0$ such that $\beta(E) \geq \varepsilon j(E)$ for all divisors E over X [Fuj17, Theorem 1.4.2]. This reads $A(\mathrm{ord}_E) - S(\mathrm{ord}_E) \geq \varepsilon(T(\mathrm{ord}_E) - S(\mathrm{ord}_E))$ for all E. Since $-K_X - B$ is ample, Proposition 5.2.11 implies $n^{-1}S(\mathrm{ord}_E) \leq T(\mathrm{ord}_E) - S(\mathrm{ord}_E) \leq nS(\mathrm{ord}_E)$, so X is uniformly K-stable iff there exists $\varepsilon' > 0$ such that $A(\mathrm{ord}_E) - S(\mathrm{ord}_E) \geq \varepsilon' S(\mathrm{ord}_E)$ for all E. But this is equivalent to $\delta(-(K_X + B)) > 1$ by Theorem 5.3.4.

When $K_X + B$ is merely $\mathbf{Q}$-Cartier, a similar argument works using Lemma 5.2.7; see Remark 5.3.5.

5.3.4 Volume estimates

The following theorem gives an upper bound on the volume of L in terms of the $\delta(L)$. When X is a $\mathbf{Q}$-Fano variety and $L = -K_X$, the theorem generalizes the volume bounds found in [Fuj15] and [Liu16], in which X is assumed K-semistable, so that $\delta(L) \geq 1$. These volume bounds were explored in [SS17] and [LiuX17].

Theorem 5.3.7. *Let L be a big line bundle. Then we have*

$$\mathrm{vol}(L) \leq \left(\frac{n+1}{n}\right)^n \delta(L)^{-n} \widehat{\mathrm{vol}}(v).$$

for any valuation v on X centered at a closed point.

This theorem is a consequence of the following proposition, first observed by Liu, and embedded in the proof of [Liu16, Theorem 21].

Proposition 5.3.8. *If $v \in \mathrm{Val}_X^*$ has linear growth and is centered at a closed point, then*

$$S(v) \geq \frac{n}{n+1} \sqrt[n]{\mathrm{vol}(L)/\mathrm{vol}(v)}.$$

Proof. We follow Liu's argument. By the exact sequence

$$0 \to H^0(X, mL \otimes \mathfrak{a}_{mt}(v)) \to H^0(X, mL) \to H^0(X, mL \otimes (\mathcal{O}_X/\mathfrak{a}_{mt}(v)),$$

we see that

$$\dim \mathcal{F}_v^{mt} H^0(X, mL) \geq \dim H^0(X, mL) - \ell(\mathcal{O}_{X,\xi}/\mathfrak{a}_{mt}(v)),$$

where $\xi \in X$ is the center of v. Diving by $m^n/n!$ and taking the limit as $m \to \infty$ gives

$$\operatorname{vol}(L; v \geq t) \geq \operatorname{vol}(L) - t^n \operatorname{vol}(v),$$

which implies the lower bound for $T(v)$. Further, integrating with respect to t shows that

$$\begin{aligned} S(v) &= \frac{1}{\operatorname{vol}(L)} \int_0^{T(v)} \operatorname{vol}(L; v \geq t)\, dt \\ &\geq \frac{1}{\operatorname{vol}(L)} \int_0^{\sqrt[n]{L^n/\operatorname{vol}(v)}} (\operatorname{vol}(L) - t^n \operatorname{vol}(v))\, dt \\ &= \frac{n}{n+1} \sqrt[n]{\operatorname{vol}(L)/\operatorname{vol}(v)}, \end{aligned}$$

which completes the proof. □

Proof of Theorem D. If $A(v) = \infty$, then $\widehat{\operatorname{vol}}(v) = \infty$ and the inequality is trivial. If $A(v) < \infty$, then v has linear growth and the previous proposition gives

$$\operatorname{vol}(L) \leq \left(\frac{n+1}{n}\right)^n S(v)^n \operatorname{vol}(v) = \left(\frac{n+1}{n}\right)^n \left(\frac{S(v)}{A(v)}\right)^n \widehat{\operatorname{vol}}(v).$$

Since $\delta(L) \leq A(v)/S(v)$ by Theorem 5.3.4, the proof is complete. □

5.3.5 Valuations computing the thresholds

We say that a valuation $v \in \operatorname{Val}_X^*$ with $A(v) < \infty$ *computes* the log-canonical threshold (resp. the stability threshold) of L if $\alpha(L) = A(v)/T(v)$ (resp. $\delta(L) = A(v)/S(v)$). In §5.5 we will prove that such valuations always exist when L is ample. Here we will describe some general properties of valuations computing one of the two thresholds.

We start by the following general result.

Proposition 5.3.9. *Let v be a nontrivial valuation on X with $A(v) < \infty$.*

(i) if v computes $\alpha(L)$ or $\delta(L)$, then v computes $\operatorname{lct}(\mathfrak{a}_\bullet(v))$*;*

(ii) if L is ample and v computes $\delta(L)$, then v is the unique valuation, up to scaling, that computes $\operatorname{lct}(\mathfrak{a}_\bullet(v))$.

Proof. First suppose $v \in \mathrm{Val}_X$ computes $\alpha(L)$. Recall that $\mathrm{lct}(\mathfrak{a}_\bullet(v)) = \inf_w \frac{A(w)}{w(\mathfrak{a}_\bullet(v))}$, where it suffices to consider the infimum over $w \in \mathrm{Val}_X^*$ normalized by $w(\mathfrak{a}_\bullet(v)) = 1$. The latter condition implies $w(\mathfrak{a}_p(v)) \geq p$ for all p, so that $w \geq v$. By Proposition 5.2.15 (i), this yields $T(w) \geq T(v)$. Since v computes $\alpha(L)$, we have $A(w)/T(w) \geq A(v)/T(v)$. Thus

$$A(v)/v(\mathfrak{a}_\bullet(v)) = A(v) \leq A(w) = A(w)/w(\mathfrak{a}_\bullet(v)),$$

so taking the infimum over w shows that v computes $\mathrm{lct}(\mathfrak{a}_\bullet(v))$. The case when v computes $\delta(L)$ is handled in the same way, and the uniqueness statement in (ii) follows from Proposition 5.2.15 (ii). □

Conjecture 5.3.10. Any valuation computing $\alpha(L)$ or $\delta(L)$ must be quasimonomial.

Note that Conjecture B in [JM12] implies Conjecture 5.3.10 in view of Proposition 5.3.9. While Conjecture 5.3.10 seems difficult in general, it is trivially true in dimension one (since all valuations are then quasimonomial). We also have the following result.

Proposition 5.3.11. *If X is a projective surface with at worst canonical singularities and $B = 0$, then:*

(i) any valuation computing $\alpha(L)$ or $\delta(L)$ must be quasimonomial;

(ii) if X is smooth, then any valuation computing $\alpha(L)$ or $\delta(L)$ must be monomial in suitable local coordinates at its center.

We expect that the statement in (i) holds for klt surfaces as well.

Proof. Suppose $v \in \mathrm{Val}_X^*$ computes $\alpha(L)$ or $\delta(L)$. By Proposition 5.3.9, v computes $\mathrm{lct}_{X,0}(\mathfrak{a}_\bullet(v))$. Let $Y \to X$ be a resolution of singularities of X. Since X has canonical singularities, the relative canonical divisor $K_{Y/X}$ is effective, and v also computes the jumping number $\mathrm{lct}_Y^{K_{Y/X}}(\mathfrak{a}_\bullet(v))$. By [JM12, §9], v is quasimonomial, proving (i). The statement in (ii) follows from [FJ05, Lemma 2.11 (i)]. □

Finally we consider the case of *divisorial* valuations computing one of the two thresholds. In [Blu16a], the author studied properties of divisorial valuations that compute log canonical thresholds of graded sequences of ideals. The following proposition follows from Proposition 5.3.9 and results in [Blu16a].

Proposition 5.3.12. *Let X be a variety with at worst klt singularities and $B = 0$.*

(i) If a divisorial valuation v computes $\alpha(L)$ or $\delta(L)$, then there exists a prime divisor E over X of log canonical type such that $v = c\,\mathrm{ord}_E$ for some $c \in \mathbf{R}_+$.

(ii) If a divisorial valuation v computes $\delta(L)$ and L is ample, then there exists a prime divisor E over X of plt type such that $v = c\,\mathrm{ord}_E$ for some $c \in \mathbf{R}_+$.

We explain some of the above terminology. Let E be a divisor over X such that there exists a projective birational morphism $\pi\colon Y \to X$ such that E is a prime divisor on Y and $-E$ is $\mathbf{Q}$-Cartier and π-ample. We say that E is of *plt* (resp., *log canonical*) *type* if the pair (Y, E) is plt (resp., log canonical) [Fuj17, Definition 1.1]. K. Fujita considered plt type divisors in [Fuj17]. Note that Proposition 5.3.12 (ii) is similar to results in [Fuj17].

Proof. We may assume $v = \mathrm{ord}_F$ for a divisor F over X. If v computes $\alpha(L)$ or $\delta(L)$, then we may apply Proposition 5.3.9 (i) to see $A(v) = \mathrm{lct}(\mathfrak{a}_\bullet(v))$. Furthermore, if v computes $\delta(L)$ and L is ample, Proposition 5.3.9 (ii) implies $A(v) < A(w)/w(\mathfrak{a}_\bullet(v))$ as long as w is not a scalar multiple of v. The statement now follows from Propositions 1.5 and 4.4 of [Blu16a]. □

5.4 Uniform Fujita approximation

In this section we prove Fujita approximation type statements for filtrations arising from valuations.[1] These results play a crucial role in the proof of Theorem F.

Related statements have appeared in the literature. See [LM09, Theorem D] for the case of graded linear series and [BC11, Theorem 1.14] for the case of filtrations. Here we specialize to filtrations defined by valuations, and the main point is to have uniform estimates in terms of the log discrepancy of the valuation. To this end we use multiplier ideals.

Throughout this section, (X, B) is a projective klt pair.

5.4.1 Approximation results

Given a valuation v on X and a line bundle L on X, we seek to understand how well $S(v)$ and $T(v)$ can be approximated by studying the filtration $\mathcal{F}_v$ restricted to $H^0(X, mL)$ for m large but fixed.

Recall that the pseudoeffective threshold of v is defined by $T(v) := \lim_{m\to\infty} T_m(v)$.

[1] The term Fujita approximation refers to the work of T. Fujita [Fuj94].

Theorem 5.4.1. *Let L be an ample line bundle on X. Then there exists a constant $C = C(X, L) > 0$ such that*

$$0 \le T(v) - T_m(v) \le \frac{C \cdot A(v)}{m}$$

for all $m \in M(L)$ and all $v \in \mathrm{Val}_X^*$ with $A(v) < \infty$.

Corollary 5.4.2. *We have $0 \le \alpha(L)^{-1} - \alpha_m(L)^{-1} \le \frac{C}{m}$ for all $m \in M(L)$.*

We also have a version of Theorem 5.4.1 for the expected order of vanishing $S(v)$, but this is in terms of a modification $\tilde{S}_m(v)$ of the invariant $S_m(v)$, which we first need to introduce.

Let $V_\bullet$ be a graded linear series of a line bundle L on X. For $m \in \mathbf{N}^*$, we write $V_{m,\bullet}$ for the graded linear series of mL defined by

$$V_{m,\ell} := H^0(X, m\ell L \otimes \overline{\mathfrak{a}^\ell}) \subset H^0(X, m\ell L),$$

where $\mathfrak{a}$ denotes the base ideal $\mathfrak{b}\big(|V_m|\big)$ and $\overline{\mathfrak{a}^\ell}$ the integral closure of the ideal $\mathfrak{a}^\ell$.

If $V_m = 0$, then it is clear that $V_{m,\ell} = 0$ for all $\ell \in \mathbf{N}^*$ and $\mathrm{vol}(V_{m,\bullet}) = 0$. When $V_m \ne 0$, we use the geometric characterization of the integral closure as in [Laz04, Remark 9.6.4] to express $V_{m,\ell}$ as follows. Let $\mu\colon Y_m \to X$ be a proper birational morphism such that Y_m is normal and $\mathfrak{b}\big(|V_m|\big) \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_m)$ for some effective Cartier divisor F_m. Then

$$V_{m,\ell} \simeq H^0(Y_m, \ell(m\mu^*(L) - F_m))$$

for all $\ell \ge 1$. Since $m\mu^*(L) - F_m$ is base point free and therefore nef,

$$\mathrm{vol}(V_{m,\bullet}) = ((m\mu^*(L) - F_m)^n)$$

by [Laz04, Corollary 1.4.41].

In the case when $V_\bullet$ contains an ample series, we have

$$\mathrm{vol}(V_\bullet) = \lim_{m\to\infty} \frac{\mathrm{vol}(V_{m,\bullet})}{m^n};$$

see [His13, Proposition 17] and also [Szé15, Appendix].

Now consider a filtration $\mathcal{F}$ of $R(X, L)$. As in §5.1.4, this gives rise to a family $V_m^t = V_m^{\mathcal{F},t}$ of graded linear series of mL, indexed by $t \in \mathbf{R}_+$, and defined by

$$V_m^t := \mathcal{F}^{mt} R_m.$$

Using the previously defined notion, we get an additional family of graded linear series $V^t_{m,\bullet}$ for each $m \in \mathbf{N}^*$. Specifically,

$$V^t_{m,\ell} := H^0(X, m\ell L \otimes \overline{\mathfrak{b}\big(|V^t_m|\big)^\ell}).$$

Clearly $\operatorname{vol}(V^t_{m,\bullet})$ is a decreasing function of t that vanishes for $t > T(\mathcal{F})$. When $\mathcal{F}$ is linearly bounded, we write

$$\tilde{S}_m(\mathcal{F}) := \frac{1}{m^n \operatorname{vol}(L)} \int_0^{T(\mathcal{F})} \operatorname{vol}\left(V^t_{m,\bullet}\right)\, dt.$$

Note that by the dominated convergence theorem,

$$S(\mathcal{F}) = \lim_{m\to\infty} \tilde{S}_m(\mathcal{F}).$$

When v is a valuation on X with linear growth, we set $\tilde{S}_m(v) := \tilde{S}_m(\mathcal{F}_v)$.

Theorem 5.4.3. *Let L be an ample line bundle on X. Then there exists a constant $C = C((X,B),L)$ such that*

$$0 \le S(v) - \tilde{S}_m(v) \le \frac{C \cdot A(v)}{m}$$

for all $m \in \mathbf{N}^*$ and all $v \in \operatorname{Val}_X$ with $A(v) < \infty$.

Theorems 5.4.1 and 5.4.3 may be viewed as global analogues of Proposition 4.1.9. Their proofs, which appear at the end of this section, use multiplier ideals and take inspiration from [DEL00] and [ELS03].

5.4.2 Multiplier ideals associated to linear series

Given a linear series V of L, we set

$$\mathcal{J}((X,B), c\cdot|V|) := \mathcal{J}(X, c\cdot \mathfrak{b}(|V|)),$$

where $\mathfrak{b}(|V|)$ is the base ideal of V. Similarly, if $V_\bullet$ is a graded linear series of L, we set

$$\mathcal{J}(X, c\cdot\|V_\bullet\|) := \mathcal{J}(X, c\cdot\mathfrak{b}_\bullet)$$

where $\mathfrak{b}_\bullet$ is the graded sequence of ideals defined by $\mathfrak{b}_m := \mathfrak{b}(|V_m|)$.

The following lemma follows from basic properties of multiplier ideals listed in Section 3.3.

Lemma 5.4.4. *Let L be a line bundle on X.*

(i) If V is a linear series of L, then $\mathfrak{b}(|V|) \subset \mathcal{J}((X,B),|V|)$.

(ii) If $V_\bullet$ is a graded linear series of L and $m \in \mathbf{N}^*$, then $\mathfrak{b}\big(|V_m|\big) \subset \mathcal{J}((X,B), m \cdot \|V_\bullet\|)$.

(iii) Fix $r \in \mathbf{N}^*$ such that $r(K_X + B)$ is Cartier. If $V_\bullet$ is a graded linear series of L and $m \in \mathbf{N}^*$, $c \in \mathbf{Q}^*_+$, then

$$(\mathrm{Jac}_X \cdot \mathcal{O}_X(-rB)^{m-1}\mathcal{J}((X,B), cm \cdot \|V_\bullet\|) \subset \mathcal{J}((X,B), c \cdot \|V_\bullet\|)^m$$

The following result is a consequence of Nadel Vanishing.

Theorem 5.4.5. *Let L be a big line bundle on X, and $V_\bullet$ a graded linear series of L.*

(i) Let M be a line bundle on X and $m \in \mathbf{N}^*$. If $M - K_X - B - mL$ is big and nef, then

$$H^i(X, M \otimes \mathcal{J}((X,B), m \cdot \|V_\bullet\|)) = 0$$

for all $i \geq 1$.

(ii) Let M and H be line bundles on X and $m \in \mathbf{N}^*$. If H is ample and globally generated, and $M - K_X - B - mL$ is big and nef, then

$$(M + jH) \otimes \mathcal{J}((X,B), m \cdot \|V_\bullet\|)$$

is globally generated for every $j \geq n = \dim(X)$.

Proof. Statement (i) is [Laz04, Theorem 11.2.12 (iii)] in the case when X is and $B = 0$ smooth. In the more general case, the statement is a consequence of [Laz04, Theorem 9.4.17 (ii)].

Statement (ii) is a well known consequence of (i) and Castelnuovo–Mumford regularity. For a similar argument, see [Laz04, Proposition 9.4.26]. □

Corollary 5.4.6. *Let L be an ample line bundle on X. There exists a positive integer $a = a((X,B),L)$ such that if $V_\bullet$ is a graded linear series of L, then*

$$(a + m)L \otimes \mathcal{J}((X,B), m \cdot \|V_\bullet\|)$$

is globally generated for all $m \in \mathbf{N}^*$. (Note that a does not depend on m or $V_\bullet$.) Furthermore, fix $r \in \mathbf{N}^*$ such that $r(K_X + B)$ is Cartier. We may choose a so that $H^0(X, aL \otimes \mathrm{Jac}_X \cdot \mathcal{O}_X(-rB))$ is nonzero.

Proof. Pick $b, c \in \mathbf{N}^*$ such that bL is globally generated and $cL - K_X - B$ is big and nef. We apply Theorem 5.4.5 (ii) with $M = (c+m)L$ and $H = bL$. Thus

$$(c + m + jb)L \otimes \mathcal{J}(X, m \cdot \|V_\bullet\|)$$

is globally generated for all $m \in \mathbf{N}^*$ and $j \geq \dim(X)$. Thus, we can set $a := c + \dim(X)b$. By increasing c further, we may assure $H^0(X, (c+\dim(X)b)L \otimes \mathrm{Jac}_X \cdot \mathcal{O}_X(-rB)) \neq 0$. $\square$

5.4.3 Applications to filtrations defined by valuations

Now let L be an *ample* line bundle on X and fix $r \in \mathbf{N}^*$ such that $r(K_X + B)$ is Cartier. Now, we fix a value $a := c + \dim(X)b$, where $c, b \in \mathbf{N}^*$ satisfy the following conditions:

(i) bL is globally generated,

(ii) $cL - K_X - B$ is big and nef, and

(iv) $H^0(X, (c + \dim(X)b)L \otimes \mathrm{Jac}_X \cdot \mathcal{O}_X(-rB)) \neq 0$.

Our choice of a satisfies the conclusion of Corollary 5.4.6. For the remainder of this section, a will always refer to this constant.

Consider a valuation $v \in \mathrm{Val}_X^*$ with $A(v) < \infty$. We proceed to study the graded linear series $V_\bullet^t = V_\bullet^{\mathcal{F}_v, t}$ of L for $t \in \mathbf{R}_+$.

Proposition 5.4.7. *If $m \in \mathbf{N}^*$ and $t \in \mathbf{Q}_+^*$ satisfies $mt \geq A(v)$, then*

$$\mathcal{J}((X, B), m \cdot \|V_\bullet^t\|) \subset \mathfrak{a}_{mt - A(v)}(v).$$

Proof. Pick $p \in \mathbf{N}^*$ such that $pt \in \mathbf{N}^*$ and $\mathcal{J}(X, m \cdot \|V_\bullet^t\|) = \mathcal{J}(X, \frac{m}{p} \cdot \mathfrak{b}(|V_p^t|))$. Then

$$\mathcal{J}(X, \tfrac{m}{p} \cdot \mathfrak{b}(|V_p^t|)) \subset \mathcal{J}(X, \tfrac{m}{p} \cdot \mathfrak{a}_{pt}(v)) \subset \mathcal{J}(X, mt \cdot \mathfrak{a}_\bullet(v)) \subset \mathfrak{a}_{mt - A(v)}(v),$$

where the first inclusion follows from the inclusion $\mathfrak{b}(|V_p^t|) \subset \mathfrak{a}_{pt}(v)$, the second from the definition of the asymptotic multiplier ideal, and the third from Proposition 3.3.12. $\square$

Proposition 5.4.8. *If $m \in \mathbf{N}^*$ and $t \in \mathbf{Q}_+^*$ satisfies $mt \geq A(v)$, then*

$$\mathcal{J}((X, B), m \cdot \|V_\bullet^t\|) \subset \mathfrak{b}(|V_{m+a}^{t'}|)$$

where $t' = (mt - A(v))/(m + a)$.

Proof. By Proposition 5.4.7, we have

$$H^0(X, (m+a)L \otimes \mathcal{J}((X,B), m \cdot \|V_\bullet^t\|)) \subset H^0(X, (m+a)L \otimes \mathfrak{a}_{mt-A(v)}(v)) = V_{m+a}^{t'}.$$

Since $(m+a)L \otimes \mathcal{J}((X,B), \|V_\bullet^t\|)$ is globally generated by Corollary 5.4.6, the desired inclusion follows by taking base ideals. □

Using the previous proposition, we can now bound $\mathrm{vol}(V_{m,\bullet}^t)$ from below.

Proposition 5.4.9. *If $m \in \mathbf{N}^*$ and $t \in \mathbf{Q}_+^*$ satisfies $mt \geq A(v)$, then*

$$\mathrm{vol}(V_\bullet^t) \leq m^{-n}\,\mathrm{vol}(V_{m+a,\bullet}^{t'}),$$

where $t' = (mt - A(v))/(a+m)$.

Proof. It suffices to show that $\dim V_{m\ell}^t \leq \dim V_{m+a,\ell}^{t'}$ for all positive integers ℓ. Indeed, diving both sides by $(m\ell)^n/n!$ and letting $\ell \to \infty$ then gives the desired inequality.

We now prove $\dim V_{m\ell}^t \leq \dim V_{m+a,\ell}^{t'}$. First, by our assumption on a, we may choose a nonzero section $s \in H^0(X, aL \otimes \mathrm{Jac}_X \cdot \mathcal{O}_X(-rB))$. Multiplication by s^ℓ gives an injective map

$$V_{\ell m}^t \longrightarrow H^0(X, (a+m)\ell L \otimes (\mathrm{Jac}_X \cdot \mathcal{O}_X(-rB))^{\ell-1} \otimes \mathfrak{b}\big(|V_{m\ell}^t|\big)).$$

Now, we have

$$\begin{aligned}
&H^0(X, (a+m)\ell L \otimes (\mathrm{Jac}_X \cdot \mathcal{O}_X(-rB))^{\ell-1} \otimes \mathfrak{b}\big(|V_{m\ell}^t|\big)) \\
&\qquad \subset H^0(X, (a+m)\ell L \otimes (\mathrm{Jac}_X \cdot \mathcal{O}_X(-rB))^{\ell-1} \otimes \mathcal{J}((X,B), m\ell \cdot \|V_\bullet^t\|)) \\
&\qquad\qquad \subset H^0(X, (a+m)\ell L \otimes \mathcal{J}((X,B), m \cdot \|V_\bullet^t\|)^\ell) \\
&\qquad\qquad\qquad \subset H^0(X, (a+m)\ell L \otimes (\mathfrak{b}(|V_{m+a}^{t'}|)^\ell) \subset V_{m+a,\ell}^{t'},
\end{aligned}$$

where the first inclusion follows from Lemma 5.4.4, the second from Corollary 3.3.7 (iii), the third from Proposition 5.4.8, and the last one from the definition of $V_{m+a,\bullet}^{t'}$. □

As an application of the previous proposition, we give bounds on $T_m(v)$ and $\tilde{S}_m(v)$.

Proposition 5.4.10. *If $m \in \mathbf{N}^*$, then*

$$T(v) - \frac{aT(v) + A(v)}{m} \leq T_m(v) \leq T(v).$$

Proof. The second inequality is trivial, since $T(v) = \sup T_m(v)$. To prove the first inequality, we may assume $m > a + \frac{A(v)}{T(v)}$. Pick $t \in \mathbf{Q}_+^*$ with $t < T(v)$ and $m > a + \frac{A(v)}{t}$. Since $V_\bullet^t$ is nontrivial (in fact, it contains an ample series), $\mathcal{J}((X,B),(m-a)\|V_\bullet^t\|)$ is nontrivial as well. Apply Proposition 5.4.8, with m replaced by $m-a$, so that $t' = t - m^{-1}(at + A)$. We get

$$\mathfrak{b}(|V_m^{t'}|) \supset \mathcal{J}((X,B),(m-a)\|V_\bullet^t\|) \neq 0.$$

In particular, $V_m^{t'} \neq \emptyset$, which implies $t' \leq T_m(v)$. Letting $t' \to T(v)$ completes the proof. $\square$

Proposition 5.4.11. *If $m \in \mathbf{N}^*$ and $m > a$, then*

$$\left(\frac{m-a}{m}\right)^{n+1}\left(S(v) - \frac{A(v)}{m-a}\right) \leq \tilde{S}_m(v). \tag{5.16}$$

Proof. To prove the inequality, we use Proposition 5.4.9 with m replaced by $m-a$ to see that

$$\left(\frac{m-a}{m}\right)^{n}\operatorname{vol}(V_\bullet^t) \leq \frac{1}{m^n}\operatorname{vol}(V_{m,\bullet}^{t'}) \tag{5.17}$$

for all $t \in \mathbf{Q}_+^*$ with $(m-a)t \geq A(v)$, where $t' = t - m^{-1}(at + A(v))$. By the continuity statement in Proposition 5.1.3, the inequality in (5.17) must hold for all $t \in [A(v)/(m-a), T(v)]$, with at most two exceptions. We can therefore integrate with respect to t from $t = A(v)/(m-a)$ to $t = (mT(v) + A(v))/(m-a)$, i.e. from $t' = 0$ to $t' = T(v)$. This yields

$$\begin{aligned}
\tilde{S}_m(v) = \int_0^{T(v)} \frac{\operatorname{vol}(V_{m,\bullet}^{t'})}{m^n \operatorname{vol}(V)}\,dt' &\geq \left(\frac{m-a}{m}\right)^{n+1}\int_{A(v)/(m-a)}^{(mT(v)+A(v))/(m-a)} \frac{\operatorname{vol}(V_\bullet^t)}{\operatorname{vol}(L)}\,dt \\
&= \left(\frac{m-a}{m}\right)^{n+1}\int_{A(v)/(m-a)}^{T(v)} \frac{\operatorname{vol}(V_\bullet^t)}{\operatorname{vol}(L)}\,dt \\
&= \left(\frac{m-a}{m}\right)^{n+1}\left(S(v) - \int_0^{A(v)/(m-a)} \frac{\operatorname{vol}(V_\bullet^t)}{\operatorname{vol}(L)}\,dt\right) \\
&\geq \left(\frac{m-a}{m}\right)^{n+1}\left(S(v) - \frac{A(v)}{m-a}\right),
\end{aligned}$$

where the second equality follows from a simple substitution and the last inequality follows since $\operatorname{vol}(V_\bullet^t) \leq \operatorname{vol}(L)$ for all t. This completes the proof. $\square$

Proof of Theorem 5.4.1. Consider any $v \in \mathrm{Val}_X^*$ with $A(v) < \infty$. By Corollary 5.3.2, we have $T(v) \le A(v)/\alpha(L)$. Proposition 5.4.10 now yields

$$T(v) - T_m(v) \le \left(\frac{a}{\alpha(L)} + 1\right)\frac{A(v)}{m}$$

for any $m \in \mathbf{N}^*$, so the theorem holds with $C = 1 + a/\alpha(L)$. $\square$

Proof of Theorem 5.4.3. Consider any $v \in \mathrm{Val}_X^*$ with $A(v) < \infty$. We claim

$$0 \le S(v) - \tilde{S}_m(v) \le \frac{a}{m}S(v) + \frac{A(v)}{m} \tag{5.18}$$

for all $m \in \mathbf{N}^*$. To prove the first inequality of (5.18), note that for $t \in \mathbf{R}_+$ and $l \in \mathbf{N}^*$ we have

$$V_{m,\ell}^t = H^0(X, m\ell L \otimes \overline{\mathfrak{b}\big(|\mathcal{F}_v^{mt} H^0(X, mL)|\big)^\ell}) \subset \mathcal{F}_v^{m\ell t} H^0(X, m\ell L) = V_{m\ell}^t.$$

Thus $\mathrm{vol}(V_{m,\bullet}^t) \le m^n \,\mathrm{vol}(V_\bullet^t)$ for $t \in \mathbf{R}_+$, and integration yields $\tilde{S}_m(v) \le S(v)$. We move on to the second inequality of (5.18). Note that the statement holds trivially for $m \le a$. Next Proposition 5.4.11 gives

$$\begin{aligned}
0 \le S(v) - \tilde{S}_m(v) &\le S(v) - \left(\frac{m-a}{m}\right)^{n+1}\left(S(v) - \frac{A(v)}{m-a}\right) \\
&= \left(1 - \left(\frac{m-a}{m}\right)^{n+1}\right) S(v) + \left(\frac{m-a}{m}\right)^n \frac{A(v)}{m} \le \frac{a}{m}S(v) + \frac{A(v)}{m}
\end{aligned}$$

for $m > a$, where the last inequality uses that $t^{n+1} \le t$ for $t \in [0, 1]$.

Finally, note that $S(v) \le T(v) \le A(v)/\alpha(L)$ by (5.6) and Corollary 5.3.2. Therefore, (5.18) implies

$$0 \le S(v) - \tilde{S}_m(v) \le \frac{CA(v)}{m},$$

where $C = 1 + a/\alpha(L)$. $\square$

5.5 Existence of valuations computing the thresholds

In this section we prove Theorem F, on the existence of valuations computing the log canonical and stability thresholds. We assume that (X, B) is a projective klt pair and L is an ample line bundle on X.

5.5.1 Linear series in families

We consider the following setup, which will arise in §5.5.3. Fix $m \in \mathbf{N}^*$ and a family of subspaces of $H^0(X, mL)$ parameterized by a variety Z. Said family is given by a submodule

$$\mathcal{W} \subset \mathcal{V} := H^0(X, mL) \otimes_k \mathcal{O}_Z.$$

For $z \in Z$ closed, we write W_z for the linear series of mL defined by

$$W_z := \operatorname{Im}\left(\mathcal{W}|_{k(z)} \to \mathcal{V}|_{k(z)} \simeq H^0(X, mL)\right).$$

Note that $\mathcal{W}$ gives rise to an ideal $\mathcal{B} \subset \mathcal{O}_{X\times Z}$ such that

$$\mathcal{B} \cdot \mathcal{O}_{X\times\{z\}} = \mathfrak{b}\big(|W_z|\big).$$

Indeed, $\mathcal{B}$ is the image of the map

$$p_2^*\mathcal{W} \otimes p_1^*(-mL) \to \mathcal{O}_{X\times Z},$$

where p_1 and p_2 denote the projection maps associated to $X \times Z$.

We need a few results on the behavior of invariants of linear series in families.

Proposition 5.5.1. *There exists a nonempty open set $U \subset Z$ such that $\operatorname{lct}(\mathfrak{b}\big(|W_z|\big))$ is constant for all closed points $z \in U$.*

Proof. Since $\operatorname{lct}(\mathfrak{b}\big(|W_z|\big)) = \operatorname{lct}(\mathcal{B} \cdot \mathcal{O}_{X\times\{z\}})$, the proposition follows from Proposition 3.4.11. □

Proposition 5.5.2. *If Z is a smooth curve and $z_0 \in Z$ a closed point, then there exists an open neighborhood U of z_0 in Z such that $\operatorname{lct}(\mathfrak{b}\big(|W_{z_0}|\big)) \leq \operatorname{lct}(\mathfrak{b}\big(|W_z|\big))$ for all $z \in U$.*

Proof. As in the proof of the previous proposition, we note that $\operatorname{lct}(\mathfrak{b}\big(|W_z|\big)) = \operatorname{lct}(\mathcal{B} \cdot \mathcal{O}_{X\times\{z\}})$ for $z \in Z$ closed. Thus, the proposition is a consequence of Proposition 3.4.12. □

Proposition 5.5.3. *There exists a nonempty open set $U \subset Z$ such that $\operatorname{vol}(W_{z,\bullet})$ is constant for all closed points $z \in U$.*

Proof. The idea is to express $\operatorname{vol}(W_{z,\bullet})$ as an intersection number. Fix a proper birational morphism $\pi\colon Y \to X \times Z$ such that Y is smooth and $\mathcal{B} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for some effective Cartier divisor on Y. For each $z \in Z$, we restrict π to get a map $\pi_z\colon Y_z \to X \times \{z\} \simeq X$.

We may find a nonempty open set $U \subset Z$ such that $Y_z \to X \times \{z\}$ is proper birational and Y_z is smooth for all $z \in U$. We then have

$$\mathrm{vol}(W_{z,\bullet}) = ((p_1^* mL - F)|_{Y_z}^n).$$

After shrinking U, we may assume $p_1^* mL - F$ is flat over U. Then $((p_1^* mL - F)|_{Y_z})^n)$ is constant on U, which concludes the proof. $\square$

Proposition 5.5.4. *Let $\mathcal{W}$ and $\mathcal{G}$ be two submodules of $\mathcal{V}$ and for $z \in Z$, let W_z and G_z denote the corresponding subspaces of V. If the function $z \mapsto \dim W_z$ is locally constant on Z, then the set $\{z \in Z \,|\, G_z \subset W_z\}$ is closed.*

Proof. We may assume Z is affine and $\dim(W_z) =: r$ is constant on Z. Choose a basis for the free $\mathcal{O}(Z)$-module $\mathcal{V}(Z)$ as well as generators for $\mathcal{W}(Z)$ and $\mathcal{G}(Z)$. Consider the matrix with entries in $\mathcal{O}(Z)$, whose rows are given by the generators of $\mathcal{W}(Z)$, followed by the generators of $\mathcal{G}(Z)$, all expressed in the chosen basis of $\mathcal{O}(Z)$. By our assumption on $\mathcal{W}$, the rank of this matrix is at least r for all $z \in Z$. Further, since $G_z \subset W_z$ if and only if $\dim(G_z + W_z) = \dim(W_z)$, the set $\{z \in Z \,|\, G_z \subset W_z\}$ is precisely the locus where this matrix has rank equal to r, and is hence closed. $\square$

5.5.2 Parameterizing filtrations

We now construct a space that parameterizes filtrations of $R(X, L)$. To have a manageable parameter space, we restrict ourselves to $\mathbf{N}$-filtrations $\mathcal{F}$ of R satisfying $T(\mathcal{F}) \leq 1$. Such a filtration $\mathcal{F}$ is given by the choice of a flag

$$\mathcal{F}^m R_m \subset \mathcal{F}^{m-1} R_m \subset \cdots \subset \mathcal{F}^1 R_m \subset \mathcal{F}^0 R_m = R_m \tag{5.19}$$

for each $m \in \mathbf{N}^*$ such that

$$\mathcal{F}^{p_1} R_{m_1} \cdot \mathcal{F}^{p_2} R_{m_2} \subset \mathcal{F}^{p_1+p_2} R_{m_1+m_2} \tag{5.20}$$

for all integers $0 \leq p_1 \leq m_1$ and $0 \leq p_2 \leq m_2$.

Let Fl_m denote the flag variety parameterizing flags of R_m of the form (5.19). In general, Fl_m may have several connected components. On each component, the signature of the flag (that is, the sequence of dimensions of the elements of the flag) is constant.

For each natural number d, we set

$$H_d := Fl_0 \times Fl_1 \times \cdots \times Fl_d$$

and, for $c \geq d$, let $\pi_{c,d} : H_c \to H_d$ denote the natural projection map. Note that a closed point $z \in H_d$ gives a collection of subspaces

$$\left(\mathcal{F}_z^m R_m \subset \mathcal{F}_z^{m-1} R_m \subset \cdots \subset \mathcal{F}_z^1 R_m \subset \mathcal{F}_z^0 R_m = R_m\right)_{0 \leq m \leq d}.$$

Furthermore, this correspondence is given by a universal flag on H_d. This means that for each $m \leq d$ on H_d there is a flag

$$\mathcal{F}^m \mathcal{R}_m \subset \mathcal{F}^{m-1} \mathcal{R}_m \subset \cdots \subset \mathcal{F}^1 \mathcal{R}_m \subset \mathcal{F}^0 \mathcal{R}_m = \mathcal{R}_m,$$

where $\mathcal{R}_m := H^0(X, mL) \otimes_k \mathcal{O}_{H_d}$. For $z \in H_d$, we have

$$\mathcal{F}_z^p R_m := \operatorname{Im}\left(\mathcal{F}^p \mathcal{R}_m|_{k(z)} \longrightarrow \mathcal{R}_m|_{k(z)} \simeq R_m\right)$$

for $0 \leq p \leq m$, where $k(z)$ denotes the residue field at z.

Since we are interested in filtrations of $R(X, L)$, consider the subset

$$J_d := \{z \in H_d \mid \mathcal{F}_z \text{ satisfies (5.20) for all } 0 \leq p_i \leq m_i \leq d\}.$$

Lemma 5.5.5. *The subset $J_d \subset H_d$ is closed.*

Proof. We consider $\mathcal{F}_z^{p_1} R_{m_1} \cdot \mathcal{F}_z^{p_2} R_{m_2}$, where $z \in H_d$, $m_1 + m_2 \leq d$, and $0 \leq p_i \leq m_i$ for $i = 1, 2$. We will realize this subspace as coming from a submodule of $\mathcal{R}_{m_1+m_2}$. Note that the natural map

$$H^0(X, m_1 L) \otimes_k H^0(X, m_2 L) \longrightarrow H^0(X, (m_1 + m_2)L)$$

induces a map $\mathcal{R}_{m_1} \otimes \mathcal{R}_{m_2} \to \mathcal{R}_{m_1+m_2}$. We define

$$\mathcal{F}^{p_1} \mathcal{R}_{m_1} \cdot \mathcal{F}^{p_2} \mathcal{R}_{m_2} := \operatorname{Im}\left(\mathcal{F}^{p_1} \mathcal{R}_{m_1} \otimes \mathcal{F}^{p_2} \mathcal{R}_{m_2} \to \mathcal{R}_{m_1+m_2}\right).$$

Since

$$\mathcal{F}_z^{p_1} R_{m_1} \cdot \mathcal{F}_z^{p_2} R_{m_2} = \operatorname{Im}\left((\mathcal{F}^{p_1} \mathcal{R}_{m_1} \otimes \mathcal{F}^{p_2} \mathcal{R}_{m_2})|_{k(z)} \longrightarrow \mathcal{R}_{m_1+m_2}|_{k(z)} \simeq R_{m_1+m_2}\right),$$

the desired statement is a consequence of Proposition 5.5.4. $\square$

Let $J_d(k)$ denote the set of closed points of J_d, and set $J := \varprojlim J_d(k)$, with respect to the inverse system induced by the maps $\pi_{c,d}$. Write π_d for the natural map $J \to J_d(k)$. By the previous discussion, there is a bijection between the elements of J and $\mathbf{N}$-filtrations $\mathcal{F}$ of $R(X, L)$ satisfying $T(\mathcal{F}) \leq 1$.

The following technical lemma will be useful for us in the next section. Its proof relies on the fact that every descending sequence of nonempty constructible subsets of a variety over an uncountable field has nonempty intersection.

Lemma 5.5.6. *For each $d \in \mathbf{N}$, let $W_d \subset J_d$ be a nonempty constructible subset, and assume $W_{d+1} \subset \pi_{d+1,d}^{-1}(W_d)$ for all d. Then there exists $z \in J$ such that $\pi_d(z) \in W_d(k)$ for all d.*

Proof. Finding such a point z is equivalent to finding a point $z_d \in W_d(k)$ for each d, such that $\pi_{d+1,d}(z_{d+1}) = z_d$ for all d. We proceed to construct such a sequence $(z_d)_d$ inductively.

We first look to find a good candidate for z_1. By assumption,

$$W_1 \supset \pi_{2,1}(W_2) \supset \pi_{3,1}(W_3) \supset \cdots$$

is a descending sequence of nonempty sets. Note that W_1 is constructible, and so are $\pi_{d,1}(W_d)$ for all d by Chevalley's Theorem. Thus,

$$W_1 \cap \pi_{2,1}(W_2) \cap \pi_{3,1}(W_3) \cap \cdots$$

is nonempty, and we may choose a closed point z_1 in this set.

Next, we look at

$$W_2 \cap \pi_{2,1}^{-1}(z_1) \supset \pi_{3,2}(W_3) \cap \pi_{2,1}^{-1}(z_1) \supset \pi_{4,2}(W_4) \cap \pi_{2,1}^{-1}(z_1) \supset \cdots$$

and note that for $d \geq 2$ the set $\pi_{d,2}(W_d) \cap \pi_{2,1}^{-1}(z_1)$ is nonempty by our choice of z_1. Thus

$$\pi_{2,1}^{-1}(z_1) \cap W_2 \cap \pi_{3,2}(W_3) \cap \pi_{4,2}(W_4) \cap \cdots$$

is nonempty, and we may choose a closed point z_2 lying in the set. Continuing in this manner, we construct a desired sequence. $\square$

5.5.3 Finding limit filtrations

The following proposition, crucial to Theorem F, is a global analogue of Proposition 4.2.2. The proofs of both results use extensions of the "generic limit" construction developed in [Kol08, dFM09, dFEM10, dFEM11].

Proposition 5.5.7. *Let $(\mathcal{F}_i)_{i \in \mathbf{N}}$ be a sequence of $\mathbf{N}$-filtrations of $R(X, L)$ with $T(\mathcal{F}_i) \leq 1$ for all i. Furthermore, fix $A, S, T \in \mathbf{R}_+$ such that*

1. $A \geq \limsup_{i\to\infty} \operatorname{lct}(\mathfrak{b}_\bullet(\mathcal{F}_i))$,

2. $S \leq \liminf_{m\to\infty} \liminf_{i\to\infty} \tilde{S}_m(\mathcal{F}_i)$, *and*

3. $T \leq \liminf_{m\to\infty} \liminf_{i\to\infty} T_m(\mathcal{F}_i)$.

Then there exists a filtration $\mathcal{F}$ of $R(X,L)$ such that

$$\operatorname{lct}(\mathfrak{b}_\bullet(\mathcal{F})) \leq A, \quad S \leq S(\mathcal{F}), \quad \textit{and} \quad T \leq T(\mathcal{F}) \leq 1.$$

Proof. We use the parameter space J from §5.5.2, parametrizing $\mathbf{N}$-filtrations of $R(X,L)$ with $T \leq 1$. Each filtration $\mathcal{F}_i$ corresponds to an element $z_i \in J$, and $\pi_m(z_i)$ correspond to the filtration $\mathcal{F}_i$ restricted to $\oplus_{d=0}^m R_d$.

Claim 1: We may choose infinite subsets

$$\mathbf{N} \supset I_0 \supset I_2 \supset I_3 \supset \cdots$$

such that for each m, the closed set

$$Z_m := \overline{\{\pi_m(z_i) \,|\, i \in I_m\}} \subset J_m$$

satisfies the property

(†) If $Y \subsetneq Z_m$ is a closed set, there are only finitely many $i \in I_m$ such that $\pi_m(z_i) \in Y$.

Note that, in particular, each Z_m is irreducible.

Indeed, we can construct the sequence $(I_m)_0^\infty$ inductively. Set $I_0 = \mathbf{N}$. Since $J_0 = Fl_0 \simeq \operatorname{Spec}(k)$, (†) is trivially satisfied for $m = 0$. Having chosen I_m, pick $I_{m+1} \subset I_m$ such that (†) is satisfied for Z_{m+1}; this is possible since J_m is Noetherian.

Claim 2: For each $m \in \mathbf{N}$, there exist a nonempty open set $U_m \subset Z_m$ and constants $a_{p,m}$, $1 \leq p \leq m$, s_m, and t_m such that if $z \in U_m$, the filtration $\mathcal{F}_z$ satisfies

(1) $p \cdot \operatorname{lct}(\mathfrak{b}_{p,m}(\mathcal{F}_z)) = a_{p,m}$ for $1 \leq p \leq m$;

(2) $\tilde{S}_m(\mathcal{F}_z) = s_m$;

(3) $T_m(\mathcal{F}_z) = t_m$.

Furthermore, $a_{p,m} \le A$ for all $1 \le p \le m$, $\liminf_{m\to\infty} s_m \ge S$, and $\liminf_{m\to\infty} t_m \ge T$.

To see this, note that there is a nonempty open set $U_m \subset Z_m$ on which the left-hand sides of (1)–(3) are constant. For (1) and (2), this is a consequence of Propositions 5.5.1 and 5.5.3. For (3), it follows from $\dim \mathcal{F}_z^p R_m$ being constant on the connected components of J_m.

Now, we let

$$I_m^\circ := \{ i \in I_m \,|\, \pi_m(z_i) \in U_m \}.$$

By (†), the set $I_m \setminus I_m^\circ$ is finite; hence, I_m° is infinite. Since

$$a_{p,m} = p \cdot \mathrm{lct}(\mathfrak{b}_{p,m}(\mathcal{F}_i)), \quad s_m = \tilde{S}_m(\mathcal{F}_i), \quad \text{and} \quad t_m = T_m(\mathcal{F}_i)$$

for all $i \in I_m^\circ$ and $1 \le p \le m$, we see that

1. $a_{p,m} \le \limsup_{i\to\infty} p \cdot \mathrm{lct}(\mathfrak{b}_{p,m}(\mathcal{F}_i)) \le \limsup_{i\to\infty} p \cdot \mathrm{lct}(\mathfrak{b}_p(\mathcal{F}_i))$,
2. $s_m \ge \liminf_{i\to\infty} \tilde{S}_m(\mathcal{F}_i)$, and
3. $t_m \ge \liminf_{i\to\infty} T_m(\mathcal{F}_i)$.

The remainder of Claim 2 follows from these three inequalities.

Claim 3: There exists a point $z \in J$ such that $\pi_m(z) \in U_m$ for all $m \in \mathbf{N}$.

Granted this claim, the filtration $\mathcal{F} = \mathcal{F}_z$ associated to $z \in J$ satisfies the conclusion of our proposition. Indeed, this is a consequence of Claim 2 and the fact that for any linearly bounded filtration $\mathcal{F}$, we have

1. $\mathrm{lct}(\mathfrak{b}_\bullet(\mathcal{F})) = \lim_{p\to\infty} \sup_{m\ge p} p \cdot \mathrm{lct}(\mathfrak{b}_{p,m}(\mathcal{F}))$;
2. $S(\mathcal{F}) = \lim_{m\to\infty} \tilde{S}_m(\mathcal{F})$;
3. $T(\mathcal{F}) = \lim_{m\to\infty} T_m(\mathcal{F})$.

We are left to prove Claim 3. To this end we apply Lemma 5.5.6. For $d \in \mathbf{N}$, set

$$W_d := U_d \cap \pi_{d,d-1}^{-1} U_{d-1} \cap \pi_{d,d-2}^{-1}(U_{d-2}) \cap \cdots \cap \pi_{d,0}^{-1}(U_0).$$

Clearly $W_d \subset J_d$ is constructible and $W_{d+1} \subset \pi_{d+1,d}^{-1}(W_d)$. We are left to check that each W_d is nonempty. But

$$\pi_d(z_i) \in W_d \text{ for all } i \in I_d^\circ \cap I_{d-1}^\circ \cdots \cap I_0^\circ,$$

and the latter index set is nonempty, since it can be written as $I_d \setminus \bigcup_{j=0}^{d}(I_j \setminus I_j^\circ)$, where I_d is infinite and each $I_j \setminus I_j^\circ$ is finite.

Applying Lemma 5.5.6 to the W_d yields a point $z \in J$ such that $\pi_d(z) \in W_d \subset U_d$ for all $d \in \mathbf{N}$. This completes the proof of the claim, as well as the proof of the proposition. $\square$

5.5.4 Proof of Theorem F

We begin by proving the following proposition.

Proposition 5.5.8. *Let $(v_i)_{i\in\mathbf{N}}$ be a sequence of valuations in Val_X^* such that $T(v_i) = 1$ and the limits $A := \lim_{i\to\infty} A(v_i)$ and $S := \lim_{i\to\infty} S(v_i)$ both exist and are finite. Then there exists a valuation v^* on X such that*

$$A(v^*) \le A, \quad S(v^*) \ge S \quad \textit{and} \quad T(v^*) \ge 1.$$

This will follow from Proposition 5.5.7 and the following lemma.

Lemma 5.5.9. *Keeping the notation and hypotheses of Proposition 5.5.8, let $\mathcal{F}_i := \mathcal{F}_{v_i,\mathbf{N}}$ denote the $\mathbf{N}$-filtration induced by $\mathcal{F}_{v_i}$ as in §5.1.7. Then we have*

1. $\limsup_{i\to\infty} \mathrm{lct}\left(\mathfrak{b}_\bullet(\mathcal{F}_i)\right) \le A$,
2. $\lim_{m\to\infty} \liminf_{i\to\infty} \tilde{S}_m(\mathcal{F}_i) = \lim_{m\to\infty} \limsup_{i\to\infty} \tilde{S}_m(\mathcal{F}_i) = S$, *and*
3. $\lim_{m\to\infty} \liminf_{i\to\infty} T_m(\mathcal{F}_i) = \lim_{m\to\infty} \limsup_{i\to\infty} T_m(\mathcal{F}_i) = 1$.

Proof. We first show that (1) holds. Note that $\mathfrak{b}_p(\mathcal{F}_i) = \mathfrak{b}_p(\mathcal{F}_{v_i})$ for all $p \in \mathbf{N}$. Indeed, this follows from the fact that $\mathcal{F}_i^p R_m = \mathcal{F}_{v_i}^p R_m$ for all $m, p \in \mathbf{N}$. Thus,

$$\mathrm{lct}(\mathfrak{b}_\bullet(\mathcal{F}_i)) = \mathrm{lct}(\mathfrak{b}_\bullet(\mathcal{F}_{v_i})) = \mathrm{lct}(\mathfrak{a}_\bullet(v_i)) \le A(v_i),$$

where the second equality follows from Lemma 5.2.19 and the last inequality is Lemma 3.4.9.

We now show (2) and (3) hold. To this end, we first claim that

$$0 \le T_m(v_i) - T_m(\mathcal{F}_i) \le \frac{1}{m} \quad \text{and} \quad 0 \le \tilde{S}_m(v_i) - \tilde{S}_m(\mathcal{F}_i) \le \frac{1}{m}. \tag{5.21}$$

Indeed, the estimates for T_m follow from Proposition 5.1.11. As for the estimates for $\tilde{S}_m$, note that $\tilde{S}_m(v_i) = \int_0^1 f_{i,m}(t)\,dt$, where $f_{i,m}(t) = \mathrm{vol}(V_{m,\bullet}^{\mathcal{F}_{v_i},t})$, whereas $\tilde{S}_m(\mathcal{F}_i)$ is a right Riemann sum approximation of this integral, obtained by subdividing $[0,1]$ into m

subintervals of equal length. Thus the estimate for $\tilde{S}_m$ in (5.21) follows, since the functions $f_{i,m}(t)$ are decreasing, with $f_{i,m}(0) = 1$ and $f_{i,m}(1) \geq 0$.

By the uniform Fujita approximation results in Theorems 5.4.1 and 5.4.3, we have

$$\lim_{m\to\infty} \sup_i |T_m(v_i) - T(v_i)| = \lim_{m\to\infty} \sup_i |\tilde{S}_m(v_i) - \tilde{S}(v_i)| = 0.$$

Together with (5.21), this yields (2) and (3), and hence completes the proof. $\square$

Proof of Proposition 5.5.8. For $i \geq 1$, consider the $\mathbf{N}$-filtrations $\mathcal{F}_i := \mathcal{F}_{v_i,\mathbf{N}}$ associated to v_i. By Lemma 5.5.9, the assumptions of Proposition 5.5.7 are satisfied with $T = 1$. Hence we may find a filtration $\mathcal{F}$ such that

$$\operatorname{lct}(\mathfrak{b}_\bullet(\mathcal{F})) \leq A, \quad S(\mathcal{F}) \geq S \quad \text{and} \quad T(\mathcal{F}) = 1.$$

Using Theorem 3.4.10, we may choose a valuation $v^* \in \mathrm{Val}_X^*$ computing $\operatorname{lct}(\mathfrak{b}_\bullet(\mathcal{F}))$. After rescaling, we may assume $v^*(\mathfrak{b}_\bullet(\mathcal{F})) = 1$. Therefore,

$$A(v^*) = \frac{A(v^*)}{v^*(\mathfrak{b}_\bullet(\mathcal{F}))} = \operatorname{lct}(\mathfrak{b}_\bullet(\mathcal{F}) \leq A.$$

By Corollary 5.2.21, $S(v^*) \geq S(\mathcal{F}) \geq S$ and $T(v^*) \geq T(\mathcal{F}) = 1$. This completes the proof. $\square$

Proof of Theorem F. We first find a valuation computing $\alpha(L)$. Choose a sequence $(v_i)_i$ in Val_X^* such that

$$\lim_{i\to\infty} \frac{A(v_i)}{T(v_i)} = \inf_v \frac{A(v)}{T(v)} = \alpha(L).$$

After rescaling, we may assume $T(v_i) = 1$ for all i. Hence, the limit $A := \lim_{i\to\infty} A(v_i)$ exists and equals $\alpha(L)$. Further, by (5.6), the sequence $(S(v_i))_i$ is bounded from above and below away from zero, so after passing to a subsequence we may assume the limit $S := \lim_{i\to\infty} S(v_i)$ exists, and is finite and positive.

By Proposition 5.5.8, there exists $v^* \in \mathrm{Val}_X^*$ with $A(v^*) \leq A$ and $T(v^*) \geq 1$. Therefore,

$$\frac{A(v^*)}{T(v^*)} \leq A = \alpha(L).$$

Since $\alpha(L) = \inf_v A(v)/T(v)$, v^* computes $\alpha(L)$.

The argument for $\delta(L)$ is almost identical. Pick a sequence $(v_i)_i$ in Val_X^* such that

$$\lim_{i\to\infty} \frac{A(v_i)}{S(v_i)} = \inf_v \frac{A(v)}{S(v)} = \delta(L).$$

Again, we rescale our valuations so that $T(v_i) = 1$ for all $i \in \mathbf{N}$. As above, we may assume that the limit $S := \lim_{i\to\infty} S(v_i)$ exists, and is finite and positive. Therefore, $A := \lim_{i\to\infty} A(v_i)$ also exists and $A/S = \delta(L)$.

We apply Proposition 5.5.8 to find a valuation v^* such that $A(v^*) \le A$ and $S(v^*) \le S$. As argued for $\alpha(L)$, we see that v^* computes $\delta(L)$. $\square$

5.6 Toric case

Let $N \simeq \mathbf{Z}^n$ be a lattice and $M = \mathrm{Hom}(N, \mathbf{Z})$ the corresponding dual lattice. We consider a projective klt pair (X, B), where $X = X(\Delta)$ is given by a rational fan $\Delta \subset N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$ and B is a torus invariant $\mathbf{Q}$-divisor on X. Since X is proper, $|\Delta| = N_{\mathbf{R}}$.

As in Section 3.5, let $v_1, \dots, v_d$ denote the primitive generators of the one-dimensional cones in Δ and $D_1, \dots, D_d$ be the corresponding torus invariant divisors on X. Hence, there exist $b_i \in \mathbf{Q}_+$ so that $B = \sum_{i=1}^d b_i D_i$. A vector $v \in \sigma$ gives rise to a valuation on X that we slightly abusively also denote by v (see Section 3.5.1). The valuation associated to the point $v_i \in N_{\mathbf{R}}$ is ord_{D_i} for $1 \le i \le d$

We fix an ample line bundle of the form $L = \mathcal{O}_X(D)$, where $D = c_1 D_1 + \cdots + c_d D_d$ is a Cartier divisor on X. Associated to D is the convex polytope

$$P = P_D = \{u \in M_{\mathbf{R}} \mid \langle u, v_i\rangle \ge -c_i \text{ for all } 1 \le i \le d\}.$$

We write $\mathrm{Vert}\, P$ for the set of vertices in P.

Recall that there is a correspondence between points in $P \cap M_{\mathbf{Q}}$ and effective torus invariant $\mathbf{Q}$-divisors $\mathbf{Q}$-linearly equivalent to D, under which $u \in P \cap M_{\mathbf{Q}}$ corresponds to

$$D_u := D + \sum_{i=1}^{d} \langle u, v_i\rangle D_i := \sum_{i=1}^{d} (\langle u, v_i\rangle + c_i) D_i.$$

Note that if $m \in \mathbf{N}^*$ is chosen so that $mu \in N$, then $D_u = D + m^{-1}\mathrm{div}(\chi^{mu})$.

Let $\psi = \psi_D \colon N_{\mathbf{R}} \to \mathbf{R}$ be the concave function that is linear on the cones of Δ and satisfies $\psi(v_i) = -c_i$ for $1 \le i \le d$. On a given cone $\sigma \in \Delta$, the linear function is given by $\psi(v) = -\langle c(\sigma), v\rangle$, where $c(\sigma) \in M$ is such that $\chi^{c(\sigma)}$ is a local equation for D on $U_\sigma \subset X$. We have $\psi(v) = \min_{u\in P}\langle u, v\rangle = \min_{u \in \mathrm{Vert}\, P}\langle u, v\rangle$ for all $v \in N_{\mathbf{R}}$.

Lemma 5.6.1. *If $u \in P \cap M_{\mathbf{Q}}$ and $v \in N_{\mathbf{R}}$, then $v(D_u) = \langle u, v\rangle - \psi(v)$.*

Proof. Pick $m \in \mathbf{N}^*$ such that $mu \in M$. Since $D_u = D + m^{-1}\mathrm{div}(\chi^{mu})$, we have

$$v(D_u) = v(D) + m^{-1}v(\chi^{mu}) = v(D) + \langle u, v\rangle,$$

and we are left to show $v(D) = -\psi(v)$. Let $\sigma \in \Delta$ be the unique cone containing v in its interior. Since $\chi^{c(\sigma)}$ is a local equation for D on U_σ, we see

$$v(D) = v(\chi^{c(\sigma)}) = \langle c(\sigma), v\rangle = -\psi(v),$$

which completes the proof. □

5.6.1 Filtrations by toric valuations

Given $v \in N_{\mathbf{R}}$, we will describe the filtration $\mathcal{F}_v$ of $R(X, L)$ and compute both $S(v)$ and $T(v)$. Recall that for each $m \in \mathbf{N}^*$,

$$H^0(X, mL) = \bigoplus_{u \in mP \cap M} k \cdot \chi^u,$$

where the rational function χ^u is viewed as a section of $\mathcal{O}_X(mD)$.

Proposition 5.6.2. *For $\lambda \in \mathbf{R}_+$ and $m \in \mathbf{N}^*$ we have*

$$\mathcal{F}_v^\lambda H^0(X, mL) = \bigoplus_{\substack{u \in mP \cap M \\ \langle u, v\rangle - m\cdot\psi(v) \geq \lambda}} k \cdot \chi^u.$$

As a consequence, the set of jumping numbers of $\mathcal{F}_v$ along $H^0(X, mL)$ is equal to the set $\{\langle u, v\rangle - m \cdot \psi(v) \mid u \in mP \cap M\}$.

Proof. It suffices to prove that $s = \sum_{u \in mP \cap M} \alpha_u \chi^u \in H^0(X, mL)$, then

$$v(s) = \min\{\langle u, v\rangle - m \cdot \psi(v) \mid \alpha_u \neq 0\}.$$

To this end, pick $\sigma \in \Delta$ such that $v \in \mathrm{Int}(\sigma)$. Note that $\chi^{-mc(\sigma)}$ is a local generator for $\mathcal{O}_X(mD)$ on U_σ. By the definition of $v(s)$, and by (3.3), we therefore have

$$v(s) = v(\sum \alpha_u \chi^{u + mc(\sigma)}) = \min\{\langle u, v\rangle + m\langle c(\sigma), v\rangle \mid \alpha_u \neq 0\},$$

which completes the proof, since $\psi(v) = -\langle c(\sigma), v\rangle$. □

Proposition 5.6.3. *For $m \in \mathbf{N}^*$, we have*

$$S_m(v) = \langle \overline{u}_m, v\rangle - \psi(v) \quad \textit{and} \quad T_m(v) = \max_{u \in P \cap m^{-1}M} \langle u, v\rangle - \psi(v),$$

where $\overline{u}_m := (\sum_{u \in P \cap m^{-1}M} u)/\#(P \cap m^{-1}M)$ *is the barycenter of the set* $P \cap m^{-1}M$.

Proof. From the description of the jumping numbers of $\mathcal{F}_{v_u}$ in Proposition 5.6.2, we see

$$S_m(v) = \frac{\sum_{u \in mP \cap M} \langle u, v \rangle - m \cdot \psi(v)}{m\#(mP \cap M)} = \left\langle \frac{\sum_{u \in mP \cap M} u}{m\#(mP \cap M)}, v \right\rangle - \psi(v),$$

and

$$T_m(v) = \frac{\max_{u \in mP \cap M} \langle u, v \rangle}{m} - \psi(v).$$

Now, multiplication by m^{-1} gives an isomorphism $mP \cap M \to P \cap m^{-1}M$. Applying said isomorphism yields the desired equalities. $\square$

Corollary 5.6.4. *We have*

$$S(v) = \langle \overline{u}, v \rangle - \psi(v) \quad \textit{and} \quad T(v) = \max_{u \in P} \langle u, v \rangle - \psi(v) = \max_{u \in \operatorname{Vert}(P)} \langle u, v \rangle - \psi(v),$$

where $\overline{u}$ *denotes the barycenter of* P *and* $\operatorname{Vert}(P)$ *denotes the set of vertices of* P.

Remark 5.6.5. One can thus think of $T(v) = \max_{u \in P} \langle u, v \rangle - \min_{u \in P} \langle u, v \rangle$ as the width of P in the direction v, see also [Amb16, §3.2].

Proof of Corollary 5.6.4. The formula for $S(v)$ is immediate from Proposition 5.6.3 since $S(v) = \lim_{m \to \infty} S_m(v)$ and $\overline{u} = \lim_{m \to \infty} \overline{u}_m$. Similarly, $T(v) = \lim_{m \to \infty} T_m(v)$, and

$$\lim_{m \to \infty} \max_{u \in P \cap m^{-1}M} \langle u, v \rangle = \max_{u \in P} \langle u, v \rangle = \max_{u \in \operatorname{Vert} P} \langle u, v \rangle,$$

where the last equality holds by linearity of $u \mapsto \langle u, v \rangle$. This completes the proof. $\square$

Remark 5.6.6. The proof shows that $T_m(v) = T(v)$ for m sufficiently divisible.

5.6.2 Deformation to the initial filtration

Given a filtration $\mathcal{F}$ of $R(X, L)$, we will construct a degeneration of $\mathcal{F}$ to a filtration whose base ideals are T-invariant. We will use this construction to show $\alpha(L)$ and $\delta(L)$ may be computed using only toric valuations. Our argument is a global analogue of [Blu16b, §7], which in turns draws on [Mus02].

First write $R(X, L)$ as the coordinate ring of an affine toric variety. Set $M' := M \times \mathbf{Z}$, $N' := \operatorname{Hom}(M', \mathbf{Z})$, $M'_{\mathbf{R}} := M \otimes_{\mathbf{Z}} \mathbf{R}$, and $N'_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$. Let σ_0 denote the cone over $P \times \{1\} \subset M_{\mathbf{R}} \times \mathbf{R}$. Then there is a canonical isomorphism $k[\sigma_0 \cap M'] \simeq R(X, L)$.

We put a $\mathbf{Z}_+^{n+1}$ order on the monomials of $k[\sigma_0 \cap M']$ using an argument in [KK14, §7]. Choose $y_1, \ldots, y_{n+1} \in \sigma_0^\vee \cap N'$ that are linearly independent in $N'_{\mathbf{R}}$. Let $\rho\colon M' \to \mathbf{Z}^{n+1}$ denote the map defined by

$$\rho(u) = (\langle u, y_1 \rangle, \ldots, \langle u, y_{n+1} \rangle)\,.$$

Then ρ is injective and has image contained in $\mathbf{Z}_+^{n+1}$.

Endowing $\mathbf{Z}_+^{n+1}$ with the lexicographic order gives an order $>$ on the monomials in $k[\sigma_0 \cap M']$. Given an element $s \in k[\sigma_0 \cap M']$ the *initial term* of s, written $\mathrm{in}_>(s)$, is the greatest monomial in s with respect to the order $>$. Given a subspace W of $H^0(X, mL)$, we set

$$\mathrm{in}_>(W) = \mathrm{span}\{\mathrm{in}_>(s) \mid s \in W\},$$

where W is viewed as a vector subspace of $k[\sigma_0 \cap M']$. Clearly, $\mathrm{in}_>(W)$ is generated by monomials in $k[\sigma_0 \cap M']$. Therefore, $\mathfrak{b}\big(|\,\mathrm{in}_>(W)|\big)$ is a T-invariant ideal on X.

Proposition 5.6.7. *If W is a subspace of $H^0(X, mL)$, then* $\dim W = \dim \mathrm{in}_>(W)$.

Proof. By construction, there exists a basis of $\mathrm{in}_>(W)$ consisting of monomials $\chi^{u_1}, \ldots, \chi^{u_r}$, where $u_i \in \sigma_0 \cap M'$, and we may assume $\chi^{u_1} > \cdots > \chi^{u_r}$. For each $1 \le i \le r$, fix $s_i \in W$ such that $\mathrm{in}_>(s_i) = \chi^{u_i}$. We claim that $s_1, \ldots, s_r$ forms a basis for W.

To show that $s_1, \ldots, s_r$ are linearly independent, we argue by contradiction, so suppose $0 = \sum_{i=1}^r c_i s_i$, with $c \in k^r \setminus \{0\}$, and pick i_0 minimal with $c_{i_0} \neq 0$. Then $0 = \mathrm{in}_{>0}(\sum c_i s_i) = c_{i_0}\chi^{u_{i_0}}$, a contradiction.

Similarly, if $s_1, \ldots, s_r$ did not span W, then there would exist an element $s \in W \setminus \mathrm{span}\{s_1, \ldots, s_r\}$ with minimal initial term. Note that $\mathrm{in}_>(s) = c\chi^{u_i}$ for some $c \in k^*$ and $i \in \{1, \ldots, r\}$. Now, $s - cs_i \in W \setminus \mathrm{span}\{s_1, \ldots, s_r\}$, but has initial term strictly smaller than $\mathrm{in}(s)$. This contradicts the minimality assumption on $\mathrm{in}_>(s)$, and the proof is complete. □

To understand $\mathrm{lct}(\mathfrak{b}\big(|\,\mathrm{in}_> W|\big))$, we construct a 1-parameter degeneration of W to $\mathrm{in}_>(W)$ essentially following [Eis95, §15.8]. Choose elements $s_1, \ldots, s_r \in W$ such that

$$W = \mathrm{span}\{s_1, \ldots, s_r\} \quad \text{and} \quad \mathrm{in}_>(W) = \mathrm{span}\{\mathrm{in}_>(s_1), \ldots, \mathrm{in}_>(s_r)\}.$$

Next, we may fix an integral weight $\mu\colon \sigma_0 \cap M \to \mathbf{Z}_+$ such that $\mathrm{in}_{>_\mu}(s_i) = \mathrm{in}_>(s_i)$ for $1 \le i \le r$ [Eis95, Exercise 15.12]. Here $>_\mu$ denotes the weight order on $\mathbf{Z}^{n+1}$ induced by μ.

We write $k[\sigma_0 \cap M'][t]$ for the polynomial ring in one variable over $k[\sigma_0 \cap M']$. For $s = \sum \beta_u \chi^u \in k[\sigma_0 \cap M']$, we write $d = \max\{\mu(u) \mid \beta_m \neq 0\}$ and set

$$\tilde{s} := t^d \sum \beta_u t^{-\mu(u)} \chi^u.$$

Next, let $\tilde{W} \subset k[\sigma_0 \cap M'][t]$ denote the $k[t]$-submodule of $k[\sigma_0 \cap M'][t]$ generated by $\tilde{s}_1, \ldots, \tilde{s}_r$. Then $\tilde{W}$ gives a family of subspaces of $H^0(X, mL)$ over $\mathbf{A}^1$. For $c \in \mathbf{A}^1(k)$, write W_c for the corresponding subspace of $H^0(X, mL)$. Clearly $W_1 = W$ and $W_0 = \mathrm{in}_>(W)$.

Lemma 5.6.8. *For $c \in k^*$, $\mathrm{lct}(\mathfrak{b}(|W_c|)) = \mathrm{lct}(\mathfrak{b}(|W|))$.*

Proof. Consider the automorphism of $R(X, L)[t^{\pm 1}]$ defined by $\chi^u \mapsto t^{\mu(u)} \chi^u$ and $t \mapsto t$. Since $X \simeq \mathrm{Proj}(R(X, L))$, this automorphism of $R(X, L)[t^{\pm 1}]$ gives an automorphism $X \times (\mathbf{A}^1 \setminus \{0\})$ over $\mathbf{A}^1 \setminus \{0\}$. For $c \in k^*$, we write ϕ_c for the corresponding automorphism of X. Since ϕ_c^* sends W_c to W, we see $\mathrm{lct}(\mathfrak{b}(|W_c|)) = \mathrm{lct}(\mathfrak{b}(|W|))$. □

Proposition 5.6.9. *If W is a subspace of $H^0(X, mL)$, then $\mathrm{lct}(\mathfrak{b}(|\,\mathrm{in}_>(W)|)) \leq \mathrm{lct}(\mathfrak{b}(|W|))$.*

Proof. Combining Proposition 5.5.2 with Lemma 5.6.8, we see $\mathrm{lct}(\mathfrak{b}(|W_0|)) \leq \mathrm{lct}(\mathfrak{b}(|W|))$. Since $\mathrm{in}_>(W) = W_0$, the proof is complete. □

Let $\mathcal{F}$ be a filtration of $R(X, L)$. We write $\mathcal{F}_{\mathrm{in}}$ for the filtration defined by

$$\mathcal{F}_{\mathrm{in}}^{\lambda} H^0(X, mL) := \mathrm{in}_> \left(\mathcal{F}^{\lambda} H^0(X, mL) \right)$$

for all $\lambda \in \mathbf{R}_+$ and $m \in \mathbf{N}$. To see that $\mathcal{F}_{\mathrm{in}}$ is indeed a filtration, first note that conditions (F1)–(F3) of §5.1.3 are trivially satisfied. Condition (F4) follows from the equality $\mathrm{in}_>(s_1 s_2) = \mathrm{in}_>(s_1)\,\mathrm{in}_>(s_2)$ for $s_1, s_2 \in R(X, L)$.

Proposition 5.6.10. *With the above setup, we have*

$$S(\mathcal{F}_{in}) = S(\mathcal{F}), \quad T(\mathcal{F}_{in}) = T(\mathcal{F}), \quad \textit{and} \quad \mathrm{lct}(\mathfrak{b}_\bullet(\mathcal{F}_{\mathrm{in}})) \leq \mathrm{lct}(\mathfrak{b}_\bullet(\mathcal{F})).$$

Proof. By Proposition 5.6.7, $\mathcal{F}$ and $\mathcal{F}_{\mathrm{in}}$ have identical jumping numbers. Thus, $S(\mathcal{F}) = S(\mathcal{F}_{\mathrm{in}})$ and $T(\mathcal{F}) = T(\mathcal{F}_{\mathrm{in}})$. By Proposition 5.6.9, $\mathrm{lct}(\mathfrak{b}_{p,m}(\mathcal{F}_{\mathrm{in}})) \leq \mathrm{lct}(\mathfrak{b}_{p,m})(\mathcal{F})$ for $m, p \in \mathbf{N}$. Letting $m \to \infty$, we get $\mathrm{lct}(\mathfrak{b}_p(\mathcal{F}_{\mathrm{in}})) \leq \mathrm{lct}_p(\mathfrak{b}_\bullet(\mathcal{F}))$ for all $p \in \mathbf{N}$, and hence $\mathrm{lct}(\mathfrak{b}_\bullet(\mathcal{F}_{\mathrm{in}})) \leq \mathrm{lct}(\mathfrak{b}_\bullet(\mathcal{F}))$. □

Proposition 5.6.11. *If w is a nontrivial valuation on X with $A(w) < \infty$, then there exists $v \in N_{\mathbf{R}} \setminus \{0\}$ such that*

$$A(v) \leq A(w), \quad T(v) \geq T(w), \quad \text{and} \quad S(v) \geq S(w).$$

Proof. Let $\mathcal{F}_{w,\mathrm{in}}$ denote the initial filtration of $\mathcal{F}_w$. Then $\mathfrak{b}_\bullet(\mathcal{F}_{w,\mathrm{in}})$ is a graded sequence of T-invariant ideals on X. Further, Proposition 5.6.10 shows that

$$\mathrm{lct}(\mathfrak{b}_\bullet(\mathcal{F}_{w,\mathrm{in}})) \leq \mathrm{lct}(\mathfrak{b}_\bullet(\mathcal{F}_w)) = \mathrm{lct}(\mathfrak{a}_\bullet(w)) \leq A(w) < \infty,$$

where the first equality Lemma 5.2.19, and the second inequality is Lemma 3.4.9.

Therefore, $\mathfrak{b}_\bullet(\mathcal{F}_{w,\mathrm{in}})$ is a nontrivial graded sequence. Proposition 3.5.2 yields a nontrivial toric valuation $v \in N_{\mathbf{R}}$ that computes $\mathrm{lct}(\mathfrak{b}_\bullet(\mathcal{F}_{w,\mathrm{in}}))$. After rescaling v, we may assume $v(\mathfrak{b}_\bullet(\mathcal{F}_{w,\mathrm{in}})) = 1$, and, thus, $A(v) = \mathrm{lct}(\mathfrak{b}_\bullet(\mathcal{F}_{w,\mathrm{in}}))$. We then have

$$A(v) = \mathrm{lct}(\mathfrak{b}_\bullet(\mathcal{F}_{w,\mathrm{in}})) \leq \mathrm{lct}(\mathfrak{b}_\bullet(\mathcal{F}_w)) = \mathrm{lct}(\mathfrak{a}_\bullet(w)) \leq A(w),$$

Next,

$$S(v) \geq S(\mathcal{F}_{w,\mathrm{in}}) = S(\mathcal{F}_w) = S(w),$$

where the inequality is Corollary 5.2.21 and the following equality is Proposition 5.6.10. A similar argument gives $T(v) \geq T(w)$ and completes the proof. □

Corollary 5.6.12. *We have the following equalities*

$$\alpha(L) = \inf_{v \in N_{\mathbf{R}} \setminus \{0\}} \frac{A(v)}{T(v)} \quad \text{and} \quad \delta(L) = \inf_{v \in N_{\mathbf{R}} \setminus \{0\}} \frac{A(v)}{S(v)}$$

Proof. This is clear from Theorem C and Proposition 5.6.11. □

5.6.3 Proof of Theorem G

We now consider the log canonical and stability thresholds of L. The following result is slightly more precise than Theorem G in the introduction.

Corollary 5.6.13. *We have*

$$\alpha(L) = \min_{u \in \mathrm{Vert}(P)} \mathrm{lct}(D_u) = \min_{u \in \mathrm{Vert}(P)} \min_{i=1,\ldots,d} \frac{1 - b_i}{\langle u, v_i \rangle + c_i} \tag{5.22}$$

and

$$\delta(L) = \mathrm{lct}(D_{\overline{u}}) = \min_{i=1,\ldots,d} \frac{1 - b_i}{\langle \overline{u}, v_i \rangle + c_i}, \tag{5.23}$$

where $\overline{u}$ denotes the barycenter of P and $\mathrm{Vert}(P)$ *the set of vertices of P. Furthermore, $\alpha(L)$ (resp. $\delta(L)$) is computed by one of the valuations $v_1, \dots, v_d$.*

Proof. Again, we will only prove the half of the corollary that concerns $\alpha(L)$. First, we combine Lemma 5.6.1, Corollary 5.6.4 and Corollary 5.6.12 to see

$$\alpha(L) = \inf_{v \in N_{\mathbf{R}} \setminus \{0\}} \min_{u \in \mathrm{Vert}(P)} \frac{A(v)}{v(D_u)} = \min_{u \in \mathrm{Vert}(P)} \inf_{v \in N_{\mathbf{R}} \setminus \{0\}} \frac{A(v)}{v(D_u)}.$$

Applying Proposition 3.5.3 to the previous expression yields (5.22).

Next, pick $u \in \mathrm{Vert}(P)$ and $i \in \{1, \dots, d\}$ such that $\alpha(L) = (1 - b_i)/(\langle u, v_i \rangle + c_i)$. Then we have $A(v_i)/T(v_i) = 1/(\langle u, v_i \rangle + c_i)$, so v_i computes $\alpha(L)$. □

5.6.4 The Fano case

Finally we consider the case when X is a toric $\mathbf{Q}$-Fano variety, that is, $-K_X$ is an ample $\mathbf{Q}$-Cartier divisor.

Corollary 5.6.14. *A toric $\mathbf{Q}$-Fano variety is K-semistable iff the barycenter of the polytope associated to $-K_X$ is equal to the origin.*

For smooth X, this result was proved by analytic methods in [BB13, Berm16], even with K-semistable replaced by K-polystable. In the general case, it can be deduced from [LiX16, Theorem 1.4].

Proof. We apply (5.23) with $b_i = 0$ and $c_i = 1$ for all i. If $\overline{u} = 0$, then $\delta(-K_X) = 1$, which by Theorem D implies that X is K-semistable. Now suppose $\overline{u} \neq 0$. Then $\langle \overline{u}, v_i \rangle < 0$ for some i, or else all the v_i would lie in a half-space, which is impossible since $|\Delta| = N_{\mathbf{R}}$. It then follows from (5.23) that $\delta(-K_X) < 1$, so by Theorem D, X is not K-semistable. □

Remark 5.6.15. The proof shows that if X is K-semistable, any toric valuation computes $\delta(-K_X) = 1$.

We now give a simple formula for $\delta(-K_X)$ in the $\mathbf{Q}$-Fano case. When X is smooth, the formula for agrees with the formula in [Li11] for $R(X)$, the greatest lower bound on the Ricci curvature of X, as defined and studied in [Tia92, Szé11].

Corollary 5.6.16. *Let X be a toric $\mathbf{Q}$-Fano variety and $\overline{u}$ denote the barycenter of the polytope $P_{-K_X} := \{u \in M_{\mathbf{R}} \mid \langle u, v_i \rangle \geq -1$ for all $1 \leq i \leq d\}$.*

(i) If X is K-semistable, then $\delta(-K_X) = 1$.

(ii) If X is not *K-semistable, then*

$$\delta(-K_X) = \frac{c}{1+c}$$

where $c > 0$ is the greatest real number such that $-c\overline{u}$ lies in P_{-K_X}.

Proof. Statement (i) follows from (5.23) and Corollary 5.6.14. For (ii), we claim that

$$0 < \langle \overline{u}, v_i \rangle + 1 \leq 1/c + 1$$

for all $i = 1, \ldots, d$ and equality holds in the last inequality for some i. Statement (ii) follows from the claim and (5.23).

We now prove the claim. Since $\overline{u}$ lies in P_{K_X}, $\langle \overline{u}, v_i \rangle \geq -1$ for all i. Since $-c\overline{u}$ lies on the boundary of P_{K_X},

$$c\langle \overline{u}, v_i \rangle = \langle -c\overline{u}, v_i \rangle \geq -1$$

for all i and equality holds in the last inequality for some i. This completes the proof. $\square$

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